Hoare logic and Model checking

Revision class

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Hoare logic and separation logic

The concept of ownership

Ownership of a heap cell is the permission to safely read/write/dispose of it. **This ownership is not duplicable**.

E.g.: use-after-free: $\mathit{dispose}(X); [X] := 5$

Separation logic:

$\{X \mapsto v\}$
dispose(X);
$\{\mathit{emp}\}$
proof fails
$\{X \mapsto v\}$
[X] := 5
$\{X \mapsto 5\}$

If ownership were duplicable:

$\{X \mapsto v\}$
$\{X \mapsto v * X \mapsto v\}$
dispose(X);
$\{X \mapsto v\}$
[X] := 5
$\{X \mapsto 5\}$

Pure assertions

$$
\begin{aligned}
[\![-]\!](=) &: \mathit{Assertion} \to \mathit{Stack} \to \mathcal{P}(\mathit{Heap}) \\
[\![\bot]\!](s) &\stackrel{\mathit{def}}{=} \emptyset \\
[\![\top]\!](s) &\stackrel{\mathit{def}}{=} \mathit{Heap} \\
[\![P \land Q]\!](s) &\stackrel{\mathit{def}}{=} [\![P]\!](s) \cap [\![Q]\!](s) \\
[\![P \lor Q]\!](s) &\stackrel{\mathit{def}}{=} [\![P]\!](s) \cup [\![Q]\!](s) \\
[\![P \Rightarrow Q]\!](s) &\stackrel{\mathit{def}}{=} \{h \in \mathit{Heap} \mid h \in [\![P]\!](s) \Rightarrow h \in [\![Q]\!](s)\} \\
&\vdots
\end{aligned}
$$

What is the meaning of pure assertions, such as $\top$ or $t_1 = t_2$? Do they implicitly require the heap to be empty?

Semantics of pure assertions

$$[\![-]\!](=) : \mathit{Assertion} \to \mathit{Stack} \to \mathcal{P}(\mathit{Heap})$$

$$[\![t_1 = t_2]\!](s) = \{h \mid [\![t_1]\!](s) = [\![t_2]\!](s)\} = \begin{cases} \mathit{Heap} & \text{if } [\![t_1]\!](s) = [\![t_2]\!](s) \\ \emptyset & \text{otherwise} \end{cases}$$

More generally, the semantics of a pure assertion in a stack s:

Informally: "check the pure assertion in s"; if it holds in s, return the set of all heaps, if not return the empty set of heaps.

Formally: don't worry about it, because we have not defined it.

Semantics of pure assertions, wrt. heap (continued).

The 2019 exam paper 8, question 7 asks:

$\{N = n \wedge N \geq 0\}$
X := null; while N > 0 do (X := alloc(N, X); N := N −1)
$\{\text{list}(X, [1, \ldots, n])\}$

(I have not checked whether that year used different definitions from ours, but) **This seems to be missing emp in the pre-condition:** $\{N = n \wedge N \geq 0 \wedge \text{emp}\}$

Why? $\{N = n \wedge N \geq 0\}$ makes no statement about the heap — if the stack has the right property, it is satisfied by any heap. But without the emp requirement, we would not be able to prove the post-condition $\{\text{list}(X, [1, \ldots, n])\}$, which asserts that the **only** ownership is that of the list predicate instance.

Another error

Related: error in 2021 Paper 8 Question 8.

The pre-condition should have

$$\cdots \wedge 1 \leq S$$

instead of

$$\cdots * 1 \leq S$$

.

Conjunction and separating conjunction

What are the differences between them and when to use which?
And how do they interact with pure assertions?

$$[\![P * Q]\!](s) \stackrel{def}{=} \left\{ h \in Heap \,\middle|\, \exists h_1, h_2.\ \begin{array}{l} h_1 \in [\![P]\!](s) \wedge \\ h_2 \in [\![Q]\!](s) \wedge \\ h = h_1 \uplus h_2 \end{array} \right\}$$

$$[\![P \wedge Q]\!](s) \stackrel{def}{=} [\![P]\!](s) \cap [\![Q]\!](s)$$

Conjunction and separating conjunction (continued)

$$[\![P * Q]\!](s) \stackrel{def}{=} \left\{ h \in Heap \;\middle|\; \exists h_1, h_2.\; \begin{array}{l} h_1 \in [\![P]\!](s) \wedge \\ h_2 \in [\![Q]\!](s) \wedge \\ h = h_1 \uplus h_2 \end{array} \right\}$$

$$[\![P \wedge Q]\!](s) \stackrel{def}{=} [\![P]\!](s) \cap [\![Q]\!](s)$$

$p_1 \mapsto v_1 * p_2 \mapsto v_2$ **vs.** $p_1 \mapsto v_1 \wedge p_2 \mapsto v_2$

