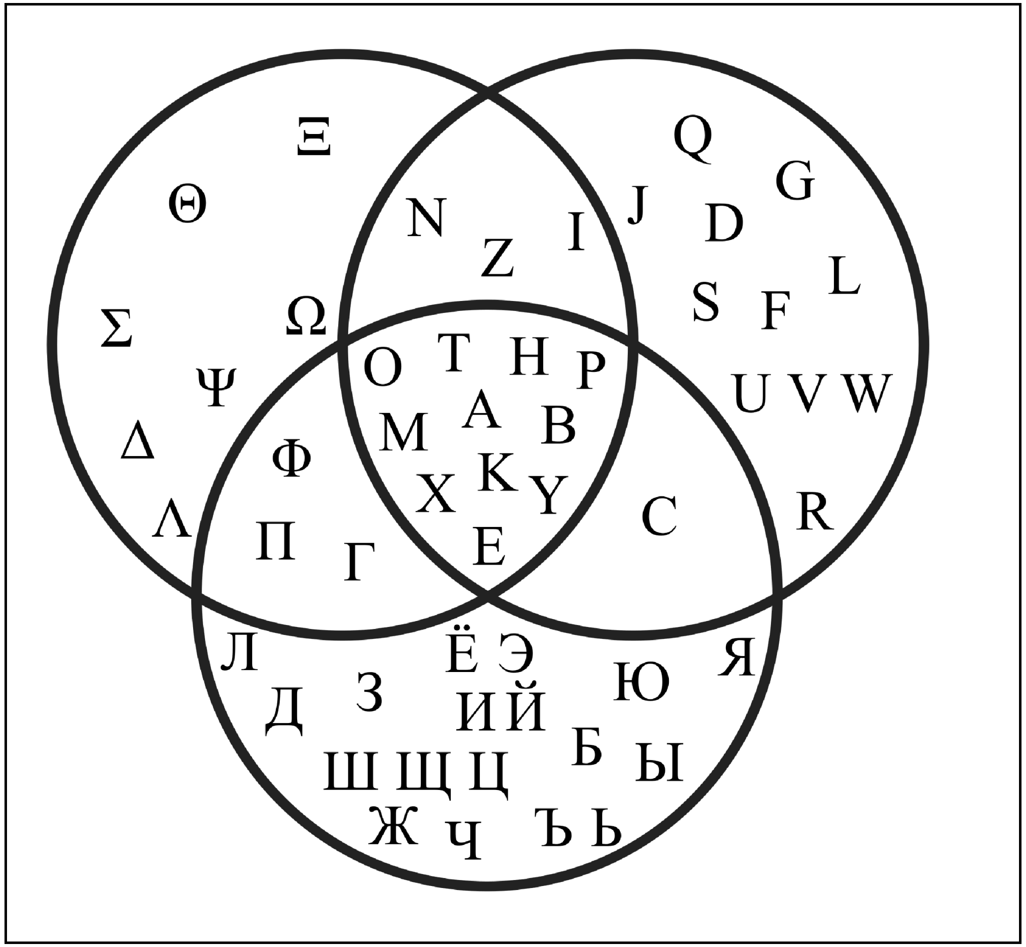
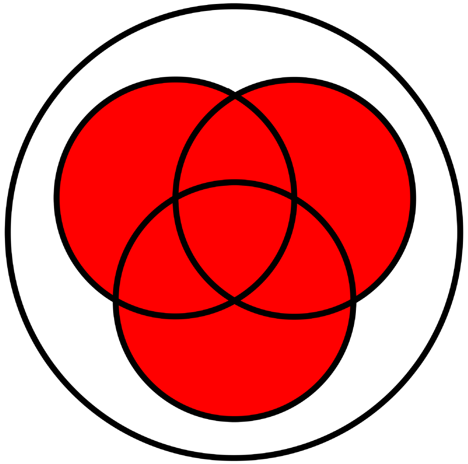
Venn diagrams[a]

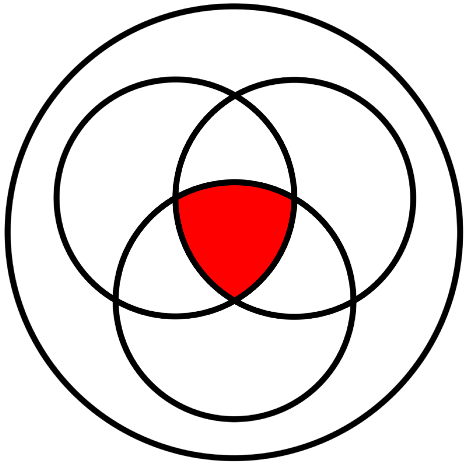


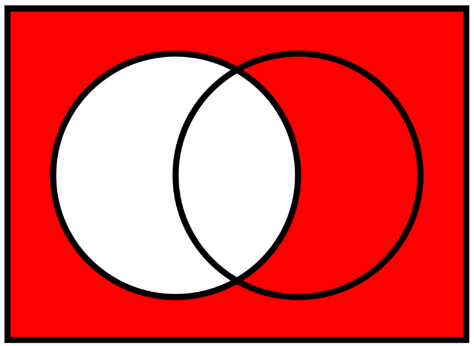
[a]From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

