Denotational Semantics

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Lectures for Part II CST 2024/2025

Practicalities

- My mail: mgapb2@cam.ac.uk. Do not hesitate to ask questions!
- Course notes will be updated, keep an eye on the course webpage.

Introduction

What is this course about?

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- Programming language semantics: what is the (mathematical) meaning of a program?

Goal: give an abstract and compositional (mathematical) model of programs.

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- **Documentation**: precise but intuitive, machine-independent specification.
- **Language design**: feedback from semantics (functional programming, monads & handlers, linearity…).
- **Rigour**: powerful way to justify formal methods.

Styles of formal semantics

- Operational
- Axiomatic
- Denotational

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- Denotational

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- **Operational**: meaning of a program in terms of the *steps of computation* it takes during execution (see Part IB Semantics).
- **Axiomatic**: meaning of a program in terms of a *program logic* to reason about it (see Part II Hoare Logic & Model Checking).
- **Denotational**: meaning of a program defined abstractly as object of some suitable *mathematical structure* (see this course).

Denotational semantics in a nutshell

$$
\begin{aligned}
\text{Syntax} &\xrightarrow{[\![-]\!]} \text{Semantics} \\
\text{Program } P &\mapsto \text{Denotation } [\![P]\!] \\
\\
\text{Arithmetic expression} &\mapsto \text{Number} \\
\text{Boolean circuit} &\mapsto \text{Boolean function} \\
\text{Recursive program} &\mapsto \text{Partial recursive function} \\
&\dots
\end{aligned}
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\text{Recursive program} &\quad \mapsto \quad \text{Partial recursive function} \\
&\quad \dots \\
\text{Type} &\quad \mapsto \quad \text{Domain} \\
\text{Program} &\quad \mapsto \quad \text{Continuous functions between domains}
\end{aligned}$$

Properties of denotational semantics

Abstraction

- mathematical object, implementation/machine independent;
- captures the concept of a programming language construct;
- should relate to practical implementations, though…

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Compositionality

- The denotation of a whole is defined using the *denotation* of its parts;
- $[\![P]\!]$ represents the contribution of P to *any* program containing P;
- More flexible and expressive than whole-program semantics.

Introduction

A basic example

Imp syntax

Programs

$$C \in \mathbf{Prog} ::= \texttt{skip} \mid L := A \mid C; C \mid \texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ B\ \texttt{do}\ C$$

Programs

ranges over a set $\mathbb{L}$ of *locations*

$$C \in \mathbf{Prog} ::= \mathtt{skip} \mid L := A \mid C; C \mid \mathtt{if}\ B\ \mathtt{then}\ C\ \mathtt{else}\ C \mid \mathtt{while}\ B\ \mathtt{do}\ C$$

Imp syntax

Arithmetic expressions

$$A \in \mathbf{Aexp} ::= \underline{n} \mid L \mid A + A \mid \ldots$$

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ranges over *integers*

Arithmetic expressions

$$A \in \mathbf{Aexp} ::= \underline{n} \mid L \mid A + A \mid \ldots$$

Programs

$$C \in \mathbf{Prog} ::= \texttt{skip} \mid L := A \mid C;C \mid \texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ B\ \texttt{do}\ C$$

Imp syntax

Arithmetic expressions

$$A \in \mathbf{Aexp} ::= \underline{n} \mid L \mid A + A \mid \ldots$$

Boolean expressions

$$B \in \mathbf{Bexp} ::= \mathtt{true} \mid \mathtt{false} \mid A = A \mid \neg B \mid \ldots$$

Programs

$$C \in \mathbf{Prog} ::= \mathtt{skip} \mid L := A \mid C; C \mid \mathtt{if}\ B\ \mathtt{then}\ C\ \mathtt{else}\ C \mid \mathtt{while}\ B\ \mathtt{do}\ C$$

Denotation functions – naïvely

$$\mathcal{A}: \quad \mathbf{Aexp} \to \mathbb{Z}$$

where

$$\mathbb{Z} \quad = \quad \{\ldots, -1, 0, 1, \ldots\}$$

Denotation functions – naïvely

$$\mathcal{A} : \quad \mathbf{Aexp} \rightarrow \mathbb{Z}$$
$$\mathcal{B} : \quad \mathbf{Bexp} \rightarrow \mathbb{B}$$

where

$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$$
$$\mathbb{B} = \{\text{true}, \text{false}\}$$

Arithmetic expressions?

$$\mathcal{A}[\![\underline{n}]\!] = n$$

$$\mathcal{A}[\![A_1 + A_2]\!] = \mathcal{A}[\![A_1]\!] + \mathcal{A}[\![A_2]\!]$$

Arithmetic expressions?

