Introduction to Probability

Lecture 11: Estimators (Part II)

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Easter 2025

Compiled: May 18, 2025 at 19:51

Outline

Estimating Population Size (First Model)

Mean Squared Error

Estimating Population Size (Second Model)

Estimating Population Size (First Model)

- Suppose we have a sample of a few serial numbers (IDs) of some product
- We assume IDs are running from 1 to an unknown parameter N (so $N = \theta$)
- Each of the IDs is drawn without replacement from the discrete uniform distribution over $\{1, 2, \ldots, N\}$
- This is also known as Tank Estimation Problem or (Discrete) Taxi Problem

$$7, 3, 10, 46, 14$$

Warning

- As before, we denote the samples $X_1, X_2, \ldots, X_n$
- Since sampling is without replacement:
  - they are not independent! (but identically distributed)
  - their number must satisfy $n \leq N$

First Estimator Based on Sample Mean

Example 1

Construct an unbiased estimator T_1 using the sample mean.

Answer

- The sample mean is

$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

- Linearity of expectation applies (even for dependent random var.!):

$$\begin{aligned} \mathbf{E}\left[\,\overline{X}_n\,\right] &= \frac{n \cdot \mathbf{E}\left[\,X_1\,\right]}{n} = \mathbf{E}\left[\,X_1\,\right] \\ &= \sum_{i=1}^{N} i \cdot \frac{1}{N} = \frac{N+1}{2}. \end{aligned}$$

- Thus we obtain an unbiased estimator by

$$T_1 := 2 \cdot \overline{X}_n - 1.$$

Example: Odd Behaviour of T_1

- Suppose $n = 5$
- Let the sample be

$$7, 3, 10, 46, 14$$

- The estimator returns:

$$T_1 = 2 \cdot \overline{X}_n - 1 = 2 \cdot \frac{80}{5} - 1 = 31 \quad ☹$$

This estimator will often unnecessarily underestimate the true value N.

Challenging exercise: Find a lower bound on $\mathbf{P}[\, T_1 < \max(X_1, X_2, \ldots, X_n) \,]$

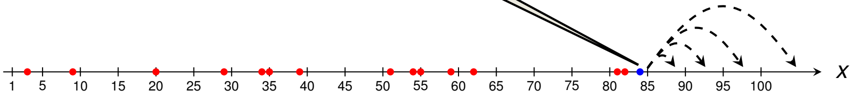
- Achieving unbiasedness alone is not a good strategy
- Improvement: find an estimator which always returns a value at least $\max(X_1, X_2, \ldots, X_n)$

Intuition: Constructing an Estimator based on Maximum Sample

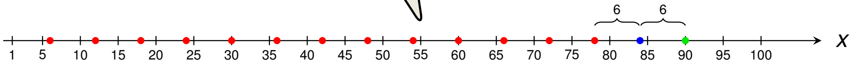
- Suppose $n = 15$
- Our samples are:

$$9, 82, 39, 35, 20, 51, 54, 62, 81, 29, {\color{blue}84}, 59, 3, 34, 55$$

How much should we add to the maximum?



Rearrange the other 14 points equi-spaced between 0 and 84.



$\max(X_1, \ldots, X_n) + \frac{\max(X_1,\ldots,X_n)}{n-1}$

This suggests $84 + 6 = 90$ as the estimate!

Deriving the Estimator Based on Maximum Sample

Example 2

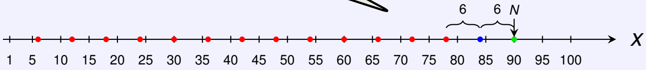
Construct an unbiased estimator T_2 using $\max(X_1, \ldots, X_n)$

Answer

- Calculate expectation of the maximum (for details see Dekking et al.)

$$\mathbf{E}\left[\max(X_1, \ldots, X_n)\right] = \ldots = \frac{n}{n+1} \cdot N + \frac{n}{n+1} = \frac{n}{n+1} \cdot (N+1).$$

Equi-spaced configuration would suggest $\max(X_1, \ldots, X_n) \approx \frac{n-1}{n} \cdot N$