- $p_1 \mapsto v_1 * p_2 \mapsto v_2$ holds for a heap h that is the disjoint union of heaplets h_1 and h_2, where h_1 contains just cell p_1, with value v_1, and h_2 just cell p_2, with value v_2. So: ownership of **two disjoint** heap cells p_1 and p_2 with $p_1 \neq p_2$.
- $p_1 \mapsto v_1 \wedge p_2 \mapsto v_2$ holds for a heap h that satisfies two assertions simultaneously (is in the intersection of their interpretations):
  (1) $p_1 \mapsto v_1$: h is a heap of just one heap cell, p_1 with value v_1
  (2) $p_2 \mapsto v_2$: h is a heap of just one heap cell, p_2 with value v_2
  So: ownership of just **one** heap cell, $p_1 = p_2$ with value $v_1 = v_2$.

Conjunction and separating conjunction (continued)

$$[\![P * Q]\!](s) \stackrel{def}{=} \left\{ h \in Heap \,\middle|\, \exists h_1, h_2.\ \begin{array}{l} h_1 \in [\![P]\!](s) \wedge \\ h_2 \in [\![Q]\!](s) \wedge \\ h = h_1 \uplus h_2 \end{array} \right\}$$

$$[\![P \wedge Q]\!](s) \stackrel{def}{=} [\![P]\!](s) \cap [\![Q]\!](s)$$

$(p \mapsto 1) * Y = 0$ **vs.** $(p \mapsto 1) \wedge Y = 0$

- $(p \mapsto 1) * Y = 0$ holds for a stack s and a heap h where h is the disjoint union of heaplets h_1 and h_2, such that h_1 contains ownership of one cell, p with value 1, and h_2 is an arbitrary heap if s satisfies $Y = 0$. So, s must map Y to 0 and h is the disjoint union of the heaplet of just p with value 1 and **an arbitrary disjoint** heap h_2.
- $(p \mapsto 1) \wedge Y = 0$ holds for a stack s and a heap h satisfying two assertion simultaneously: $p \mapsto 1$ and $Y = 0$. This means s must map Y to 0 and h must be the heap consisting of just that one cell.

emp in the alloc rule

Q: Why have the 'emp' in the precondition of the alloc rule?

$$\frac{}{\vdash \{X = x \wedge emp\}\ X := \mathbf{alloc}(E_0, ..., E_n)\ \{X \mapsto E_0[x/X], ..., E_n[x/X]\}}$$

A: This is needed for soundness. Otherwise the alloc rule would allow us to silently drop ownership of other heap cells.

Use of frame rule (to obtain "$\cdots \wedge emp$") in Lecture 5, slide 28

$$\{list(Y,\alpha) \wedge X = x\}$$

$$\{\exists z. (list(Y,\alpha) \wedge X = x) \wedge HEAD = z\}$$

$$\{(list(Y,\alpha) \wedge X = x) \wedge HEAD = z\}$$

$$\{(list(Y,\alpha) \wedge X = x) * (HEAD = z \wedge emp)\}$$

$$\{HEAD = z \wedge emp\}$$

$$HEAD := \textbf{alloc}(X, Y)$$

$$\{HEAD \mapsto X[z/HEAD], Y[z/HEAD]\}$$

$$\{HEAD \mapsto X, Y\}$$

$$\{(list(Y,\alpha) \wedge X = x) * HEAD \mapsto X, Y\}$$

$$\{(list(Y,\alpha) * HEAD \mapsto X, Y) \wedge X = x)\}$$

$$\{\exists z. (list(Y,\alpha) * HEAD \mapsto X, Y) \wedge X = x)\}$$

$$\{(list(Y,\alpha) * HEAD \mapsto X, Y) \wedge X = x\}$$

It is good to be careful of the possibly unexpected behaviour of the new separation logic assertions!