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\mathcal{A}[\![\underline{n}]\!] &= n \\
\mathcal{A}[\![A_1 + A_2]\!] &= \mathcal{A}[\![A_1]\!] + \mathcal{A}[\![A_2]\!] \\
\mathcal{A}[\![L]\!] &= ???
\end{aligned}$$

Denotation functions – less naïvely

$$\text{State} = (\mathbb{L} \rightarrow \mathbb{Z})$$

Denotation functions – less naïvely

$$\text{State} = (\mathbb{L} \to \mathbb{Z})$$

$$\mathcal{A} : \mathbf{Aexp} \to (\text{State} \to \mathbb{Z})$$
$$\mathcal{B} : \mathbf{Bexp} \to (\text{State} \to \mathbb{B})$$

where

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$$
$$\mathbb{B} = \{\text{true}, \text{false}\}\,.$$

$$\text{State} = (\mathbb{L} \to \mathbb{Z})$$

$$\begin{aligned}
\mathcal{A} &: \textbf{Aexp} \to (\text{State} \to \mathbb{Z}) \\
\mathcal{B} &: \textbf{Bexp} \to (\text{State} \to \mathbb{B}) \\
\mathcal{C} &: \textbf{Prog} \to (\text{State} \to \text{State})
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{Z} &= \{\dots, -1, 0, 1, \dots\} \\
\mathbb{B} &= \{\text{true}, \text{false}\}\,.
\end{aligned}$$

Semantics of arithmetic expressions

$$\mathcal{A}[\![\underline{n}]\!] = \lambda s \in \text{State}.\, n$$

$$\mathcal{A}[\![A_1 + A_2]\!] = \lambda s \in \text{State}.\, \mathcal{A}[\![A_1]\!](s) + \mathcal{A}[\![A_2]\!](s)$$

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\mathcal{A}\left[\!\left[\underline{n}\right]\!\right] &= \lambda s \in \text{State}.\, n \\
\mathcal{A}[\![A_1 + A_2]\!] &= \lambda s \in \text{State}.\, \mathcal{A}[\![A_1]\!]\,(s) + \mathcal{A}[\![A_2]\!]\,(s) \\
\mathcal{A}[\![L]\!] &= \lambda s \in \text{State}.\, s(L)
\end{aligned}
$$

Semantics of boolean expressions

$$
\begin{aligned}
\mathcal{B}[\![\texttt{true}]\!] &= \lambda s \in \text{State}.\ \text{true} \\
\mathcal{B}[\![\texttt{false}]\!] &= \lambda s \in \text{State}.\ \text{false} \\
\mathcal{B}[\![A_1 = A_2]\!] &= \lambda s \in \text{State}.\ \text{eq}\,(\mathcal{A}[\![A_1]\!]\,(s), \mathcal{A}[\![A_2]\!]\,(s)) \\
& \quad \text{where } \text{eq}(a, a') = \begin{cases} \text{true} & \text{if } a = a' \\ \text{false} & \text{if } a \neq a' \end{cases}
\end{aligned}
$$

Semantics of programs

$$\mathcal{C}[\![\texttt{skip}]\!] \quad = \quad \lambda s \in \text{State}.\ s$$

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$$\mathcal{C}[\![\texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C']\!] = \lambda s \in \text{State}.\ \text{if}\,(\mathcal{B}[\![B]\!]\,(s), \mathcal{C}[\![C]\!]\,(s), \mathcal{C}[\![C']\!]\,(s))$$

$$\text{where}\ \text{if}(b, x, x') = \begin{cases} x & \text{if } b = \text{true} \\ x' & \text{if } b = \text{false} \end{cases}$$

$$\mathcal{C}[\![\texttt{skip}]\!] = \lambda s \in \text{State}.\ s$$

This is compositionality!

$$\mathcal{C}[\![\texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C']\!] = \lambda s \in \text{State}.\ \text{if}\,(\mathcal{B}[\![B]\!]\,(s), \mathcal{C}[\![C]\!]\,(s), \mathcal{C}[\![C']\!]\,(s))$$

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Semantics of programs

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$$\mathcal{C}[\![L := A]\!] = \lambda s \in \text{State}.\ s[L \mapsto \mathcal{A}[\![A]\!](s)]$$

