Introduction to Probability

Lecture 10: Estimators (Part I)

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Outline

Introduction

Defining and Analysing Estimators

More Examples

Introduction

Setting: We can take random samples in the form of i.i.d. random variables $X_1, X_2, \ldots, X_n$ from an unknown distribution.

- Taking enough samples allows us to estimate the mean (WLLN, CLT)
- Using indicator variables, we can estimate $\mathbf{P}[X \leq a]$ for any $a \in \mathbb{R}$
  $\rightsquigarrow$ in principle we can reconstruct the entire distribution

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  $\rightsquigarrow$ bias (this lecture) and mean squared error (next lecture)

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$$\text{Measurement} = \text{Quantity of Interest} + \text{Measurement Error}$$

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Empirical Distribution Functions

Definition of Empirical Distribution Function (Empirical CDF)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. samples, and F be the corresponding distribution function. For any $a \in \mathbb{R}$, define

$$F_n(a) := \frac{\text{number of } X_i \in (-\infty, a]}{n}.$$

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Thus by taking enough samples, we can estimate the entire distribution (including its expectation and variance).

Empirical Distribution Functions (Example 1/2)

Example 1

Consider throwing an unbiased dice 8 times, and let the realisation be:

$$(x_1, x_2, \ldots, x_8) = (4, 1, 4, 3, 1, 6, 4, 1).$$

What is the Empirical Distribution Function $F_8(a)$?

Answer

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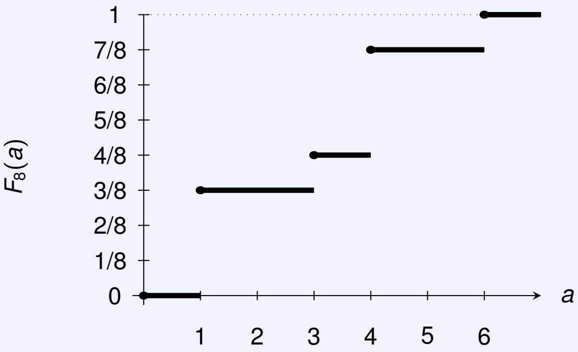

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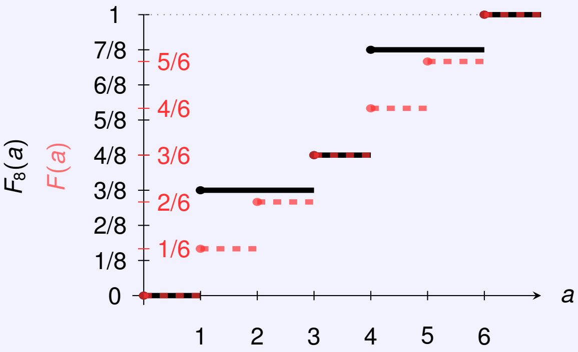

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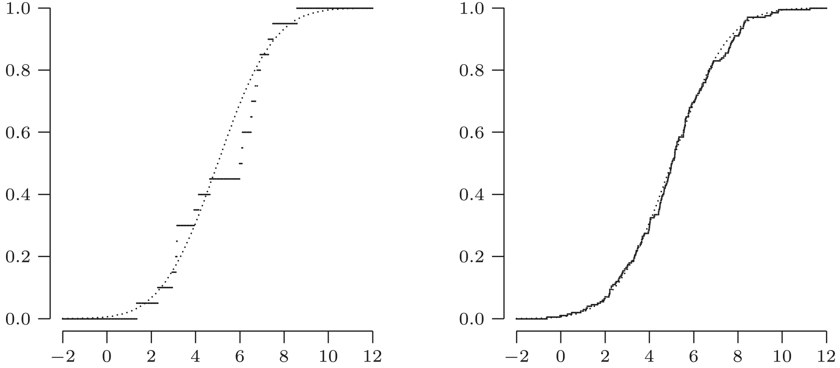

Source: Modern Introduction to Statistics

Figure: Empirical Distribution Functions of samples from a Normal Distribution $\mathcal{N}(5, 4)$ ($n = 20$ left, $n = 200$ right)

An Example of an Estimation Problem

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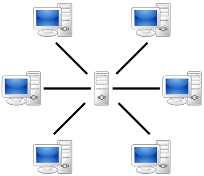

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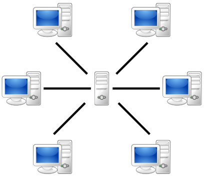

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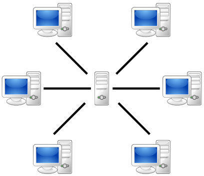