Example: 2019-p08-q07, e

Give a loop invariant for the following list concatenation triple:

$\{\mathrm{list}(X, \alpha) * \mathrm{list}(Y, \beta)\}$

```
if X = null then
   Z:=Y
else (
   Z := X; U := Z; V := [Z + 1];
   while V ≠ null do (U := V ; V := [V + 1]);
   [U + 1] := Y
)
```

$\{\mathrm{list}(Z, \alpha +\!\!+ \beta)\}$

Example: 2019-p08-q07, e

$\{\mathrm{list}(X, \alpha) * \mathrm{list}(Y, \beta)\}$

if X = null then

    Z:=Y

else (

    Z := X; U := Z; V := [Z + 1];

    while V ≠ null do (U := V ; V := [V + 1]);

    [U + 1] := Y

)

$\{\mathrm{list}(Z, \alpha +\!\!+ \beta)\}$

Example: 2019-p08-q07, e

$$\{(\mathrm{list}(X, \alpha) * \mathrm{list}(Y, \beta)) \wedge X \neq \mathsf{null}\}$$

$\mathsf{Z := X;\ U := Z;\ V := [Z + 1];}$

$\mathsf{while\ V \neq null\ do\ (U := V\ ;\ V := [V + 1]);}$

$\mathsf{[U + 1] := Y}$

$$\{\mathrm{list}(Z, \alpha +\!\!+ \beta)\}$$

$\{(\mathrm{list}(X,\alpha) * \mathrm{list}(Y,\beta)) \wedge X \neq \mathsf{null}\}$

$\{\exists t, p, \delta.\ \alpha = [t] +\!\!+\ \delta \wedge (X \mapsto t, p * \mathrm{list}(p,\delta) * \mathit{list}(Y,\beta))\}$

Z := X;

$\{\exists t, p, \delta.\ \alpha = [t] +\!\!+\ \delta \wedge (Z \mapsto t, p * \mathrm{list}(p,\delta) * \mathit{list}(Y,\beta))\}$

U := Z;

$\{\exists t, p, \delta.\ \alpha = [t] +\!\!+\ \delta \wedge U = Z \wedge (Z \mapsto t, p * \mathrm{list}(p,\delta) * \mathit{list}(Y,\beta))\}$

V := [Z + 1];

$\{\exists t, \delta.\ \alpha = [t] +\!\!+\ \delta \wedge U = Z \wedge (Z \mapsto t, V * \mathrm{list}(V,\delta) * \mathit{list}(Y,\beta))\}$

$I : \{\exists \gamma, t, \delta.\ \alpha = \gamma +\!\!+\ [t] +\!\!+\ \delta \wedge (\mathrm{plist}(Z,\gamma,U) * \mathrm{plist}(U,[t],V) * \mathrm{list}(V,\delta) * \mathit{list}(Y,\beta))\}$

while V $\neq$ null do (U := V ; V := [V + 1]);

$\{\exists \gamma, t, \delta.\ \alpha = \gamma +\!\!+\ [t] +\!\!+\ \delta \wedge (\mathrm{plist}(Z,\gamma,U) * \mathrm{plist}(U,[t],V) * \mathrm{list}(V,\delta) * \mathit{list}(Y,\beta))$

$\quad \wedge \neg(V \neq \mathsf{null})\}$

[U + 1] := Y

$\{\exists \gamma, t, \delta.\ \alpha = \gamma +\!\!+\ [t] +\!\!+\ \delta \wedge (\mathrm{plist}(Z,\gamma,U) * \mathrm{plist}(U,[t],Y) * \mathrm{list}(V,\delta) * \mathit{list}(Y,\beta))$

$\quad \wedge \neg(V \neq \mathsf{null})\}$

$\{\mathrm{list}(Z, \alpha +\!\!+\ \beta)\}$

Proof outlines + loop invariants

Q: How much detail to give in proof outline in exam?

Q: If asked to provide a loop invariant, do you need to provide the full proof?

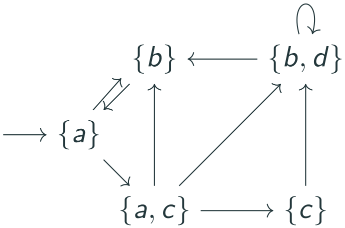
A: The exam text will be clear about that.

Model Checking

Temporal operators, e.g. in CTL

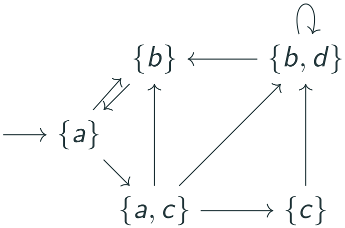
- $AX\psi$ and $EX\psi$:
  - Does the state satisfying ψ have to be different from the starting state?
  - Does ψ have to continue holding?
- $A(\psi_1 U \psi_2)$ and $E(\psi_1 U \psi_2)$:
  - Does ψ_1 have to continue holding?
  - What about ψ_2?