$$\text{where } s[L \mapsto n](L') = \begin{cases} n & \text{if } L' = L \\ s(L) & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
\mathcal{C}[\![\texttt{skip}]\!] &= \lambda s \in \mathrm{State}.\, s \\[1em]
\mathcal{C}\left[\!\left[\texttt{if } B \texttt{ then } C \texttt{ else } C'\right]\!\right] &= \lambda s \in \mathrm{State}.\, \mathrm{if}\,(\mathcal{B}[\![B]\!]\,(s), \mathcal{C}[\![C]\!]\,(s), \mathcal{C}\left[\!\left[C'\right]\!\right](s)) \\
&\ \text{where } \mathrm{if}(b, x, x') = \begin{cases} x & \text{if } b = \mathrm{true} \\ x' & \text{if } b = \mathrm{false} \end{cases} \\[1em]
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&\ \text{where } s[L \mapsto n](L') = \begin{cases} n & \text{if } L' = L \\ s(L) & \text{otherwise} \end{cases} \\[1em]
\mathcal{C}\left[\!\left[C; C'\right]\!\right] &= \mathcal{C}\left[\!\left[C'\right]\!\right] \circ \mathcal{C}[\![C]\!] \\
&= \lambda s \in \mathrm{State}.\, \mathcal{C}\left[\!\left[C'\right]\!\right](\mathcal{C}[\![C]\!]\,(s))
\end{aligned}
$$

Introduction

A semantics for loops

Semantics of loops?

This is all very nice, but...

$$[\![\texttt{while}\ B\ \texttt{do}\ C]\!] = ???$$

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$$[\![\texttt{while}\ B\ \texttt{do}\ C]\!] = ???$$

Remember:

- $(\texttt{while}\ B\ \texttt{do}\ C, s) \rightsquigarrow (\texttt{if}\ B\ \texttt{then}\ (C; \texttt{while}\ B\ \texttt{do}\ C)\ \texttt{else}\ \texttt{skip}, s)$
- we want a *compositional* semantic: $[\![\texttt{while}\ B\ \texttt{do}\ C]\!]$ in terms of $[\![C]\!]$ and $[\![B]\!]$

$$\begin{aligned}\llbracket \texttt{while}\ B\ \texttt{do}\ C \rrbracket &= \llbracket \texttt{if}\ B\ \texttt{then}\ (C; \texttt{while}\ B\ \texttt{do}\ C)\ \texttt{else}\ \texttt{skip} \rrbracket \\ &= \lambda s \in \mathrm{State}.\ \mathrm{if}(\llbracket B \rrbracket, \llbracket \texttt{while}\ B\ \texttt{do}\ C \rrbracket \circ \llbracket C \rrbracket (s), s)\end{aligned}$$

$$
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[\![\texttt{while}\ B\ \texttt{do}\ C]\!] &= [\![\texttt{if}\ B\ \texttt{then}\ (C;\texttt{while}\ B\ \texttt{do}\ C)\ \texttt{else}\ \texttt{skip}]\!] \\
&= \lambda s \in \mathrm{State}.\ \mathrm{if}([\![B]\!], [\![\texttt{while}\ B\ \texttt{do}\ C]\!] \circ [\![C]\!](s), s)
\end{aligned}
$$

Not a direct definition for $[\![\texttt{while}\ B\ \texttt{do}\ C]\!]$... But a fixed point equation!

$$
[\![\texttt{while}\ B\ \texttt{do}\ C]\!] = F_{[\![B]\!],[\![C]\!]}([\![\texttt{while}\ B\ \texttt{do}\ C]\!])
$$

$$
\text{where} \quad F_{b,c} : \begin{array}{rcl} (\mathrm{State} \to \mathrm{State}) & \to & (\mathrm{State} \to \mathrm{State}) \\ w & \mapsto & \lambda s \in \mathrm{State}.\ \mathrm{if}(b(s), w \circ c(s), s). \end{array}
$$

Now we have a goal

- Why/when does $w = F_{b,c}(w)$ have a solution?
- What if it has several solutions? Which one should be our $[\![\texttt{while}\ B\ \texttt{do}\ C]\!]$?

Introduction

A taste of domain theory

Total functions are not enough

Forget about State for a second, consider these equations ($f \in \mathbb{Z} \to \mathbb{Z}$):

$$f(x) = f(x) + 1 \tag{1}$$

$$f(x) = f(x) \tag{2}$$

What about their fixed points?

Forget about State for a second, consider these equations ($f \in \mathbb{Z} \to \mathbb{Z}$) :

$$f(x) = f(x) + 1 \tag{1}$$
$$f(x) = f(x) \tag{2}$$

What about their fixed points?

- No function satisfies Eq. (1)!
- All functions satisfy Eq. (2)!

Both functions should diverge!

Partial functions to the rescue

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New rule: partial functions $f \in \mathbb{Z} \rightharpoonup \mathbb{Z}$

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has a unique solution: the nowhere-defined function $\bot$

Partial functions to the rescue

Both functions should diverge!

New rule: partial functions $f \in \mathbb{Z} \rightharpoonup \mathbb{Z}$

$$f(x) = f(x) + 1$$

has a unique solution: the nowhere-defined function $\bot$

But

$$f(x) = f(x)$$

Has even more solutions now...