$\mathcal{P}(U) = \{ X \mid X \subseteq U \}$

The powerset Boolean algebra

$$\Big(\quad \mathcal{P}(U) \ , \quad \underset{\substack{\| \\ \{x \in U \mid \underline{false}\}}}{\emptyset} \ , \quad U = \{x \in U \mid \underline{true}\} \ , \quad \cup \ , \quad \cap \ , \quad (\cdot)^c \quad \Big)$$

For all $A, B \in \mathcal{P}(U)$,

$$A \cup B \ = \ \{x \in U \mid x \in A \vee x \in B\} \ \in \mathcal{P}(U)$$

$$A \cap B \ = \ \{x \in U \mid x \in A \wedge x \in B\} \ \in \mathcal{P}(U)$$

$$A^c \ = \ \{x \in U \mid \neg(x \in A)\} \qquad \in \mathcal{P}(U)$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathcal{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A \ = \ A \ = \ \mathcal{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad , \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) \ = \ A \ = \ A \cap (A \cup B)$$

$$A \cup (A \cap B) \stackrel{?}{=} A$$

$$\Leftrightarrow \left(A \cup (A \cap B) \stackrel{?}{\subseteq} A\right) \wedge \left(A \stackrel{?}{\subseteq} A \cup (A \cap B)\right)$$

(1) $A \subseteq A \cup (A \cap B)$ $\Longleftarrow$ **Lemma** $X \subseteq X \cup Y$

iff $\forall x .\ x \in X \Rightarrow x \in X \cup Y$

Assume $x \in X$.

RTP: $x \in X \vee x \in Y$

which follows by assumption.

(2) $A \cup (A \cap B) \subseteq A$

iff $\forall x .\ x \in A \cup (A \cap B) \Rightarrow x \in A$

Assume $x \in A \cup (A \cap B) \Leftrightarrow (x \in A \vee x \in A \cap B)$

RTP: $x \in A$. (*)

Assume $x \in A$

So (*) follows.

Assume $x \in A \cap B$

Then $x \in A$ and $x \in B$

so (*) follows.

► The complement operation $(\cdot)^c$ satisfies complementation laws.

$$A \cup A^c = \mathcal{U} \ , \quad A \cap A^c = \emptyset$$

NB: $(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

$(A^c)^c = A$

Proposition 105 *Let* U *be a set and let* $A, B \in \mathcal{P}(U)$.

1. $\forall X \in \mathcal{P}(U).\ A \cup B \subseteq X \iff (A \subseteq X \wedge B \subseteq X)$.
2. $\forall X \in \mathcal{P}(U).\ X \subseteq A \cap B \iff (X \subseteq A \wedge X \subseteq B)$.

PROOF: Let $X \in \mathcal{P}(U)$.

($\Longrightarrow$) Assume $A \cup B \subseteq X$

RTP: $A \subseteq X$ and $B \subseteq X$

Since $A \subseteq A \cup B$ and $A \cup B \subseteq X$, and $\subseteq$ is transitive

Then $A \subseteq X$. And similarly for $B \subseteq X$.

($\Longleftarrow$) Assume: $A \subseteq X$ and $B \subseteq X$

RTP: $A \cup B \subseteq X$

Let $x \in A \cup B$; that is, ($x \in A$ or $x \in B$)

($\Leftarrow$) Assume: (1) $A \subseteq X$ and (3) $B \subseteq X$

RTP: $A \cup B \subseteq X$

Let $x \in A \cup B$; That is, ($x \in A$ or $x \in B$)

RTP: $x \in X$

Assume (2) $x \in A$ RTP: $x \in X$ By (1) and (2), we are done.	Assume (4) $x \in B$ RTP: $x \in X$ By (3) and (4), we are done.

Corollary 106 *Let* U *be a set and let* $A, B, C \in \mathcal{P}(U)$.

1. $C = A \cup B$

   iff

   ① $[A \subseteq C \wedge B \subseteq C]$

   $\wedge$

   ② $[\forall X \in \mathcal{P}(U).\ (A \subseteq X \wedge B \subseteq X) \implies C \subseteq X]$

   The union of A and B is the smallest set that contains both A and B.

