Introduction to Probability

Lecture 11: Estimators (Part II)

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Outline

Estimating Population Size (First Model)

Mean Squared Error

Estimating Population Size (Second Model)

Estimating Population Size (First Model)

- Suppose we have a sample of a few serial numbers (IDs) of some product
- We assume IDs are running from 1 to an unknown parameter N (so $N = \theta$)
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  - their number must satisfy $n \leq N$

First Estimator Based on Sample Mean

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Challenging exercise: Find a lower bound on $\mathbf{P}[\, T_1 < \max(X_1, X_2, \ldots, X_n) \,]$

- Achieving unbiasedness alone is not a good strategy
- Improvement: find an estimator which always returns a value at least $\max(X_1, X_2, \ldots, X_n)$

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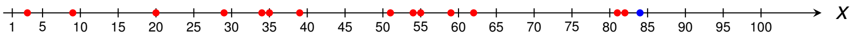

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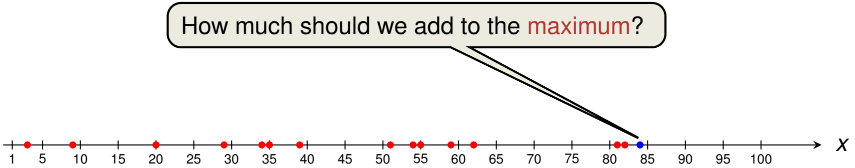

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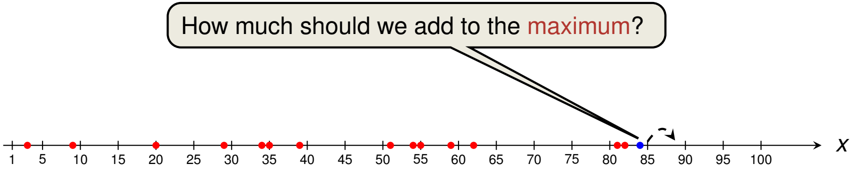

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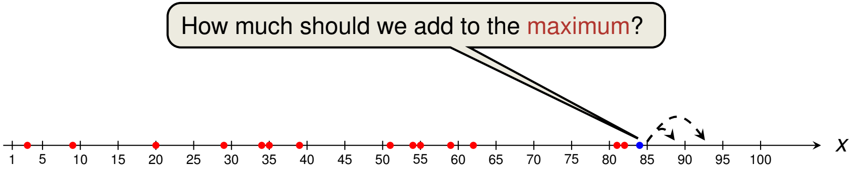

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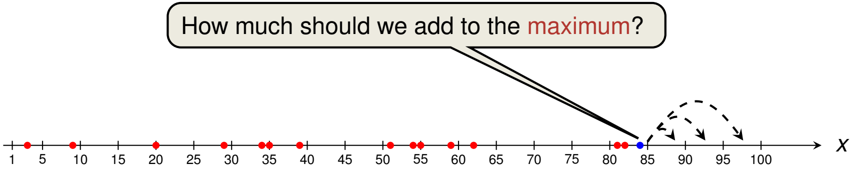

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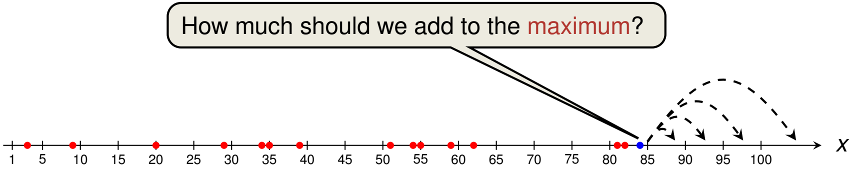

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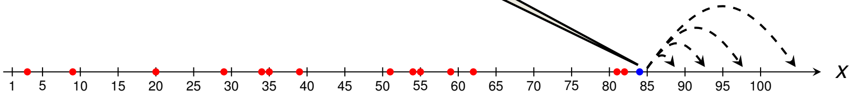