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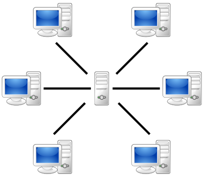

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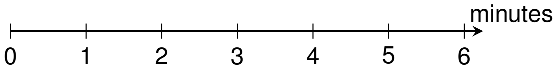

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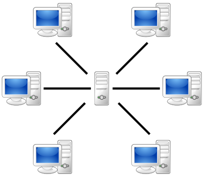

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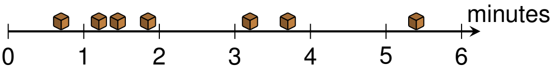

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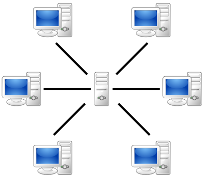

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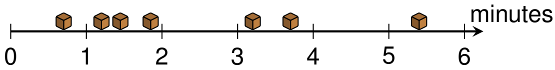

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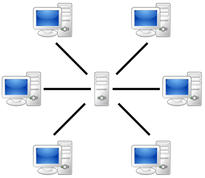

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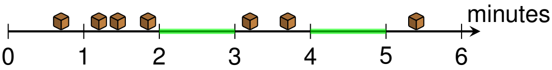

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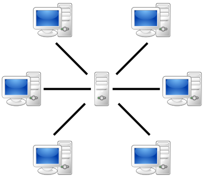

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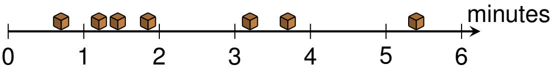

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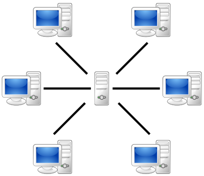

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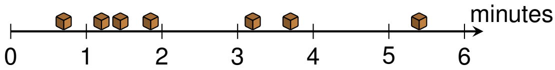

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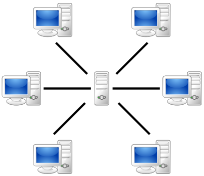

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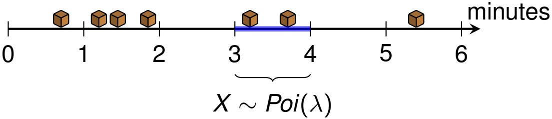

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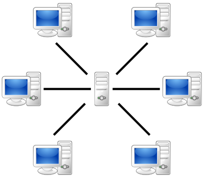

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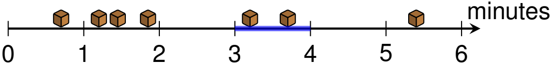

$X \sim Poi(\lambda)$

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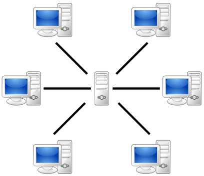

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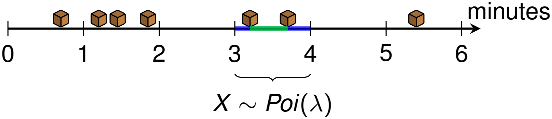

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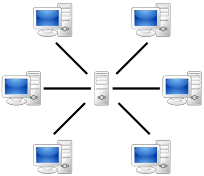

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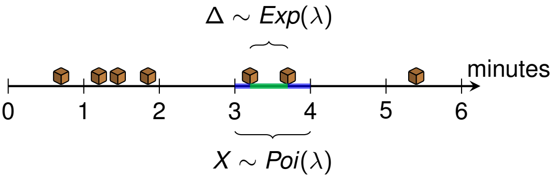

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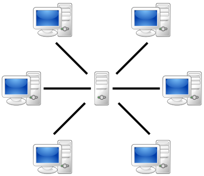

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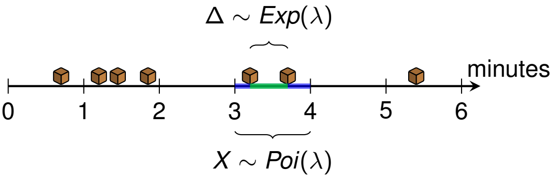
Waiting Time (Lecture 5, Slide 22)



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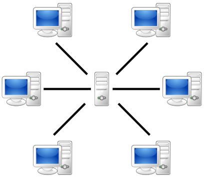

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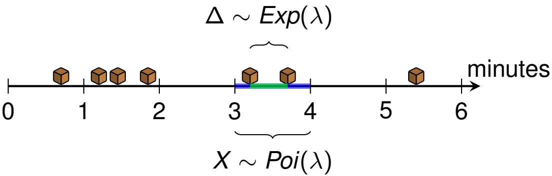

