Introduction to Probability

Lecture 11: Estimators (Part II)

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Outline

Estimating Population Size (First Model)

Mean Squared Error

Estimating Population Size (Second Model)

Estimating Population Size (First Model)

- Suppose we have a sample of a few serial numbers (IDs) of some product
- We assume IDs are running from 1 to an unknown parameter N (so $N = \theta$)
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  - their number must satisfy $n \leq N$

First Estimator Based on Sample Mean

Example 1

Construct an unbiased estimator T_1 using the sample mean.

Answer

Example: Odd Behaviour of T_1

- Suppose $n = 5$

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Challenging exercise: Find a lower bound on $\mathbf{P}[\, T_1 < \max(X_1, X_2, \ldots, X_n) \,]$

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Challenging exercise: Find a lower bound on $\mathbf{P}[\, T_1 < \max(X_1, X_2, \ldots, X_n) \,]$

- Achieving unbiasedness alone is not a good strategy
- Improvement: find an estimator which always returns a value at least $\max(X_1, X_2, \ldots, X_n)$

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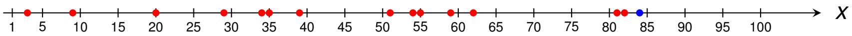

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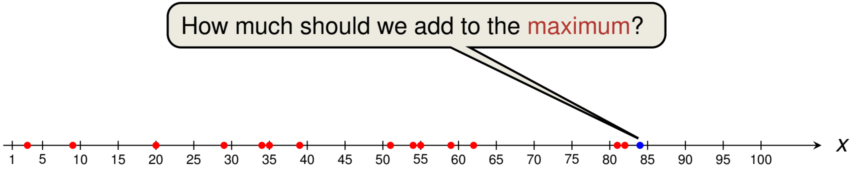

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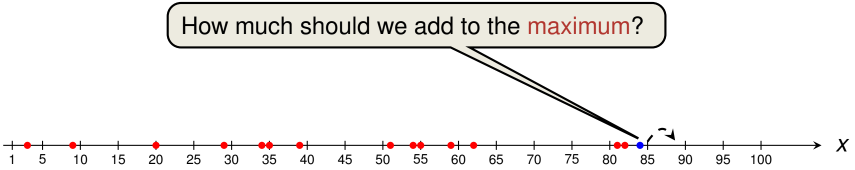

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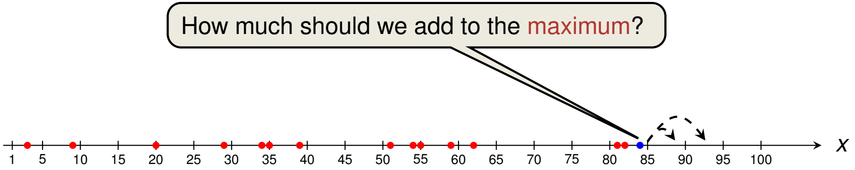

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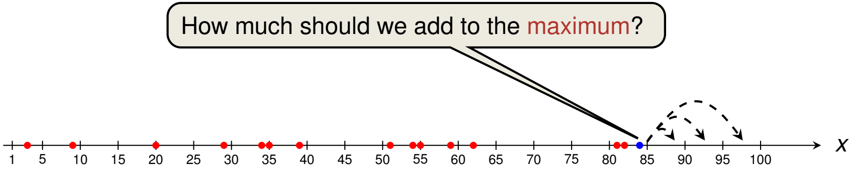

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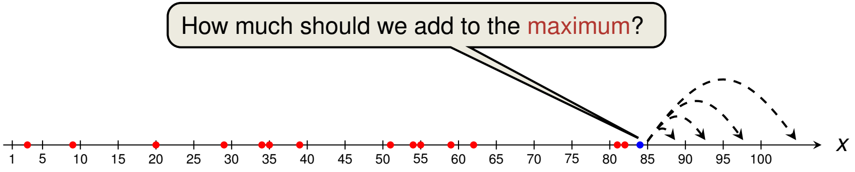

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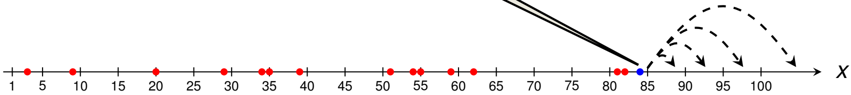
How much should we add to the maximum?



