Proposition 110 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

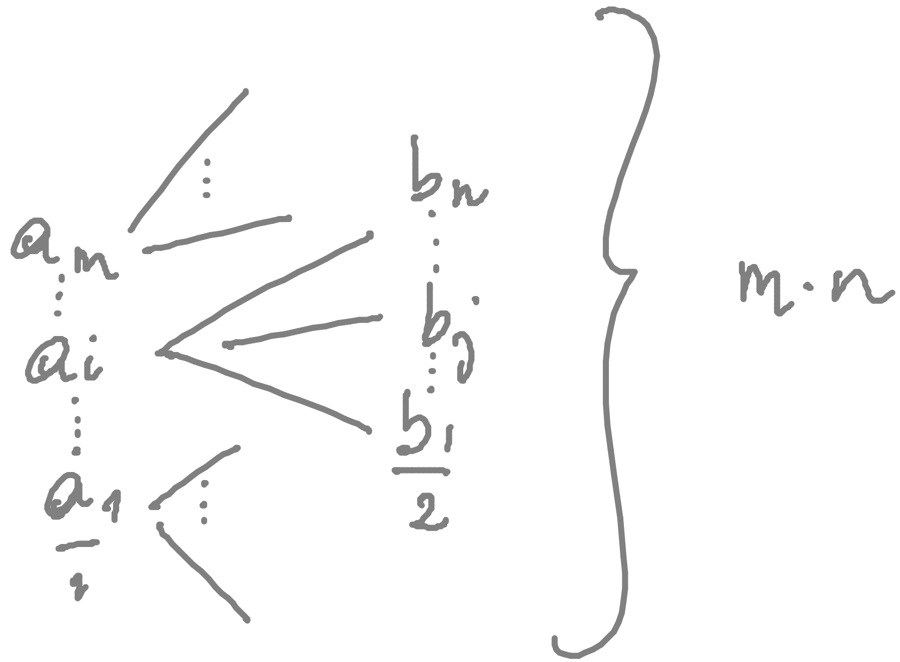
PROOF IDEA:

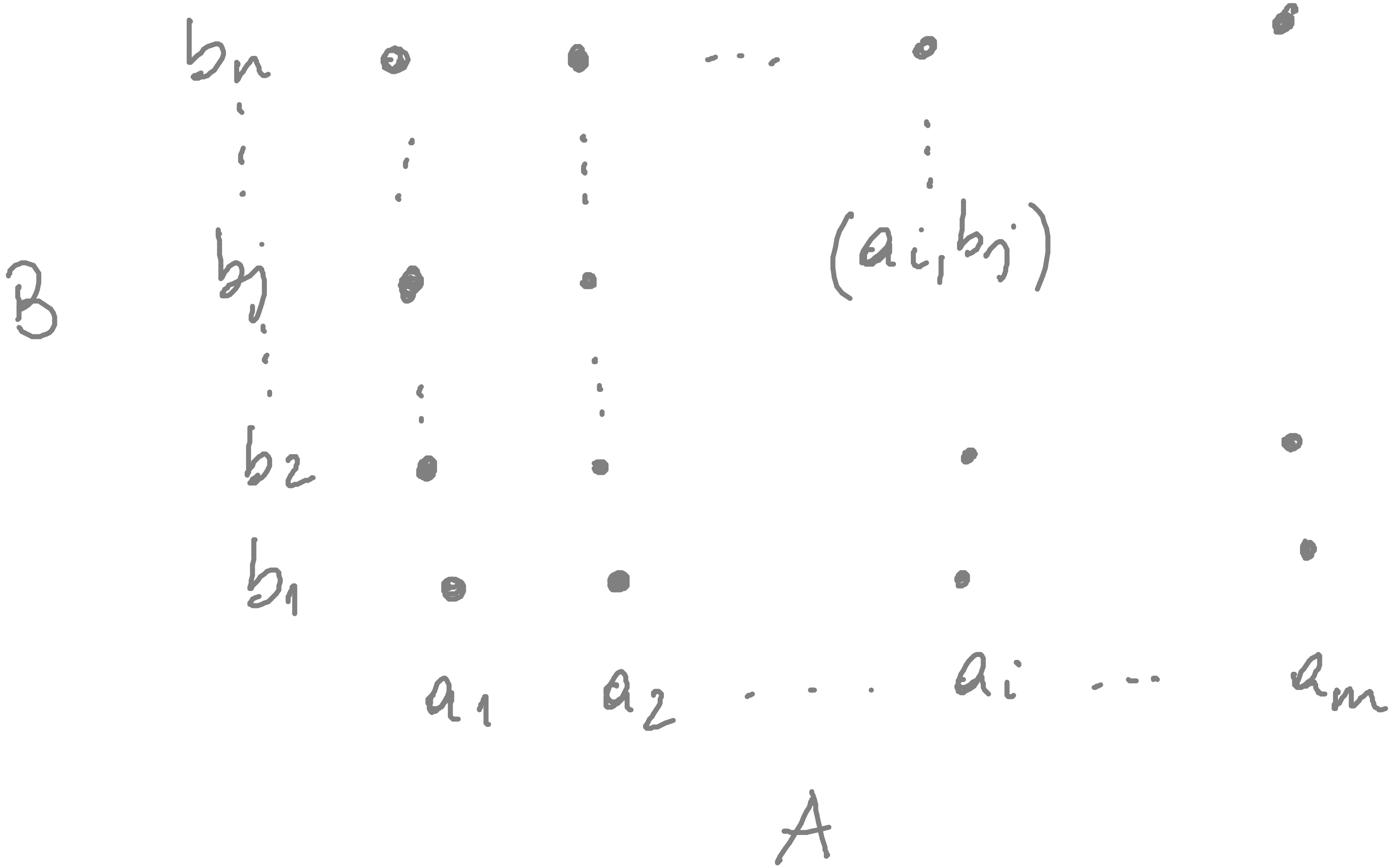
$$A \times B = \{ (a, b) \mid a \in A,\ b \in B \}$$