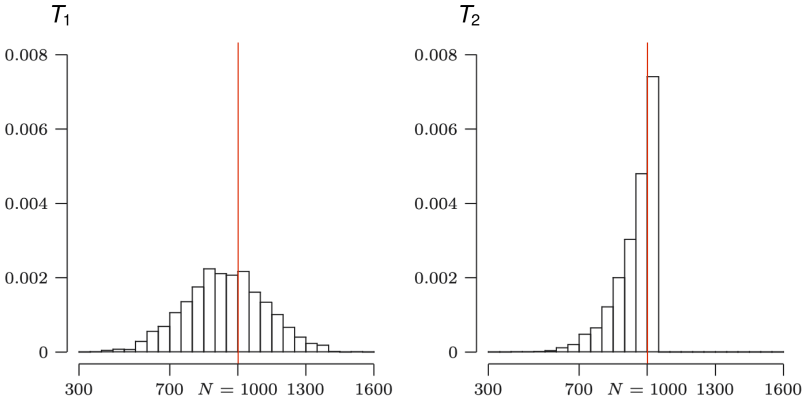


- Hence we obtain an unbiased estimator by

$$T_2 := \frac{n+1}{n} \cdot \max(X_1, \ldots, X_n) - 1.$$

- For our samples before, we get $t_2 = \frac{16}{15} \cdot 84 - 1 = 88.6$.

Empirical Analysis of the two Estimators



Source: Modern Introduction to Statistics

Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

Can we find a quantity that captures the superiority of T_2 over T_1?

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Mean Squared Error

Estimating Population Size (Second Model)

Mean Squared Error

Mean Squared Error Definition

Let T be an estimator for a parameter θ. The mean squared error of T is

$$\mathbf{MSE}\left[\,T\,\right] = \mathbf{E}\left[\,(T - \theta)^2\,\right].$$

- According to this, estimator T_1 better than T_2 if $\mathbf{MSE}\left[\,T_1\,\right] < \mathbf{MSE}\left[\,T_2\,\right]$.

Bias-Variance Decomposition

The mean squared error can be decomposed into:

$$\mathbf{MSE}\left[\,T\,\right] = \underbrace{(\mathbf{E}\left[\,T\,\right] - \theta)^2}_{=\,\text{Bias}^2} + \underbrace{\mathbf{V}\left[\,T\,\right]}_{=\,\text{Variance}}$$

- If T_1 and T_2 are both unbiased, T_1 is better than T_2 iff $\mathbf{V}\left[\,T_1\,\right] < \mathbf{V}\left[\,T_2\,\right]$.

Bias-Variance Decomposition: Derivation

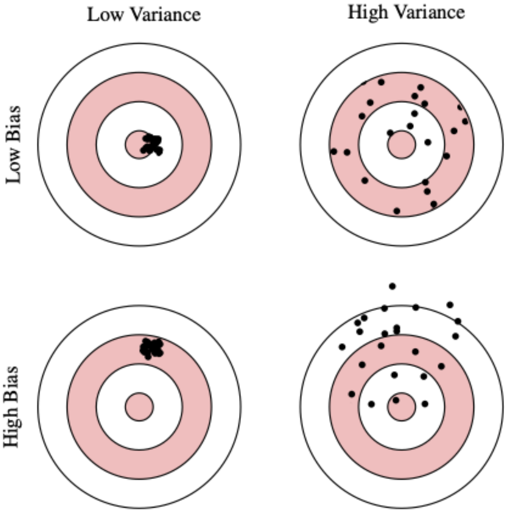
Example 3

We need to prove: $\mathbf{MSE}[\,T\,] = (\mathbf{E}[\,T\,] - \theta)^2 + \mathbf{V}[\,T\,]$.