Partial order on $\mathbb{Z} \rightharpoonup \mathbb{Z}$:

$w \sqsubseteq w'$ if for all $s \in \mathbb{Z}$, if w is defined at s so is w' and moreover $w(s) = w'(s)$.
if the graph of w is included in the graph of w'.

An order on partial functions

Partial order on $\mathbb{Z} \rightharpoonup \mathbb{Z}$:

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Least element $\bot \in \mathbb{Z} \rightharpoonup \mathbb{Z}$:

$\bot$ = totally undefined partial function

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Least element $\bot \in \mathbb{Z} \rightharpoonup \mathbb{Z}$:

$\bot$ = totally undefined partial function

$\bot$ is the least solution to $f(x) = f(x)$, making it "canonical".

Back to loops

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$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

Back to loops

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$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

should be some w such that:

$$w = F_{[\![X>0]\!],[\![Y:=X*Y;X:=X-1]\!]}(w).$$

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should be some w such that:

$$w = F_{[\![X>0]\!],[\![Y:=X*Y;X:=X-1]\!]}(w).$$

That is, we are looking for a fixed point of the following F:

$$\begin{aligned} F : (\text{State} \rightharpoonup \text{State}) &\to (\text{State} \rightharpoonup \text{State}) \\ w &\mapsto \lambda[X \mapsto x, Y \mapsto y].\begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w\,([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0 \end{cases} \end{aligned}$$

Approximating the least fixed point

Define $w_n = F^n(w)$, that is $\begin{cases} w_0 & = \bot \\ w_{n+1} & = F(w_n) \end{cases}$.

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$$w_1[X \mapsto x, Y \mapsto y] = F(\bot)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ \text{undefined} & \text{if } x \geq 1 \end{cases}$$

Approximating the least fixed point

Define $w_n = F^n(w)$, that is $\begin{cases} w_0 & = \bot \\ w_{n+1} & = F(w_n) \end{cases}$.

$$w_2[X \mapsto x, Y \mapsto y] = F(w_1)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto y] & \text{if } x = 1 \\ \text{undefined} & \text{if } x \geq 2 \end{cases}$$

Define $w_n = F^n(w)$, that is $\begin{cases} w_0 & = \bot \\ w_{n+1} & = F(w_n) \end{cases}$.

$$w_n[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } 0 \leq x < n \\ \text{undefined} & \text{if } x \geq n \end{cases}$$

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$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots$$

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$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots \sqsubseteq w_\infty?$$

Approximating the least fixed point

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$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots \sqsubseteq w_\infty$$

$$w_\infty[X \mapsto x, Y \mapsto y] = \bigsqcup_{i \in \mathbb{N}} w_i = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x \geq 0 \end{cases}$$

We have our semantics

$$F(w_\infty)[X \mapsto x, Y \mapsto y]$$

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$$F(w_\infty)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w_\infty[X \mapsto x - 1, Y \mapsto x \cdot y] & \text{if } x > 0 \end{cases} \qquad \text{(definition of } F\text{)}$$

We have our semantics

$$F(w_\infty)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w_\infty[X \mapsto x-1, Y \mapsto x \cdot y] & \text{if } x > 0 \end{cases} \qquad \text{(definition of } F\text{)}$$

$$= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x-1)! \cdot x \cdot y] & \text{if } x > 0 \end{cases} \qquad \text{(definition of } w_\infty\text{)}$$

$$
\begin{aligned}
F(w_\infty)[X \mapsto x, Y \mapsto y] &= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w_\infty[X \mapsto x-1, Y \mapsto x \cdot y] & \text{if } x > 0 \end{cases} && \text{(definition of } F\text{)} \\
&= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x-1)! \cdot x \cdot y] & \text{if } x > 0 \end{cases} && \text{(definition of } w_\infty\text{)} \\
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\begin{aligned}
F(w_\infty)[X \mapsto x, Y \mapsto y] &= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w_\infty[X \mapsto x-1, Y \mapsto x \cdot y] & \text{if } x > 0 \end{cases} && \text{(definition of } F\text{)} \\
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&= w_\infty[X \mapsto x, Y \mapsto y]
\end{aligned}
$$

- $F(w_\infty) = w_\infty$ *i.e.* w_∞ is a fixed point of F;
- actually, the least fixed point;
- which agrees with the operational semantics (!)

The rest of this course

Part I domain theory → building mathematical tools

Part II denotational semantics for PCF

Least Fixed Points

Least Fixed Points

Posets and monotone functions

Partially ordered set

A partial order on a set D is a binary relation $\sqsubseteq$ that is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

antisymmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Partially ordered set

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reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

antisymmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

$$\textsc{Refl}\ \frac{}{x \sqsubseteq x} \qquad \textsc{Trans}\ \frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z} \qquad \textsc{ASym}\ \frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions $X \rightharpoonup Y$

Underlying set: partial functions f with domain of definition $\mathbf{dom}(f) \subseteq X$ and taking values in Y;

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Proof!