$\#A = m$

$\#B = n$

$(\underset{\text{from } A}{\underline{1}}, \underset{\text{from } B}{\underline{2}})$




b_n
(a_i, b_j)
B
b_j
b_2
b_1
a_1 a_2 ... a_i ... a_m
A

Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{\,\mathbf{false}\,,\ \mathbf{true}\,\}$
$\emptyset$	$\mathbf{false}$
$\mathcal{U}$	$\mathbf{true}$
$\cup$	$\vee$
$\cap$	$\wedge$
$(\,\cdot\,)^{\mathrm{c}}$	$\neg(\,\cdot\,)$
$\bigcup$	$\exists$
$\bigcap$	$\forall$

big

NB: $\bigcup\{A,B\} = A \cup B$ $\bigcup\{A_1, A_2, \ldots, A_n\} = A_1 \cup A_2 \cup \cdots \cup A_n$

Big unions

Example: $[5] = \{0,1,2,3,4\}$

- Consider the family of sets

$$\mathcal{T} = \left\{ \mathrm{T} \subseteq [5] \;\middle|\; \begin{array}{l} \text{the sum of the elements of} \\ \mathrm{T} \text{ is less than or equal } 2 \end{array} \right\}$$

$$= \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}, \{2\}\}$$

- The *big union* of the family $\mathcal{T}$ is the set $\bigcup \mathcal{T}$ given by the union of the sets in $\mathcal{T}$:

$$n \in \bigcup \mathcal{T} \iff \exists \mathrm{T} \in \mathcal{T}.\, n \in \mathrm{T} \quad .$$

Hence, $\bigcup \mathcal{T} = \{0,1,2\}$.

$\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$

$\Updownarrow$

Definition 111 *Let* $\mathcal{U}$ *be a set. For a collection of sets* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$*, we let the* big union *(relative to* $\mathcal{U}$*) be defined as*

$$\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}.\, x \in A \} \in \mathcal{P}(\mathcal{U}) \quad .$$

$\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ $\quad \bigcup\{\bigcup A_1, \bigcup A_2, \ldots, \bigcup A_n\}$

$\bigcup \mathcal{F} = A_1 \cup A_2 \cup \ldots \cup A_n$

$\bigcup(\bigcup \mathcal{F}) = \bigcup(A_1 \cup A_2 \cup \ldots \cup A_n)$ // claim

$\mathcal{F} \subseteq \mathcal{P}(\mathcal{P}(U))$

$\Updownarrow$

Proposition 112 *For all* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(U)))$,

$$\bigcup(\bigcup \mathcal{F}) = \bigcup\left\{ \bigcup \mathcal{A} \in \mathcal{P}(U) \;\middle|\; \mathcal{A} \in \mathcal{F} \right\} \in \mathcal{P}(U) \quad .$$

PROOF: $x \in \bigcup(\bigcup \mathcal{F})$

$\iff \exists A.\ A \in \bigcup \mathcal{F} \wedge x \in A$

$\iff \exists A.\ (\exists \mathcal{A}.\ \mathcal{A} \in \mathcal{F} \wedge A \in \mathcal{A}) \wedge x \in A$