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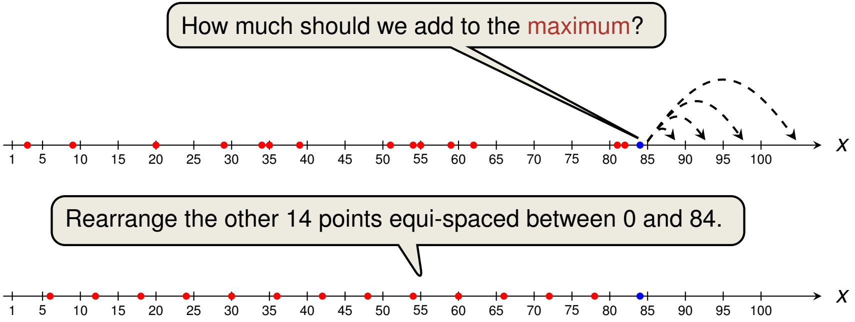

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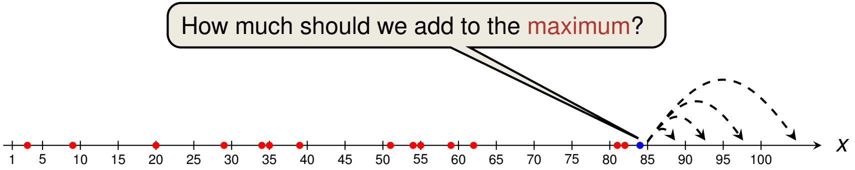

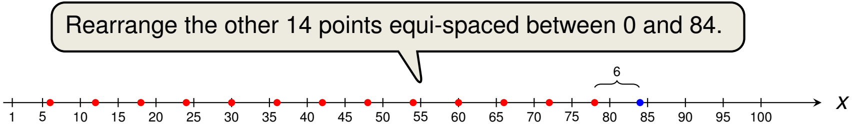

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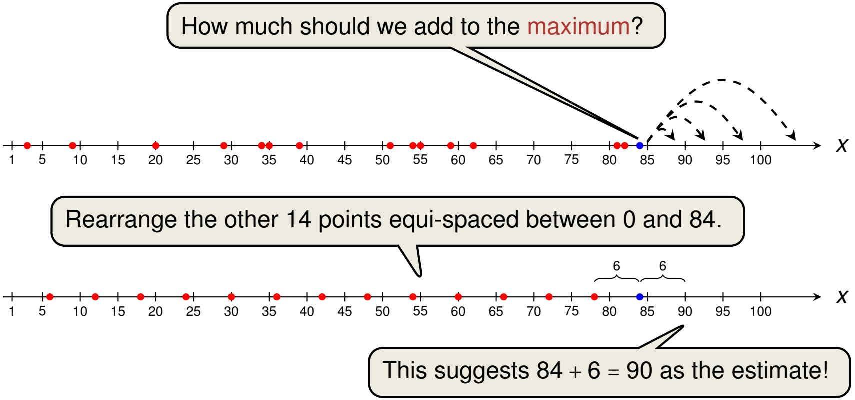

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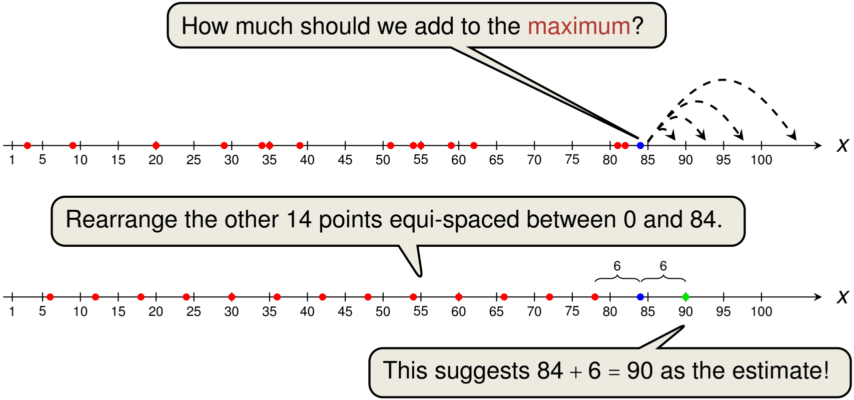

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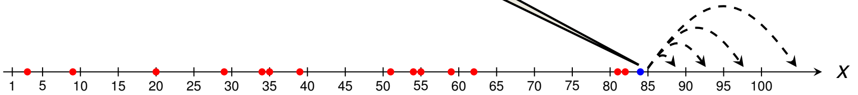

Rearrange the other 14 points equi-spaced between 0 and 84.