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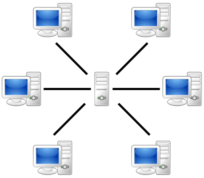
Estimator for $e^{-\lambda}$



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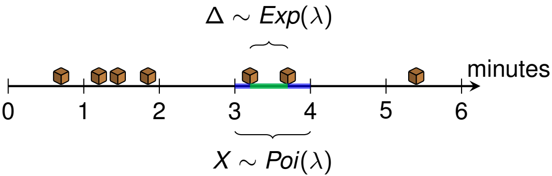

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Questions:

- What makes an estimator suitable? unbiased (later: mean squared error)
- Does an unbiased estimator always exist? How to compute it?
- If there are several unbiased estimators, which one to choose?

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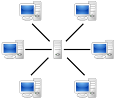
More Examples

Example: Arrival of Packets (1/3)



- Samples: Given $X_1, X_2, \ldots, X_n$ i.i.d., $X_i \sim Pois(\lambda)$
- Meaning: X_i is the number of packets arriving in minute i

Example 2

Estimate λ by using the sample mean $\overline{X}_n$.

Answer

Example: Arrival of Packets (2/3)

Example 3a

Define an estimator h_1 for the probability of zero arrivals, $e^{-\lambda}$.

Answer

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Example 3b

Define an estimator h_2 for $e^{-\lambda}$ based on $\overline{X}_n$.

Answer

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- Suppose we have $n = 30$ and we want to estimate $e^{-\lambda}$
- Consider the two estimators $h_1(X_1, \ldots, X_n)$ and $h_2(X_1, \ldots, X_n)$.

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$\Rightarrow$ The first estimator can only attain values $0, \frac{1}{30}, \frac{2}{30}, \ldots, 1$

$\Rightarrow$ The second estimator can only attain values $1, e^{-1/30}, e^{-2/30}, \ldots$

Behaviour of the Estimators

- Suppose we have $n = 30$ and we want to estimate $e^{-\lambda}$
- Consider the two estimators $h_1(X_1, \ldots, X_n)$ and $h_2(X_1, \ldots, X_n)$.

How **good** are these two estimators?

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For most values of λ, both estimators will never return the exact value of $e^{-\lambda}$ on the basis of 30 observations.

Simulation of the two Estimators

- The unknown parameter is $p = e^{-\lambda} = 0.1$ (i.e., $\lambda = \ln 10 \approx 2.30\ldots$)

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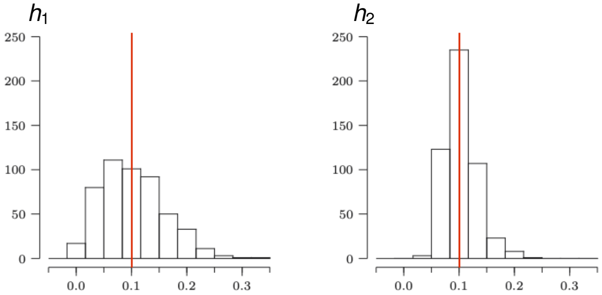

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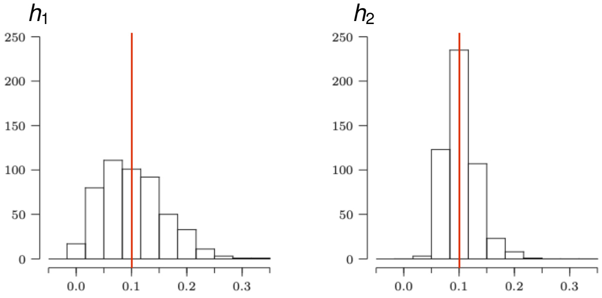

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> Both estimators concentrate around the true value 0.1, but the second estimator appears to be more concentrated.

Unbiased Estimators and Bias

Definition

An estimator T is called an unbiased estimator for the parameter θ if

$$\mathbf{E}[T] = \theta,$$

irrespective of the value θ.

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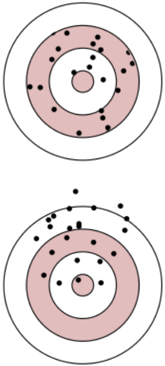

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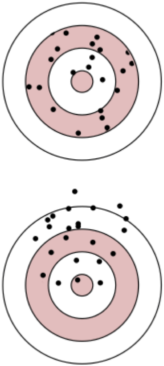

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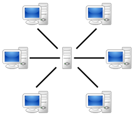
Which of the two estimators h_1, h_2 are unbiased?