2. $C = A \cap B$

   iff

   ① $[C \subseteq A \wedge C \subseteq B]$

   $\wedge$

   ② $[\forall X \in \mathcal{P}(U).\ (X \subseteq A \wedge X \subseteq B) \implies X \subseteq C]$

   The intersection of A and B is the largest set that is contained in both A and B.

$\underline{NB}$: $\underline{\gcd}(m,n) \mid m \;\wedge\; \underline{\gcd}(m,n) \mid n$

$\wedge$

$\forall d.\; d \mid m \wedge d \mid n \Rightarrow d \mid \underline{\gcd}(m,n)$

Sets and logic

$\mathcal{P}(\mathbb{U})$	$\{\,\mathbf{false}\,,\ \mathbf{true}\,\}$
$\emptyset$	$\mathbf{false}$
$\mathbb{U}$	$\mathbf{true}$
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{c}$	$\neg(\cdot)$

Pairing axiom

$\boxed{\text{For every } a \text{ and } b\text{, there is a set with } a \text{ and } b \text{ as its only elements.}}$

$$\{ a , b \}$$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \lor x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{ a \}$, and referred to as a *singleton*.

Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

Proposition 107 *For all* a, b, c, x, y,

1. $\{x, y\} \subseteq \{a\} \implies x = y = a$

2. $\{c, x\} = \{c, y\} \implies x = y$

PROOF:

(1) Assume $\{x, y\} \subseteq \{a\}$.

Since $x \in \{x, y\} \Rightarrow x \in \{a\} \Rightarrow x = a$

$y \in \{x, y\} \Rightarrow y \in \{a\} \Rightarrow y = a$.

(2) Assume: $\{c, x\} = \{c, y\}$

$$\left.\begin{array}{l} \text{Since } x \in \{c, x\} = \{c, y\} \Longrightarrow \text{①} (x = c \vee x = y) \\ \text{Since } y \in \{c, y\} = \{c, x\} \Longrightarrow \text{②} (y = c \vee y = x) \end{array}\right\} \Rightarrow x = y$$

$\boxtimes$

KURATOWSKI PAIRING

$$K(a,b) := \{\{a\},\{a,b\}\}$$

<u>Assume</u>: $K(a,b) = K(x,y)$

That is, $\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$ (*)

<u>Then</u> $\{a\} \in \{\{x\},\{x,y\}\}$

and so (1) $\{a\} = \{x\}$ or (2) $\{a\} = \{x,y\}$

<u>Case (1)</u>: Then $a = x \Rightarrow \{\{a\},\{a,b\}\} = \{\{a\},\{a,y\}\}$

$\Rightarrow \{a,b\} = \{a,y\} \Rightarrow b = y.$

Case 2 : $\{a\} = \{x, y\} \Rightarrow a = x = y$

Then $\{\{a\}, \{a, b\}\} = \{\{a\}, \{a, a\}\}$

$\Rightarrow \{a, b\} = \{a, a\} = \{a\}$

$\Rightarrow a = b$.

We have shown:

$K(a, b) = K(x, y) \Rightarrow (a = x \text{ and } b = y)$.

Ordered pairing

Notation:

$$(a, b) \quad \text{or} \quad \langle a, b\rangle$$

Fundamental property:

$$(a, b) = (x, y) \implies a = x \wedge b = y$$

A construction:

For every pair a and b,

$$\langle a, b \rangle = \{ \{ a \}, \{ a, b \} \}$$

defines an *ordered pairing* of a and b.

Proposition 108 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff \big(a = x \wedge b = y\big) \quad .$$

PROOF:
This holds for $\langle a, b\rangle = K(a, b)$.

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b) \,\}$$

where

$$\begin{array}{l} \forall a_1, a_2 \in A, b_1, b_2 \in B. \\ \quad (a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \;\wedge\; b_1 = b_2) \end{array} \quad .$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

Pattern-matching notation

Example: The subset of ordered pairs from a set A with equal components is formally

$$\{ x \in A \times A \mid \exists\, a_1 \in A.\, \exists\, a_2 \in A.\, x = (a_1, a_2) \ \wedge \ a_1 = a_2 \}$$

but often abbreviated using *pattern-matching notation* as

$$\{ (a_1, a_2) \in A \times A \mid a_1 = a_2 \} \ .$$

Notation: For a property $P(a, b)$ with a ranging over a set A and b ranging over a set B,

$$\{ (a, b) \in A \times B \mid P(a, b) \}$$

abbreviates

$$\{ x \in A \times B \mid \exists\, a \in A.\, \exists\, b \in B.\, x = (a, b) \ \wedge \ P(a, b) \} \ .$$

Proposition 110 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

PROOF IDEA:

$A = \{a_1, \ldots, a_m\}$ $\quad$ $B = \{b_1, \ldots, b_n\}$

$A \times B = \{(a_1, b_1), \ldots, (a_i, b_j), \ldots, (a_m, b_n)\}$