Proof by contrapositive

Corollary 40 *For all statements P and Q,*

$$(P \implies Q) \iff (\neg Q \implies \neg P) \quad .$$

Btw Using the above equivalence to prove an implication is known as *proof by contrapositive*.

Corollary 41 *For every positive irrational number x, the real number $\sqrt{x}$ is irrational.*

Lemma 42 *A positive real number* x *is rational iff*

$$\exists \textit{ positive integers } m, n : \qquad (\dagger)$$
$$x = m/n \wedge \neg(\exists \textit{ prime } p : p \mid m \wedge p \mid n)$$

PROOF: ($\Leftarrow$) Assume: $\exists$ pos. int. m, n.
$(x = m/n) \wedge \neg(\exists \text{prime } p.\ p \mid m \wedge p \mid n)$

From assumption, let m_0 and n_0 be pos. int.
s.t. ① $x = m_0/n_0$ and $\neg(\exists \text{prime } p.\ p \mid m_0 \wedge p \mid n_0)$.
By ①, x is rational and we are done.

($\Rightarrow$) Assume: x is a positive rational; Then,
① $x = m_0/n_0$ for pos. int. m_0 and n_0

RTP:

$\exists$ pos. int. m and n.

$$x = m/n \wedge \neg(\exists \text{ prime } p.\ p \mid m \wedge p \mid n)$$

By contradiction, assume

$$\neg\left(\exists \text{ pos. int. } m \text{ and } n.\ x = m/n \wedge \neg(\exists \text{ prime } p.\ p \mid m \wedge p \mid n)\right)$$

$\iff$ $\forall$ pos. int. m and n.

$$\neg\left(x = m/n \wedge \neg(\exists \text{ prime } p.\ p \mid m \wedge p \mid n)\right)$$

$\iff$ $\forall$ pos. int. m and n.

$$\neg(x = m/n) \vee (\exists \text{ prime } p.\ p \mid m \wedge p \mid n)$$

$\Leftrightarrow$ (2) $\forall$ pos. int m and n.

$$\left(x = m/n \Rightarrow \exists \text{ prime } p.\ p \mid m \wedge p \mid n\right)$$

<u>Recall</u> (1) $x = m_0/n_0$ for pos. int m_0 and n_0.

By (2), we have (3) $\left(x = m_0/n_0 \Rightarrow \exists \text{ prime } p.\ p \mid m_0 \wedge p \mid n_0\right)$

From (1) and (3), we have (4) $\exists$ prime p. $p \mid m_0 \wedge p \mid n_0$

So let p_0 be a prime s.t. $p_0 \mid m_0$ and $p_0 \mid n_0$; That is,

$m_0 = p_0 \cdot m_1$ and $n_0 = p_0 \cdot n_1$ for pos. int. m_1 and n_1.

Then: (5) $x = m_0/n_0 = p_0 \cdot m_1 / p_0 \cdot n_1 = m_1/n_1$.

By (2), $\left(x = m_1/n_1 \Rightarrow \exists \text{ prime } p.\ p \mid m_1 \wedge p \mid n_1\right)$.

From 2 and 5 ; 6 $\exists$ prime p. $p \mid m_1 \wedge p \mid n_1$.

Let p_1 be a prime s.t. $p_1 \mid m_1 \wedge p_1 \mid n_1$; that is,
$m_1 = p_1 \cdot m_2$ and $n_1 = p_1 \cdot n_2$ for pos. int. m_2 and n_2.