$\max(X_1, \ldots, X_n) + \frac{\max(X_1,\ldots,X_n)}{n-1}$

This suggests $84 + 6 = 90$ as the estimate!

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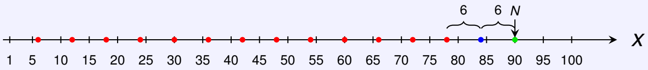

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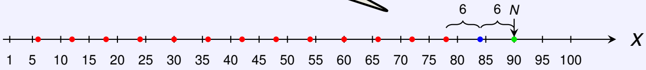

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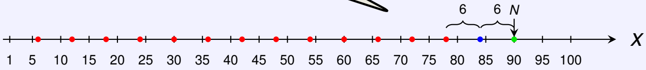

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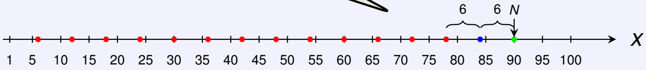

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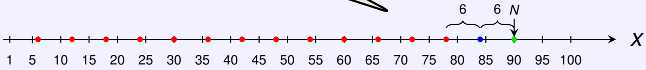

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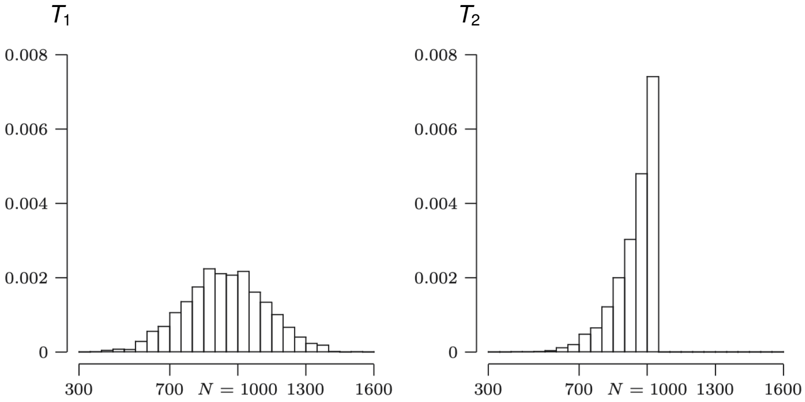
- For our samples before, we get $t_2 = \frac{16}{15} \cdot 84 - 1 = 88.6$.

Empirical Analysis of the two Estimators



Source: Modern Introduction to Statistics

Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

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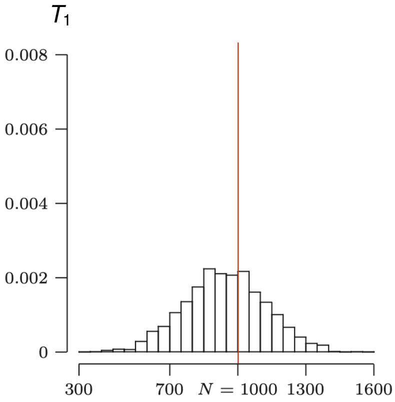

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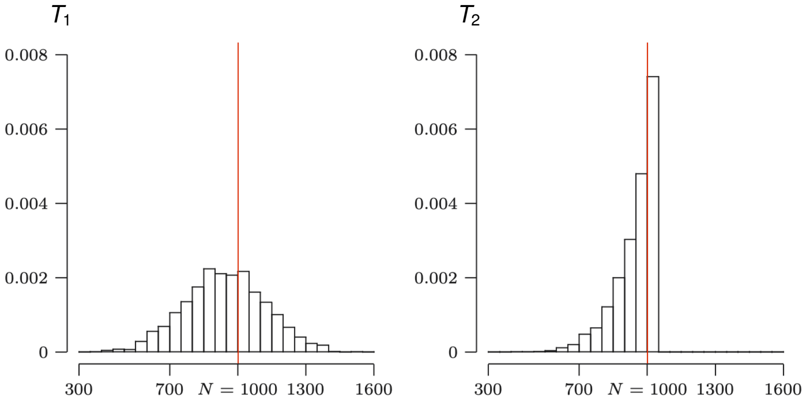

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Can we find a quantity that captures the superiority of T_2 over T_1?