Monotonicity

A function $f\colon D \to E$ between posets is monotone if

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$$\textsc{Mon}\ \frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)}$$

Least Fixed Points

Least elements and pre-fixed points

Least element

An element $d \in S$ is the least element of S if it satisfies

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If it exists, it is unique , and is written $\bot_S$, or simply $\bot$.

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An element $d \in S$ is the **least** element of S if it satisfies

$$\forall x \in S.\ d \sqsubseteq x.$$

If it exists, it is unique , and is written $\bot_S$, or simply $\bot$.

$$\textsc{Least}\ \frac{x \in S}{\bot_S \sqsubseteq x} \qquad \textsc{Asym}\ \frac{\textsc{Least}\ \dfrac{\bot'_S \in S}{\bot_S \sqsubseteq \bot'_S} \qquad \textsc{Least}\ \dfrac{\bot_S \in S}{\bot'_S \sqsubseteq \bot_S}}{\bot_S = \bot'_S}$$

Fixed point

A fixed point for a function $f\colon D \to D$ is an element $d \in D$ satisfying $f(d) = d$.

(Least) pre-fixed point

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

(Least) pre-fixed point

An element $d \in D$ is a **pre-fixed point** of f if it satisfies $f(d) \sqsubseteq d$.

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$$\mathrm{fix}(f)$$

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The **least pre-fixed point** of f, if it exists, will be written

$$\mathrm{fix}(f)$$

It is thus (uniquely) specified by the two properties:

$$\textsc{lfp-fix}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \textsc{lfp-least}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

Proofs with least fixed points

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)}$$

The least pre-fixed point is a pre-fixed point.

Proofs with least fixed points

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \text{LFP-LEAST}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

To prove $\mathrm{fix}(f) \sqsubseteq d$, it is enough to show $f(d) \sqsubseteq d$.

$$\textsc{lfp-fix}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \textsc{lfp-least}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

Application: least pre-fixed points of monotone functions are (least) fixed points.

$$\textsc{Asym}\ \frac{\textsc{lfp-fix}\ \dfrac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \dfrac{}{\mathrm{fix}(f) \sqsubseteq f(\mathrm{fix}(f))}}{f(\mathrm{fix}(f)) = \mathrm{fix}(f)}$$

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Application: least pre-fixed points of monotone functions are (least) fixed points.

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Least Fixed Points

Least upper bounds

Least upper bound of a chain

The **least upper bound** of countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$, written $\bigsqcup_{n\geq 0} d_n$, satisfies the two following properties:

$$\text{LUB-BOUND} \quad \frac{}{x_i \sqsubseteq \bigsqcup_{n\geq 0} x_n} \qquad\qquad \text{LUB-LEAST} \quad \frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n\geq 0} x_n \sqsubseteq x}$$

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- Other names: supremum, limit...
- Might write simply $\bigsqcup_n d_n$ or even $\bigsqcup d_n$
- Only lubs of chains – but can be generalized
- $\bigsqcup_{i \geq 0} d_i$ need not be one of the d_i – this is the interesting case!

Properties of lubs

Lubs are unique.

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Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

Properties of lubs

Lubs are unique.

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\textsc{lub-mon}\ \frac{\forall i.\ d_i \sqsubseteq e_i}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

Properties of lubs

Lubs are unique.

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

For any d, $\bigsqcup_n d = d$.

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For any chain and $N \in \mathbb{N}$, $\bigsqcup_n d_n = \bigsqcup_n d_{n+N}$.

Properties of lubs

Lubs are unique (if they exist).

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$ (if they exist).

For any d, $\bigsqcup_n d = d$ (and in particular it exists).

For any chain and $N \in \mathbb{N}$, $\bigsqcup_n d_n = \bigsqcup_n d_{n+N}$ (if any of the two exists).

Assume $d_{m,n} \in D$ $(m, n \geq 0)$ satisfies

$$m \leq m' \wedge n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}.$$

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Then, assuming they exist, the lubs form two chains

$$\bigsqcup_{n\geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n\geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n\geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m\geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m\geq 0} d_{m,2} \sqsubseteq \ldots$$

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Moreover, again assuming the lubs of these chains exist,

$$\bigsqcup_{m\geq 0}\left(\bigsqcup_{n\geq 0} d_{m,n}\right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0}\left(\bigsqcup_{m\geq 0} d_{m,n}\right) .$$

Least Fixed Points

Complete partial orders and domains

A chain complete poset/cpo is a poset $(D, \sqsubseteq)$ in which all chains have least upper bounds.