Rearrange the other 14 points equi-spaced between 0 and 84.

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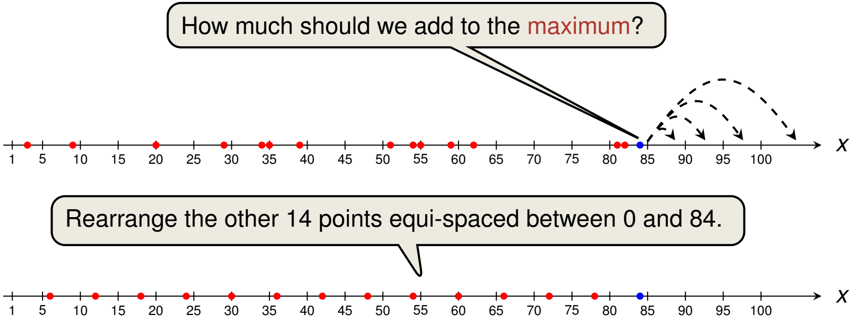

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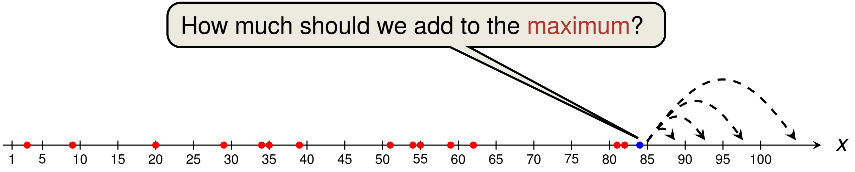

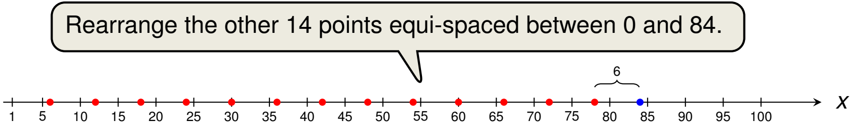

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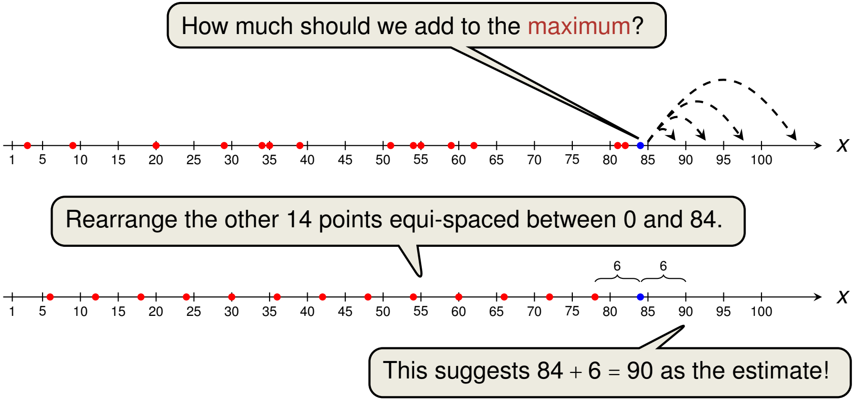