$\iff \exists \mathcal{A} \in \mathcal{F}.\ \exists A \in \mathcal{A}.\ x \in A$

$\iff \exists \mathcal{A} \in \mathcal{F}.\ x \in \bigcup \mathcal{A} \iff x \in \bigcup\{\bigcup \mathcal{A} \mid \mathcal{A} \in \mathcal{F}\}$

$\boxtimes$

NB: $\bigcap\{A,B\} = A \cap B \quad \bigcap\{A_1, A_2, \ldots, A_n\} = A_1 \cap A_2 \cap \cdots \cap A_n$

Big intersections

Example:

- Consider the family of sets

$$\mathcal{S} = \left\{ S \subseteq [5] \;\middle|\; \text{the sum of the elements of } S \text{ is } 6 \right\}$$

$$\{0,1,2,3\}$$

$$= \{\{2,4\}, \{0,2,4\}, \{1,2,3\}\}$$

- The *big intersection* of the family $\mathcal{S}$ is the set $\bigcap \mathcal{S}$ given by the intersection of the sets in $\mathcal{S}$:

$$n \in \bigcap \mathcal{S} \iff \forall S \in \mathcal{S}.\, n \in S \quad .$$

Hence, $\bigcap \mathcal{S} = \{2\}$.

Definition 113 *Let* $\mathcal{U}$ *be a set. For a collection of sets* $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$*, we let the* big intersection *(relative to* $\mathcal{U}$*) be defined as*

$$\bigcap \mathcal{F} = \{ x \in \mathcal{U} \mid \forall A \in \mathcal{F}.\, x \in A \} \ .$$

NB: $\forall A \in \mathcal{F}.\ \bigcap \mathcal{F} \subseteq A$

$(A \cap B \subseteq A \ \wedge \ A \cap B \subseteq B)$

Theorem 114 *Let* closure property.

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \;\middle|\; (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

PROOF:

$\mathbb{N} \in \mathcal{F} \overset{(i)}{\implies} \bigcap \mathcal{F} \subseteq \mathbb{N}$

$\mathbb{Z} \in \mathcal{F}$

$\mathbb{R} \in \mathcal{F}$

$\mathbb{Q} \in \mathcal{F}$

$S \in \mathcal{F} \implies S$ infinite

(ii) $\forall n \in \mathbb{N}.\, n \in \bigcap \mathcal{F}$

$\iff \forall n \in \mathbb{N}.\, \forall S \in \mathcal{F}.\, n \in S$ by induction

$\boxtimes$

Proposition 115 *Let U be a set and let $\mathcal{F} \subseteq \mathcal{P}(U)$ be a family of subsets of U.*

To show S is $\bigcup \mathcal{F}$ proceed to show:

1. *For all* $S \in \mathcal{P}(U)$,

$$S = \bigcup \mathcal{F}$$

iff

$$① \, [\forall A \in \mathcal{F}. A \subseteq S]$$
$$\wedge ② \, [\forall X \in \mathcal{P}(U). (\forall A \in \mathcal{F}. A \subseteq X) \Rightarrow S \subseteq X]$$

To show T is $\bigcap \mathcal{F}$ proceed to show:

2. *For all* $T \in \mathcal{P}(U)$,

$$T = \bigcap \mathcal{F}$$

iff

$$① \, [\forall A \in \mathcal{F}. T \subseteq A]$$
$$\wedge ② \, [\forall Y \in \mathcal{P}(U). (\forall A \in \mathcal{F}. Y \subseteq A) \Rightarrow Y \subseteq T]$$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}.\, x \in X$$

$x \in \bigcup \emptyset \iff \exists X.\ X \in \emptyset \wedge x \in X \iff$ false

So $\bigcup \emptyset = \emptyset$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \big(\forall X \in \mathcal{F}.\, x \in X\big) \quad .$$

NB: $(\{1\} \times A) \cap (\{2\} \times B) = \emptyset$

Disjoint unions

Definition 116 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \;=\; \big(\{1\} \times A\big) \cup \big(\{2\} \times B\big) \quad .$$

Thus,

$$\forall x.\, x \in (A \uplus B) \iff \big(\, \exists a \in A.\, x = (1, a) \,\big) \vee \big(\, \exists b \in B.\, x = (2, b) \,\big) \,.$$

Proposition 118 *For all finite sets* A *and* B*,*

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA:

$$\underbrace{\overbrace{a_1, a_2, \ldots, a_m}^{m}}_{A}, \underbrace{\overbrace{b_1, b_2, \ldots, b_n}^{n}}_{B}$$

Corollary 119 *For all finite sets* A *and* B*,*

$$\#(A \uplus B) = \#A + \#B \quad .$$