Answer

$$\begin{aligned}
\mathbf{MSE}[\,T\,] &= \mathbf{E}\left[\,(T-\theta)^2\,\right] \\
&= \mathbf{E}\left[\,T^2 - 2T\theta + \theta^2\,\right] \\
&= \mathbf{E}[\,T\,]^2 - 2\cdot\mathbf{E}[\,T\,]\cdot\theta + \theta^2 + \mathbf{E}\left[\,T^2\,\right] - \mathbf{E}[\,T\,]^2 \\
&= (\mathbf{E}[\,T\,] - \theta)^2 + \mathbf{V}[\,T\,].
\end{aligned}$$

Bias-Variance Decomposition: Illustration



Source: Edwin Leuven (Point Estimation)

Example 4

It holds that $\mathbf{MSE}[\,T_1\,] = \Theta\left(\frac{N^2}{n}\right)$, where $T_1 = 2 \cdot \overline{X}_n - 1$.

Answer

- Since T_1 is unbiased, $\mathbf{MSE}[\,T_1\,] = (\mathbf{E}[\,T_1\,] - \theta)^2 + \mathbf{V}[\,T_1\,] = \mathbf{V}[\,T_1\,]$, and

$$\mathbf{V}[\,T_1\,] = \mathbf{V}\left[2 \cdot \overline{X}_n - 1\right] = 4 \cdot \mathbf{V}\left[\overline{X}_n\right] = \frac{4}{n^2} \cdot \mathbf{V}[\,X_1 + \cdots + X_n\,]$$

- Note: The X_i's are not independent!
- Use generalisation of $\mathbf{V}[\,X_1 + X_2\,] = \mathbf{V}[\,X_1\,] + \mathbf{V}[\,X_2\,] + 2 \cdot \mathbf{Cov}[\,X_1, X_2\,]$ (Exercise Sheet) to n r.v.'s, and then that the X_i's are identically distributed, and also the (X_i, X_j), $i \neq j$:

$$\begin{aligned}\mathbf{V}[\,X_1 + \cdots + X_n\,] &= \sum_{i=1}^{n} \mathbf{V}[\,X_i\,] + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbf{Cov}[\,X_i, X_j\,] \\ &= n \cdot \mathbf{V}[\,X_1\,] + 2\binom{n}{2} \cdot \mathbf{Cov}[\,X_1, X_2\,].\end{aligned}$$

- By definition of the discrete uniform distribution, $\mathbf{V}[\,X_1\,] = \frac{(N+1)(N-1)}{12}$
- Intuitively, X_1 and X_2 are negatively correlated, which would be sufficient to complete the proof. For a more rigorous and precise derivation (see Dekking et al.):

$$\mathbf{Cov}[\,X_1, X_2\,] = -\frac{1}{12}(N+1).$$

- Rearranging and simplifying gives

$$\mathbf{V}[\,T_1\,] = \frac{(N+1)(N-n)}{3n}.$$

Analysis of the MSE for T_2 (non-examinable)

Example 5

It holds that $\mathbf{MSE}[\, T_2 \,] = \Theta\left(\frac{N^2}{n^2}\right)$, where $T_2 = \frac{n+1}{n} \cdot \max(X_1, \ldots, X_n) - 1$.

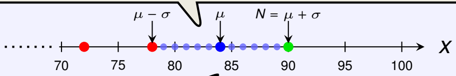
Answer

- T_2 is unbiased $\Rightarrow$ need $\mathbf{V}[\, T_2 \,]$ which reduces to $\mathbf{V}[\, \max(X_1, \ldots, X_n) \,]$
- One can prove:

For details see Dekking et al.

$$\mathbf{V}[\, \max(X_1, \ldots, X_n) \,] = \cdots = \frac{n(N+1)(N-n)}{(n+2)(n+1)^2} = \Theta\left(\frac{N^2}{n^2}\right)$$

Equi-spaced (idealised) configuration suggests a standard deviation of $\sigma \approx \frac{N}{n}$



Maximum could have equally likely taken any value between 79 and 90

- $\mathbf{MSE}[\, T_2 \,]$ is much lower than $\mathbf{MSE}[\, T_1 \,] = \Theta\left(\frac{N^2}{n}\right)$, i.e., $\frac{\mathbf{MSE}[\, T_1 \,]}{\mathbf{MSE}[\, T_2 \,]} = \frac{n+2}{3}$
- $\Rightarrow$ confirms simulations suggesting that T_2 is better than T_1!
- can be shown T_2 is the best unbiased estimator, i.e., it minimises MSE.