Cpos and domains

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Beware: the lub need only exist if the x_i form a chain!

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A domain is a cpo with a least element $\perp$.

Domain of partial functions

Least element: $\perp$ is the totally undefined function.

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Lub of a chain: $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ has lub f such that

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \mathbf{dom}(f_n) \text{ for some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

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Beware: the definition of $\bigsqcup_{n \geq 0} f_n$ is unambiguous only if the f_i form a chain!

Finite cpos

Finite posets are always cpos – why?

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Are they always domains?

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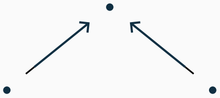
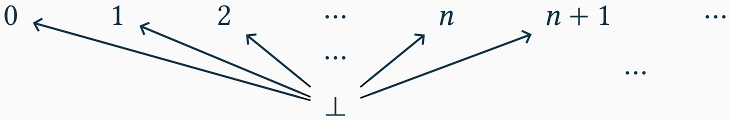


The flat natural numbers $\mathbb{N}_\perp$



$$\begin{array}{c} n+1 \\ \uparrow \\ n \\ \vdots \\ 1 \\ \uparrow \\ 0 \end{array}$$

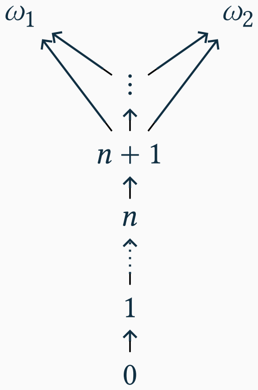
No! (Why?)

Vertical natural numbers

$$\begin{array}{c} \omega \\ \uparrow \\ \vdots \\ n+1 \\ \uparrow \\ n \\ \uparrow \\ \vdots \\ 1 \\ \uparrow \\ 0 \end{array}$$

Yes!

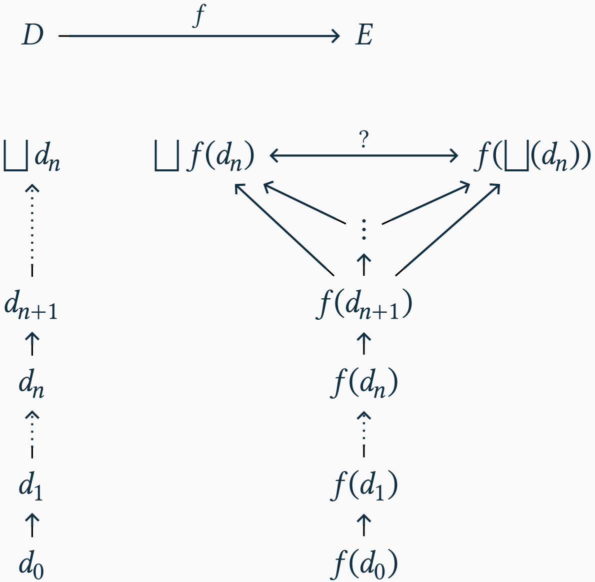
Vertical natural numbers

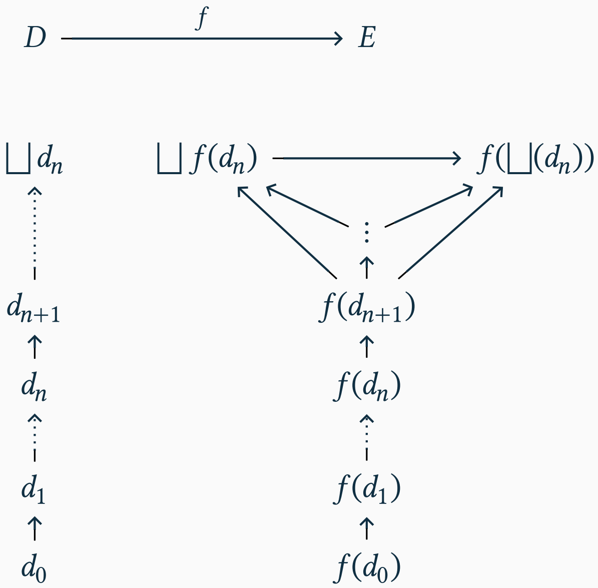


No! (Why?)

Least Fixed Points

Continuous functions


D
f
E
$\bigsqcup d_n$
$\bigsqcup f(d_n)$
?
$f(\bigsqcup(d_n))$
d_{n+1}
$f(d_{n+1})$
d_n
$f(d_n)$
d_1
$f(d_1)$
d_0
$f(d_0)$


D
f
E
⊔ d_n
⊔ f(d_n)
f(⊔(d_n))
d_{n+1}
f(d_{n+1})
d_n
f(d_n)
d_1
f(d_1)
d_0
f(d_0)

Continuity and strictness

Given two cpos D and E, a function $f : D \to E$ is continuous if

- it is monotone, and
- it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, we have

$$f\left(\bigsqcup_{n \geq 0} d_n\right) = \bigsqcup_{n \geq 0} f(d_n)$$

Note: one direction is automatic.