LTL examples



ϕ	$M \vDash \phi$
a	yes
Xa	no
Fb	yes
Fc	no
$(a \vee b)Uc$	no
dUa	yes
$G(a \vee b \vee c)$	yes
GFb	yes
FGb	no

CTL examples



ψ	$M \vDash \psi$
$EX(b \wedge \neg c)$	yes
AFd	no
EFd	yes
$E(aUd)$	yes
$AGEFd$	yes
$AFEGd$	no
$EFEGd$	yes
$E((a \vee c)U(EGb))$	yes

LTL/CTL expressivity

An elevator property: **"If it is possible to answer a call to some level in the next step, then the elevator does that"**

CTL formula ψ: A G $((\mathsf{Call}_2 \wedge \mathsf{E\ X\ Loc}_2) \rightarrow \mathsf{A\ X\ Loc}_2)$

Q: Can we express the same in LTL with formula ϕ:
G $(\mathsf{Call}_2 \wedge (\mathsf{Loc}_1 \vee \mathsf{Loc}_3)) \rightarrow \mathsf{X\ Loc}_2$**?**

This depends on the details of the elevator temporal model.[1] In any case, ψ and ϕ are not generally equivalent. The point is: expressing properties of the tree of possible paths out of a given state — such as asserting the **existence** of some path — is not possible with LTL.

[1] I think — the way we have sketched the elevator in lecture 7 — this will not work: $\mathsf{Loc}_1 \vee \mathsf{Loc}_3$ does not imply there exists a next step such that Loc_2 holds.

LTL/CTL expressivity

An LTL formula not expressible in CTL: $\phi = (\mathsf{F}\ p) \to (\mathsf{F}\ q)$.

a) CTL formula $\psi_1 = (\mathsf{A}\ \mathsf{F}\ p) \to (\mathsf{A}\ \mathsf{F}\ q)$.
ϕ does not hold, ψ_1 does.

$$\begin{tikzcd}
 & {} \arrow[d] & \\
3 : \{\} \arrow[loop above] & 1 : \{\} \arrow[l] \arrow[r] & 2 : \{p\} \arrow[loop above]
\end{tikzcd}$$

b) CTL formula $\psi_2 = \mathsf{A}\ \mathsf{G}\ (p \to (\mathsf{A}\ \mathsf{F}\ q))$.
ϕ holds, ψ_2 does not.

$$\begin{tikzcd}
{} \arrow[r] & 4 : \{q\} \arrow[r] & 5 : \{p\} \arrow[loop above]
\end{tikzcd}$$

LTL/CTL expressivity

Why are F G p **in LTL and** A F A G p **in CTL not equivalent?**

$$
\begin{tikzcd}
\arrow[r] & 1 : \{p\} \arrow[r] \arrow[loop below] & 2 : \{\} \arrow[r] & 3 : \{p\} \arrow[loop below]
\end{tikzcd}
$$

Two kinds of infinite paths: (L1) loop in 1 forever, (L2) loop in 3 forever. Both kinds of paths **eventually** reach a state in which p holds **generally** (1 or 3, respectively). So F G p holds.

Informally: A F A G p holds if (check CTL (CTL*) semantics):

- all paths π from 1 satisfy F A G p, so
- all paths π from 1 eventually reach a state where A G p holds

But path kind (L1) does not: never leaves 1, and in 1, A G p is not satisfied, because there exists a path π_2 that goes to 2 from there.

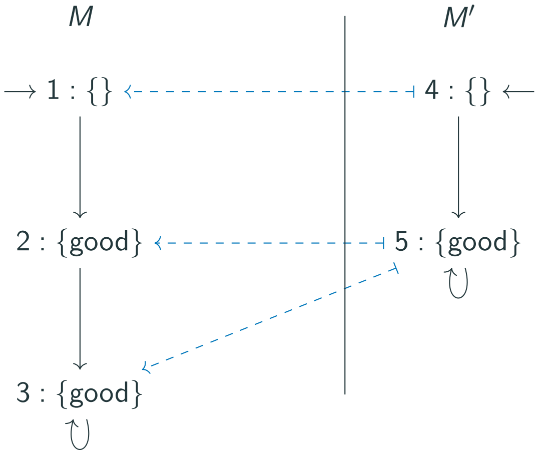
Q: LTL vs ACTL?

It is good to be careful about the unexpected interaction of the temporal operators, with other temporal operators and with path quantifiers.

Simulation relations

Q: Why simulation relations and not simulation functions?

Example: $AP = AP' = \{\text{good}\}$. M simulates M'



Good luck!