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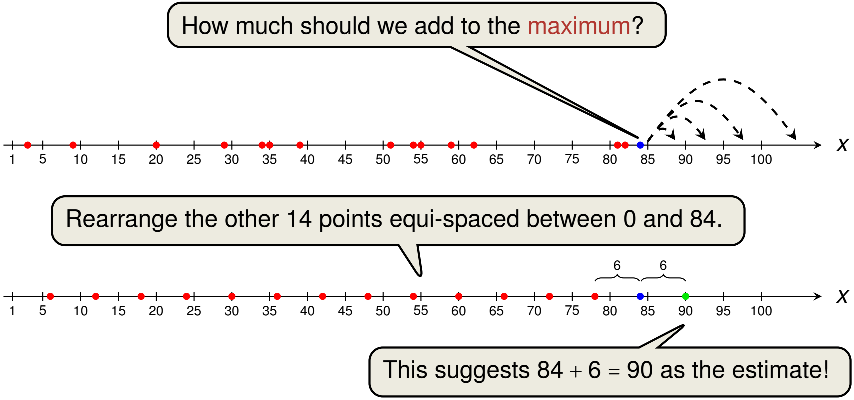

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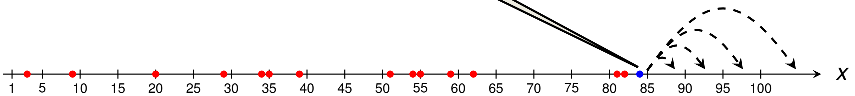

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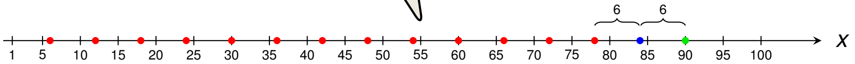


$$\max(X_1, \ldots, X_n) + \frac{\max(X_1, \ldots, X_n)}{n-1}$$

This suggests $84 + 6 = 90$ as the estimate!

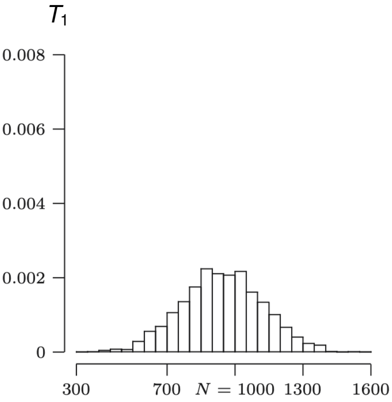
Deriving the Estimator Based on Maximum Sample

Example 2

Construct an unbiased estimator T_2 using $\max(X_1, \ldots, X_n)$

Answer

Empirical Analysis of the two Estimators



Source: Modern Introduction to Statistics

Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

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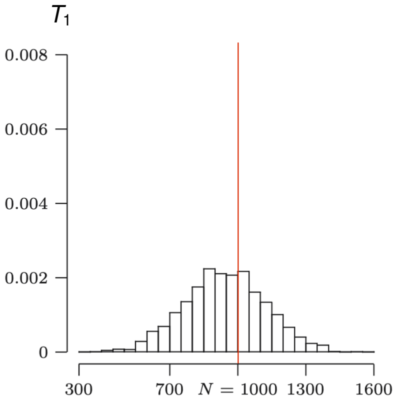

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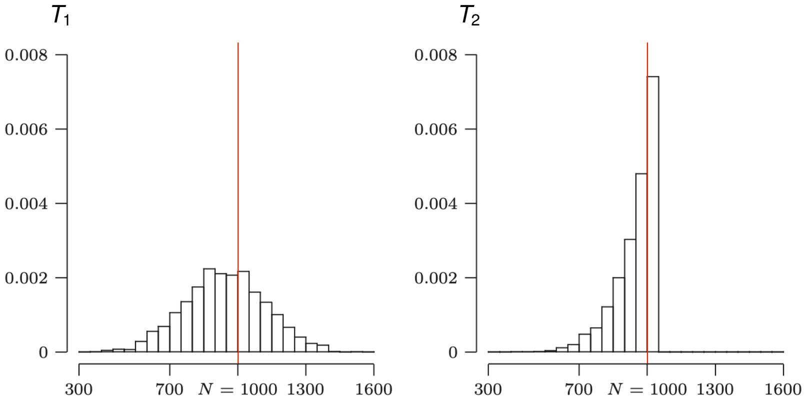

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Can we find a quantity that captures the superiority of T_2 over T_1?