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Note: one direction is automatic.

A function f is strict if $f(\bot_D) = \bot_E$.

Thesis

All computable functions are continuous.

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Typical non-continuous function: "is a sequence the constant 0"? $(\mathbb{N} \rightharpoonup \mathbb{B}) \rightharpoonup \mathbb{B}$

$$
\begin{array}{llllllll}
0 & 0 & \bot & \ldots & & & & \mapsto \bot \\
0 & 0 & 0 & 0 & 1 & \ldots & & \mapsto 1 \\
\\
0 & 0 & 0 & 0 & 0 & \overline{0} & & \mapsto 0
\end{array}
$$

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0 & 0 & 0 & 0 & 0 & \ldots & & \mapsto ? \\
 & & & & & & & \\
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Intuition: non-continuity ≈ "jump at infinity" ≈ non-computability

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$$
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0 & 0 & \bot & \dots & & & & & & & \mapsto \bot \\
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\end{array}
$$

Intuition: non-continuity $\approx$ "jump at infinity" $\approx$ non-computability

Later in the course: show the thesis… by giving a denotational semantics.

Least Fixed Points

Kleene's fixed point theorem

Let $f : D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

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$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

It is thus also the least fixed point of f!

Constructions on Domains

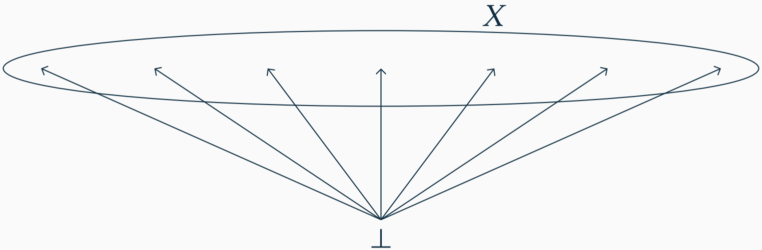
Constructions on Domains

Flat domains

Flat domain on X

The flat domain on a set X is defined by:

- its underlying set $X \biguplus \{\bot\}$;
- $x \sqsubseteq x'$ if either $x = \bot$ or $x = x'$.



Flat domain lifting

Let $f : X \rightharpoonup Y$ be a partial function between two sets. Then

$$
\begin{aligned}
f_\perp : \quad X_\perp \quad &\to \quad Y_\perp \\
d \quad &\mapsto \begin{cases} f(d) & \text{if } d \in X \text{ and } f \text{ is defined at } d \\ \perp & \text{if } d \in X \text{ and } f \text{ is not defined at } d \\ \perp & \text{if } d = \perp \end{cases}
\end{aligned}
$$

defines a strict continuous function between the corresponding flat domains.

Constructions on Domains

Products of domains

Binary product

The product of two posets $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \wedge d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \stackrel{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \wedge d_2 \sqsubseteq_2 d_2'$$

Binary product

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$$\textsc{pox}\ \frac{d_1 \sqsubseteq_1 d_1' \qquad d_2 \sqsubseteq_2 d_2'}{(d_1, d_2) \sqsubseteq (d_1', d_2')}$$

lubs of chains are computed componentwise:

$$\bigsqcup_{n \geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}).$$

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If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ have least elements, so does $(D_1 \times D_2, \sqsubseteq)$ with

$$\bot_{D_1\times D_2} = (\bot_{D_1}, \bot_{D_2})$$

Componentwise lubs and least elements

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$$\bot_{D_1 \times D_2} = (\bot_{D_1}, \bot_{D_2})$$

Products of cpos (domains) are cpos (domains).

Functions of two arguments

A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e').$$

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$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e').$$

Moreover, it is continuous if and only if it preserves lubs in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m\, ,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$
$$f(d\, ,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

Derived rules for functions of two arguments

$$\text{MON}\times \; \frac{f \text{ monotone} \qquad x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x, y) \sqsubseteq f(x', y')}$$

$$f\left(\bigsqcup_m x_m, \bigsqcup_n y_n\right) = \bigsqcup_m \bigsqcup_n f(x_m, y_n) = \bigsqcup_k f(x_k, y_k)$$

Projection and pairing

Let D_1 and D_2 be cpos. The projections

$$\begin{aligned} \pi_1 : \quad D_1 \times D_2 &\rightarrow D_1 \\ (d_1, d_2) &\mapsto d_1 \end{aligned} \qquad\qquad \begin{aligned} \pi_2 : \quad D_1 \times D_2 &\rightarrow D_2 \\ (d_1, d_2) &\mapsto d_2 \end{aligned}$$

are continuous functions.