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A New Estimation Problem

Previous Model

- Population/ID space $S = \{1, 2, \ldots, N\}$
- We take uniform samples from S without replacement
- **Goal:** Find estimator for N

Similar idea applies to situations where elements are not labelled before we see them first time (Mark & Recapture Method)

New Model

- Population/ID space of size $|S| = N$
- We take uniform samples from S with replacement
- **Goal:** Find estimator for N

- Suppose $n = 6$, $N = 11$, $S = \{3, 4, 7, 8, 10, 15.83356, 20, 21, 56, 81, 10000\}$
- Let the sample be

$$10, 81, 20, 3, 81, 10000$$

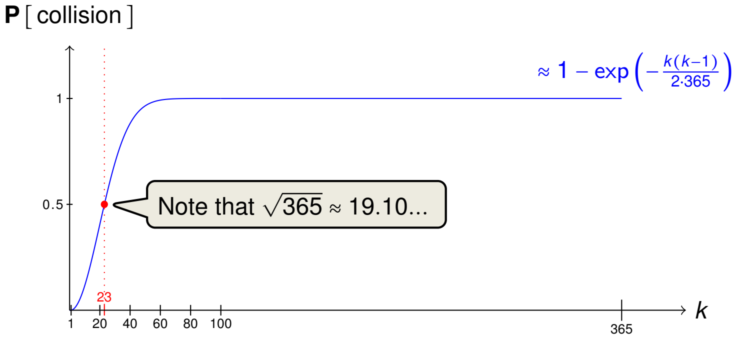
Let us call this a **collision**

As we do not know S, our only clue are elements that **were sampled twice.**

Birthday Problem

Birthday Problem: Given a set of k people

- What is the probability of having two with the same birthday (i.e., having at least one collision)?
- What is the expected number of people one needs to ask until the first collision occurs?



Estimation via Collision: The Algorithm

Recall: As we do not know S, our only information are **collisions**.

FIND-FIRST-COLLISION(S)

$C = \emptyset$
For $i = 1, 2, \ldots$
    Take next i.i.d. sample X_i from S
    If $X_i \notin C$ **then** $C \leftarrow C \cup \{X_i\}$
    else return $T(i)$
End For

$T(i)$ will be the value of the estimator if algo returns after i rounds. (We want T unbiased)

- Running Time: The expected time until the algorithm stops is:
  $=$ the expected number of samples until a collision...

Same as the birthday problem, but now with $|S| = N$ days... ☺

Expected Running Time (Knuth, Ramanujan)

$$\sqrt{\frac{\pi N}{2}} - \frac{1}{3} + O\left(\frac{1}{\sqrt{N}}\right).$$

Exercise: Prove a bound of $\leq 2 \cdot \sqrt{N}$

Estimation via Collision: Getting the Estimator Unbiased

Example 6

One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[\,T\,] = |S|$ for any finite, non-empty set S.

Answer

- We outline a construction by induction.
- Case $|S| = 1$: Algo always stops after $i = 2$ rounds and returns $T(2)$. We want
$$1 = \mathbf{E}[\,T\,] = T(2) \qquad \Rightarrow \qquad T(2) = 1.$$
- Case $|S| = 2$: Algo stops after 2 or 3 rounds (w.p. 1/2 each). We want
$$2 = \mathbf{E}[\,T\,] = \frac{1}{2} \cdot T(2) + \frac{1}{2} \cdot T(3) \qquad \Rightarrow \qquad T(3) = 3.$$
- Case $|S| = 3$: gives $3 = \mathbf{E}[\,T\,] = \frac{1}{3} \cdot T(2) + \frac{4}{9} \cdot T(3) + \frac{2}{9} \cdot T(4)$ $\Rightarrow T(4) = 6$, similarly, $T(5) = 10$ etc.
- can continue to define $T(i)$ inductively in this way (note T is unique) (a proof that $T(i) = \binom{i}{2}$ is harder)