Iterating the above we have,

$$m_0 = p_0 \cdot p_1 \cdot p_2 \cdot \dots \cdot p_k \cdot m_{k+1}$$
$$n_0 = p_0 \cdot p_1 \cdot p_2 \cdot \dots \cdot p_k \cdot n_{k+1}$$
for pos. int m_{k+1} and n_{k+1}

Then,
$$m_0 \geq \underbrace{2 \cdot 2 \cdot \dots \cdot 2}_{k \text{ times}} \cdot m_{k+1} > 2^k$$

For $k = m_0$, we have $m_0 > 2^{m_0}$, which is absurd. $\boxtimes$

Numbers

Objectives

- Get an appreciation for the abstract notion of number system, considering four examples: natural numbers, integers, rationals, and modular integers.
- Prove the correctness of three basic algorithms in the theory of numbers: the division algorithm, Euclid's algorithm, and the Extended Euclid's algorithm.
- Exemplify the use of the mathematical theory surrounding Euclid's Theorem and Fermat's Little Theorem in the context of public-key cryptography.
- To understand and be able to proficiently use the Principle of Mathematical Induction in its various forms.

Natural numbers

In the beginning there were the *natural numbers*

$$\mathbb{N} : \quad 0 \,, \quad 1 \,, \quad \ldots \,, \quad n \,, \quad n+1 \,, \quad \ldots$$

generated from *zero* by successive increment; that is, put in ML:

```
datatype
  N = zero | succ of N
```

The basic operations of this number system are:

- Addition

$$\underbrace{\overbrace{* \cdot \cdot \cdot *}^{m}\,\overbrace{* \cdot \cdot \cdot \cdot \cdot \cdot *}^{n}}_{m+n}$$

- Multiplication

$$m\left\{\begin{array}{c}\overbrace{* \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot *}^{n}\\ \vdots \qquad m \cdot n \qquad \vdots\\ * \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot *\end{array}\right.$$

NB: $l+m+n$ stand for $(l+m)+n = l+(m+n)$

The *additive structure* $(\mathbb{N}, 0, +)$ of natural numbers with zero and addition satisfies the following:

- Monoid laws

$$0 + n = n = n + 0 \quad , \quad (l + m) + n = l + (m + n)$$

- Commutativity law

$$m + n = n + m$$

and as such is what in the mathematical jargon is referred to as a *commutative monoid*.

Commutative monoid laws

- Neutral element laws

$$\overbrace{}^{0}\overbrace{* \cdots\cdots *}^{n} = \overbrace{* \cdots\cdots *}^{n} = \overbrace{* \cdots\cdots *}^{n}\overbrace{}^{0}$$

- Associativity law

$$\overbrace{* \cdots * * \cdots\cdots *}^{\ell+m}\overbrace{* \cdots\cdots\cdots *}^{n} = \overbrace{* \cdots *}^{\ell}\overbrace{* \cdots\cdots * * \cdots\cdots\cdots *}^{m+n}$$

- Commutativity law

$$\overbrace{* \cdots *}^{m}\overbrace{* \cdots\cdots *}^{n} = \overbrace{* \cdots\cdots *}^{n}\overbrace{* \cdots *}^{m}$$

Monoids

Definition 43 *A* monoid *is an algebraic structure with*

- *a* neutral element, *say* e,
- *a* binary operation, *say* $\bullet$,

satisfying

- neutral element laws*:* $e \bullet x = x = x \bullet e$
- associativity law*:* $(x \bullet y) \bullet z = x \bullet (y \bullet z)$

A monoid is commutative *if:*

- commutativity*:* $x \bullet y = y \bullet x$

is satisfied.

Ex:

(list α, nil, @)

is a monoid

That is

commutative

iff α has 1

or 0 elements.

Also the *multiplicative structure* $(\mathbb{N}, 1, \cdot)$ of natural numbers with one and multiplication is a commutative monoid:

- Monoid laws

$$1 \cdot n = n = n \cdot 1 \quad , \quad (l \cdot m) \cdot n = l \cdot (m \cdot n)$$

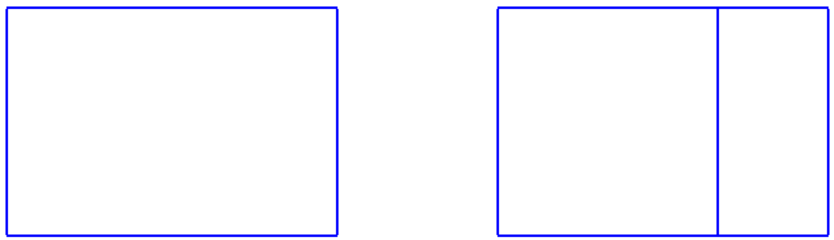
- Commutativity law

$$m \cdot n = n \cdot m$$

The additive and multiplicative structures interact nicely in that they satisfy the