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Let T be an estimator for a parameter θ. The mean squared error of T is

$$\mathbf{MSE}[\,T\,] = \mathbf{E}\left[\,(T - \theta)^2\,\right].$$

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- If T_1 and T_2 are both unbiased, T_1 is better than T_2 iff $\mathbf{V}[\,T_1\,] < \mathbf{V}[\,T_2\,]$.

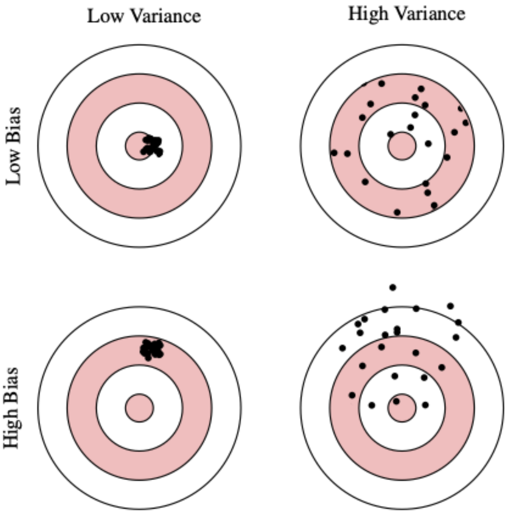
Bias-Variance Decomposition: Derivation

Example 3

We need to prove: $\mathbf{MSE}[\,T\,] = (\mathbf{E}[\,T\,] - \theta)^2 + \mathbf{V}[\,T\,]$.

Answer

Bias-Variance Decomposition: Illustration



Source: Edwin Leuven (Point Estimation)

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It holds that **MSE** $[\ T_2\ ] = \Theta\left(\frac{N^2}{n^2}\right)$, where $T_2 = \frac{n+1}{n} \cdot \max(X_1, \ldots, X_n) - 1$.

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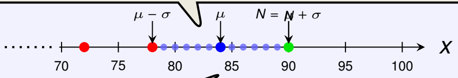
- T_2 is unbiased $\Rightarrow$ need $\mathbf{V}[\,T_2\,]$ which reduces to $\mathbf{V}[\,\max(X_1, \ldots, X_n)\,]$
- One can prove:

For details see Dekking et al.

$$\mathbf{V}[\,\max(X_1, \ldots, X_n)\,] = \cdots = \frac{n(N+1)(N-n)}{(n+2)(n+1)^2} = \Theta\left(\frac{N^2}{n^2}\right)$$

Equi-spaced (idealised) configuration suggests a standard deviation of $\sigma \approx \frac{N}{n}$



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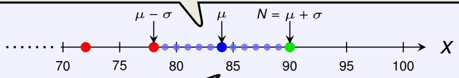

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- $\mathbf{MSE}[\,T_2\,]$ is much lower than $\mathbf{MSE}[\,T_1\,] = \Theta\left(\frac{N^2}{n}\right)$, i.e., $\frac{\mathbf{MSE}[\,T_1\,]}{\mathbf{MSE}[\,T_2\,]} = \frac{n+2}{3}$
- $\Rightarrow$ confirms simulations suggesting that T_2 is better than T_1!
- can be shown T_2 is the best unbiased estimator, i.e., it minimises MSE.

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Let us call this a **collision**

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Similar idea applies to situations where elements are not labelled before we see them first time (Mark & Recapture Method)

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- We take uniform samples from S with replacement
- **Goal:** Find estimator for N

- Suppose $n = 6$, $N = 11$, $S = \{3, 4, 7, 8, 10, 15.83356, 20, 21, 56, 81, 10000\}$
- Let the sample be

$$10, 81, 20, 3, 81, 10000$$

Let us call this a **collision**

As we do not know S, our only clue are elements that **were sampled twice.**

Birthday Problem

Birthday Problem: Given a set of k people

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- What is the probability of having two with the same birthday (i.e., having at least one collision)?