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- If T_1 and T_2 are both unbiased, T_1 is better than T_2 iff $\mathbf{V}[\,T_1\,] < \mathbf{V}[\,T_2\,]$.

Bias-Variance Decomposition: Derivation

Example 3

We need to prove: $\mathbf{MSE}[T] = (\mathbf{E}[T] - \theta)^2 + \mathbf{V}[T]$.

Answer

Bias-Variance Decomposition: Derivation

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We need to prove: $\textbf{MSE}[\,T\,] = (\textbf{E}[\,T\,] - \theta)^2 + \textbf{V}[\,T\,]$.

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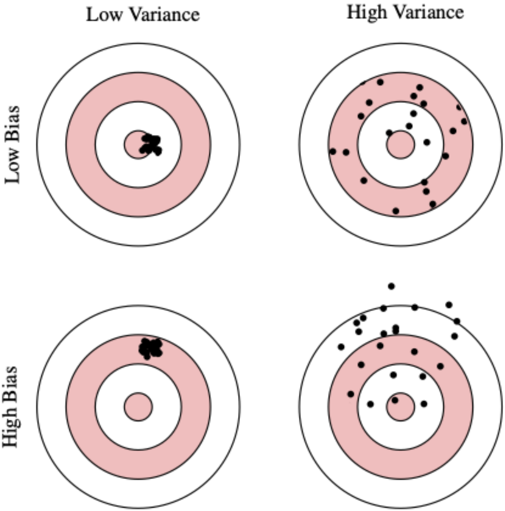
Bias-Variance Decomposition: Illustration



Source: Edwin Leuven (Point Estimation)

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It holds that **MSE** $[\, T_1 \,] = \Theta\left(\frac{N^2}{n}\right)$, where $T_1 = 2 \cdot \overline{X}_n - 1$.

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- Rearranging and simplifying gives

$$\mathbf{V}[\, T_1 \,] = \frac{(N+1)(N-n)}{3n}.$$

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It holds that **MSE** $[\, T_2 \,] = \Theta\left(\frac{N^2}{n^2}\right)$, where $T_2 = \frac{n+1}{n} \cdot \max(X_1, \ldots, X_n) - 1$.

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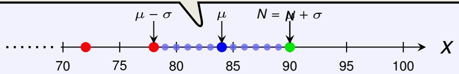

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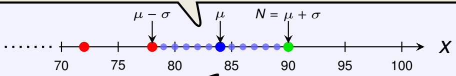

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- $\mathbf{MSE}[\,T_2\,]$ is much lower than $\mathbf{MSE}[\,T_1\,] = \Theta\left(\frac{N^2}{n}\right)$, i.e., $\frac{\mathbf{MSE}[\,T_1\,]}{\mathbf{MSE}[\,T_2\,]} = \frac{n+2}{3}$
- $\Rightarrow$ confirms simulations suggesting that T_2 is better than T_1!
- can be shown T_2 is the best unbiased estimator, i.e., it minimises MSE.

Outline

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Similar idea applies to situations where elements are not labelled before we see them first time (Mark & Recapture Method)

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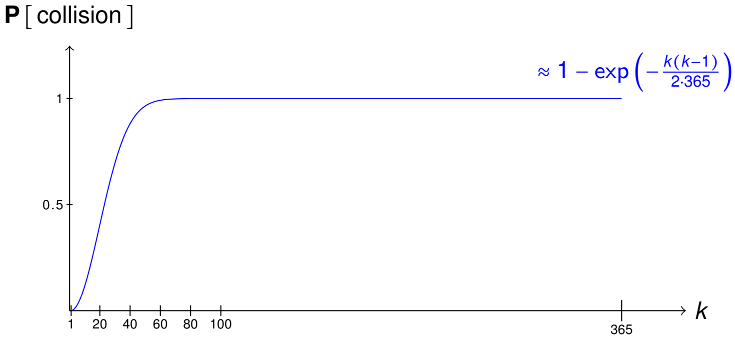