- Distributive laws

$$
\begin{aligned}
l \cdot 0 &= 0 \\
l \cdot (m + n) &= l \cdot m + l \cdot n
\end{aligned}
$$



and make the overall structure $(\mathbb{N}, 0, +, 1, \cdot)$ into what in the mathematical jargon is referred to as a *commutative semiring*.

Semirings

Definition 44 *A* semiring *(or* rig*) is an algebraic structure with*

- *a* commutative monoid *structure, say* $(0, \oplus)$*,*
- *a* monoid structure*, say* $(1, \otimes)$*,*

satifying the distributivity laws*:*

- $0 \otimes x = 0 = x \otimes 0$
- $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$, $(y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x)$

A semiring is commutative *whenever* $\otimes$ *is.*

Cancellation

The additive and multiplicative structures of natural numbers further satisfy the following laws.

- Additive cancellation

  For all natural numbers k, m, n,

  $$k + m = k + n \implies m = n \quad .$$

- Multiplicative cancellation

  For all natural numbers k, m, n,

  $$\text{if } k \neq 0 \text{ then } k \cdot m = k \cdot n \implies m = n \quad .$$

Definition 45 *A binary operation* $\bullet$ *allows* cancellation *by an element* c

- on the left*: if* $c \bullet x = c \bullet y$ *implies* $x = y$
- on the right*: if* $x \bullet c = y \bullet c$ *implies* $x = y$

Example: The append operation on lists allows cancellation by any list on both the left and the right.

Inverses

Definition 46 *For a monoid with a neutral element* e *and a binary operation* $\bullet$*, and element* x *is said to admit an*

- inverse on the left *if there exists an element* ℓ *such that* $\ell \bullet x = e$
- inverse on the right *if there exists an element* r *such that* $x \bullet r = e$
- inverse *if it admits both left and right inverses*

Proposition 47 *For a monoid* $(e, \bullet)$ *if an element admits an inverse then its left and right inverses are equal.*

PROOF: Let x an inverse; so there is ℓ s.t. $\ell \cdot x = e$ and there is r s.t. $x \bullet r = e$. Then, $\ell = r$.

Indeed,

$$r = e \cdot r = (\ell \cdot x) \cdot r = \ell \cdot x \bullet r = \ell \cdot (x \cdot r) = \ell \cdot e = \ell \qquad \boxtimes$$

Groups

Definition 49 *A* group *is a monoid in which every element has an inverse.*

An Abelian group *is a group for which the monoid is commutative.*

Inverses

Definition 50

1. *A number x is said to admit an* additive inverse *whenever there exists a number y such that $x + y = 0$.*

2. *A number x is said to admit a* multiplicative inverse *whenever there exists a number y such that $x \cdot y = 1$.*

Extending the system of natural numbers to: (i) admit all additive inverses and then (ii) also admit all multiplicative inverses for non-zero numbers yields two very interesting results:

(i) the *integers*

$$\mathbb{Z} \ : \quad \ldots -n\,,\ \ldots\,,\ -1\,,\ 0\,,\ 1\,,\ \ldots\,,\ n\,,\ \ldots$$

which then form what in the mathematical jargon is referred to as a *commutative ring*, and

(ii) the *rationals* $\mathbb{Q}$ which then form what in the mathematical jargon is referred to as a *field*.

Rings

Definition 51 *A* ring *is a semiring* $(0, \oplus, 1, \otimes)$ *in which the commutative monoid* $(0, \oplus)$ *is a group.*

A ring is commutative *if so is the monoid* $(1, \otimes)$*.*

Fields

Definition 52 *A* field *is a commutative ring in which every element besides* 0 *has a reciprocal (that is, and inverse with respect to* $\otimes$*).*