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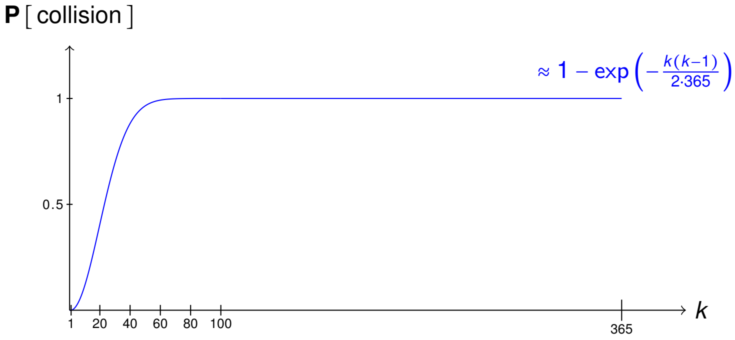

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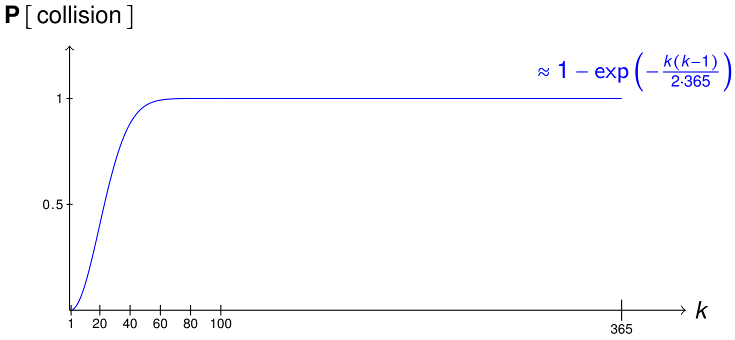
Birthday Problem: Given a set of k people

- What is the probability of having two with the same birthday (i.e., having at least one collision)?
- What is the expected number of people one needs to ask until the first collision occurs?



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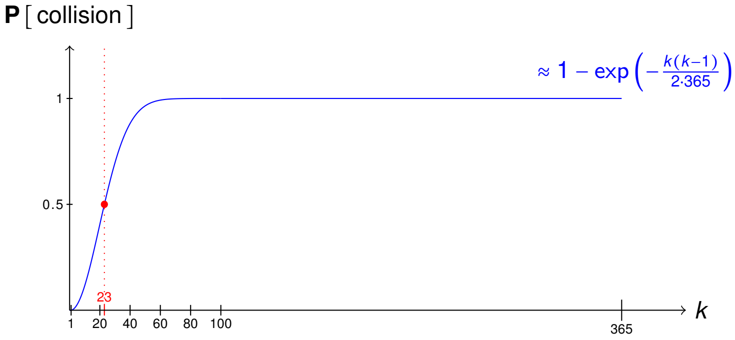

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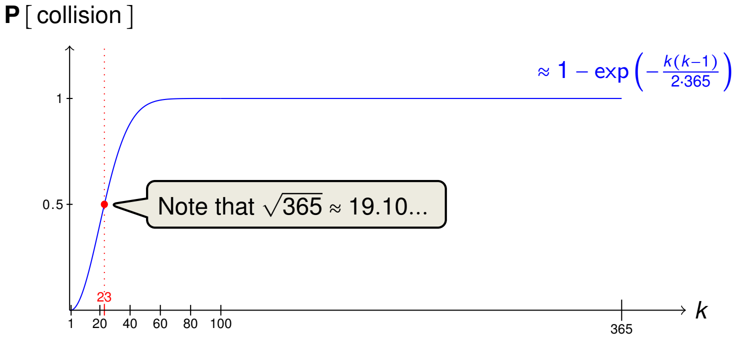

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Recall: As we do not know S, our only information are **collisions**.

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$C = \emptyset$
For $i = 1, 2, \ldots$
  Take next i.i.d. sample X_i from S
  If $X_i \notin C$ **then** $C \leftarrow C \cup \{X_i\}$
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Exercise: Prove a bound of $\leq 2 \cdot \sqrt{N}$

Estimation via Collision: Getting the Estimator Unbiased

Example 6

One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[\,T\,] = |S|$ for any finite, non-empty set S.

Answer