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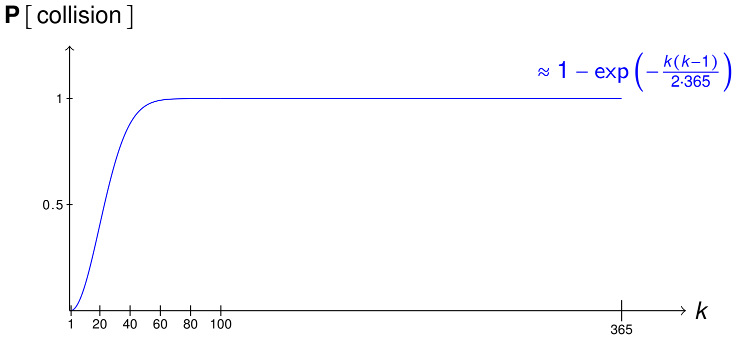

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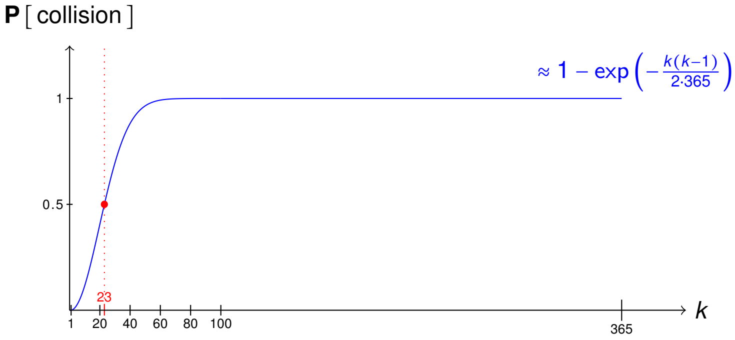

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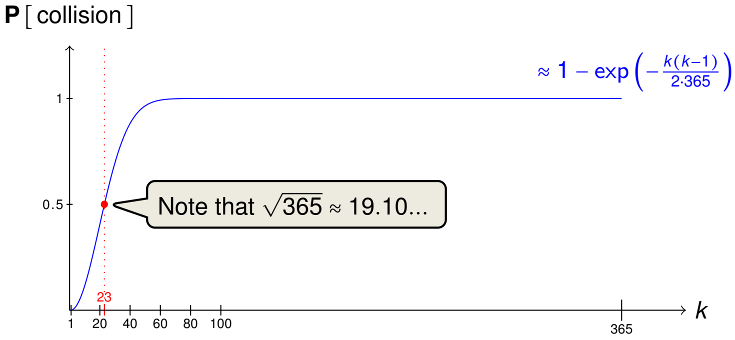

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FIND-FIRST-COLLISION(S)
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For i = 1, 2, ...
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Exercise: Prove a bound of $\leq 2 \cdot \sqrt{N}$

Estimation via Collision: Getting the Estimator Unbiased

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One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[\,T\,] = |S|$ for any finite, non-empty set S.

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Answer

- We outline a construction by induction.
- Case $|S| = 1$: Algo always stops after $i = 2$ rounds and returns $T(2)$.

Estimation via Collision: Getting the Estimator Unbiased

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One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[T] = |S|$ for any finite, non-empty set S.

Answer

- We outline a construction by induction.
- Case $|S| = 1$: Algo always stops after $i = 2$ rounds and returns $T(2)$. We want

$$1 = \mathbf{E}[T] = T(2)$$

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- Case $|S| = 3$: gives $3 = \mathbf{E}[T] = \frac{1}{3} \cdot T(2) + \frac{4}{9} \cdot T(3) + \frac{2}{9} \cdot T(4)$ $\Rightarrow T(4) = 6$, similarly, $T(5) = 10$ etc.

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- can continue to define $T(i)$ inductively in this way (note T is unique) (a proof that $T(i) = \binom{i}{2}$ is harder)