Analysis of the Bias of the First Estimator

Example 4a

Is $h_1(X_1, X_2, \ldots, X_n) = \frac{Y_1+Y_2+\cdots+Y_n}{n}$ an unbiased estimator for $e^{-\lambda}$?

Answer

Bias of the Second Estimator (and Jensen's Inequality)

Example 4b

Is $h_2(X_1, X_2, \ldots, X_n) = e^{-\overline{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

Answer

Bias of the Second Estimator (and Jensen's Inequality)

Example 4b

Is $h_2(X_1, X_2, \ldots, X_n) = e^{-\overline{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

Answer

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Example 4b

Is $h_2(X_1, X_2, \ldots, X_n) = e^{-\overline{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

Answer

Jensen's Inequality

For any random variable X, and any convex function $g : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbf{E}\left[\, g(X) \,\right] \geq g(\mathbf{E}\left[\, X \,\right]).$$

If g is strictly convex and X is not constant, then the inequality is strict.

Bias of the Second Estimator (and Jensen's Inequality)

Example 4b

Is $h_2(X_1, X_2, \ldots, X_n) = e^{-\overline{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

Answer

$\lambda g(a) + (1-\lambda) g(b) \; \geq \; g(\lambda a + (1-\lambda) b)$

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Asymptotic Bias of the Second Estimator (non-examinable)

Example 4c

$\mathbf{E}[\,h_2(X_1,\ldots,X_n)\,] \stackrel{n\to\infty}{\longrightarrow} e^{-\lambda}$ (hence it is asymptotically unbiased).

Answer

- Recall $h_2(X_1,\ldots,X_n) = e^{-\overline{X}_n}$. For any $0 \leq k \leq n$,

$$\mathbf{P}\left[\,h_2(X_1,\ldots,X_n) = e^{-k/n}\,\right] = \mathbf{P}\left[\,\sum_{i=1}^{n} X_i = k\,\right] = \mathbf{P}[\,Z = k\,],$$

where $Z \sim Pois(n\cdot\lambda)$ (since $Pois(\lambda_1) + Pois(\lambda_2) = Pois(\lambda_1+\lambda_2)$)

$$\Rightarrow \qquad \mathbf{P}\left[\,h_2(X_1,\ldots,X_n) = e^{-k/n}\,\right] = \frac{e^{-n\lambda}\cdot(n\lambda)^k}{k!}$$

$$\Rightarrow \qquad \mathbf{E}[\,h_2(X_1,\ldots,X_n)\,] = \sum_{k=0}^{\infty} e^{-n\lambda}\cdot\frac{(n\lambda^k)}{k!}\cdot e^{-k/n}$$

By LOTUS

$$= e^{-n\lambda}\cdot e^{n\lambda e^{-1/n}}\sum_{k=0}^{\infty} e^{-n\lambda e^{-1/n}}\cdot\frac{(n\lambda e^{-1/n})^k}{k!}$$

$$= e^{-n\lambda\cdot(1-e^{-1/n})}\cdot 1$$

$$\stackrel{n\to\infty}{\approx} e^{-n\lambda\cdot(1-1+1/n+O(1/n^2))} = e^{-\lambda+O(\lambda/n)}.$$

since $e^x = 1 + x + O(x^2)$ for small x

Hence in the limit, the positive bias of h_2 diminishes.

Outline

Introduction

Defining and Analysing Estimators

More Examples

Unbiased Estimator for Expectation and Variance

Unbiased Estimators for Expectation and Variance

Let $X_1, X_2, \ldots, X_n$ be identically distributed samples from a distribution with finite expectation μ and finite variance σ^2.

- Then
$$\overline{X}_n := \frac{X_1 + X_2 + \cdots + X_n}{n}$$
is an unbiased estimator for μ.
- Furthermore, for $n \geq 2$,
$$S_n = S_n(X_1, \ldots, X_n) := \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(X_i - \overline{X}_n \right)^2$$
is an unbiased estimator for σ^2.

Example 5

We need to prove: $\mathbf{E}[S_n] = \sigma^2$.

Answer

An Unbiased Estimator may not always exist

Example 6

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

Answer

An Unbiased Estimator may not always exist (cntd. - non-examinable)

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  $\Rightarrow \mathbf{E}[T(X)]$ can be equal to $1/p$ for at most $n + 1$ values of p, and thus cannot be an unbiased.