Projection and pairing

Let D_1 and D_2 be cpos. The projections

$$\begin{array}{rrcl} \pi_1 : & D_1 \times D_2 & \to & D_1 \\ & (d_1, d_2) & \mapsto & d_1 \end{array} \qquad \begin{array}{rrcl} \pi_2 : & D_1 \times D_2 & \to & D_2 \\ & (d_1, d_2) & \mapsto & d_2 \end{array}$$

are continuous functions.

If $f_1 : D \to D_1$ and $f_2 : D \to D_2$ are continuous functions from a cpo D, then the pairing function

$$\begin{array}{rrcl} \langle f_1, f_2 \rangle : & D & \to & D_1 \times D_2 \\ & d & \mapsto & (f_1(d), f_2(d)) \end{array}$$

is continuous.

For any domain D, the conditional function

$$\begin{aligned} \mathrm{if} : \quad \mathbb{B}_\perp \times (D \times D) \quad &\to \quad D \\ (x, d) \qquad\qquad &\mapsto \quad \begin{cases} \pi_1(d) & \text{if } x = \mathrm{true} \\ \pi_2(d) & \text{if } x = \mathrm{false} \\ \perp_D & \text{if } x = \perp \end{cases} \end{aligned}$$

is continuous.

General product

Given a set I, suppose that for each $i \in I$ we are given a set X_i. The (cartesian) product of the X_i is

$$\prod_{i \in I} X_i$$

Two ways to see it:

- tuples: $(\dots, x_i, \dots)_{i \in I}$ such that $x_i \in X_i$;

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Projections (for any $i \in I$):

$$\pi_i : \left(\prod_{i \in I} X_i\right) \to X_i$$

General product of domains

Given a set I, suppose that for each $i \in I$ we are given a cpo $(D_i, \sqsubseteq_i)$. The product of this whole family of cpos has

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I-indexed products of cpos (domains) are cpos (domains), and projections are continuous.

Constructions on Domains

Function domains

Given two cpos $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the **function cpo** $(D \to E, \sqsubseteq)$ has underlying set

$$\{f : D \to E \mid \text{ is a } \textit{continuous} \text{ function}\}$$

equipped with the pointwise order:

$$f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D.\, f(d) \sqsubseteq_E f'(d).$$

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$$\frac{f \sqsubseteq_{D \to E} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq_E g(y)}$$

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Argumentwise least elements and lubs:

$$\bot_{D \to E}(d) = \bot_E \qquad \left(\bigsqcup_{n \geq 0} f_n\right)(d) = \bigsqcup_{n \geq 0} f_n(d)$$

Evaluation, currying ($f : (D' \times D) \to E$) and composition

$$\begin{aligned} \text{eval} : \quad (D \to E) \times D &\to E \\ (f, d) &\mapsto f(d) \end{aligned}$$

$$\begin{aligned} \text{cur}(f) : \quad D' &\to (D \to E) \\ d' &\mapsto \lambda d \in D.\ f(d', d) \end{aligned}$$

$$\begin{aligned} \circ : \quad \big((E \to F) \times (D \to E)\big) &\longrightarrow (D \to F) \\ (f, g) &\mapsto \lambda d \in D.\ g(f(d)) \end{aligned}$$

are all well-defined and continuous.

Continuity of the fixed point operator

$$\text{fix:} \quad (D \to D) \quad \to \quad D$$

is continuous.

Constructions on Domains

Back to the introduction

$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\mathbf{State} \rightharpoonup \mathbf{State})$:

$$F(w)([X \mapsto x, Y \mapsto y]) \quad = \quad \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0. \end{cases}$$

The semantics of a while loop

$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\text{State}_\bot \to \text{State}_\bot)$:

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$$F(\bot) = \bot$$

$\text{State}_\bot \to \text{State}_\bot$ is a domain!

Kleene's fixed point theorem

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is the least fixed point of F, and in particular a fixed point.

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We can compute explicitly

$$w_\infty[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x \geq 0 \end{cases}$$

And check this agrees with the operational semantics.

Scott Induction

Reasoning on fixed points: Scott induction

Let D be a domain, $f\colon D \to D$ be a continuous function and $S \subseteq D$ be a subset of D. If the set S

(i) contains $\bot$,

(ii) is chain-closed, *i.e.* the lub of any chain of elements of S is also in S,

(iii) is stable for f, *i.e.* $f(S) \subseteq S$,

then $\mathrm{fix}(f) \in S$.

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$$\textsc{ScottInd}\ \frac{\Phi(\bot) \qquad \Phi(x) \Rightarrow \Phi(f(x)) \qquad (\forall i \in \mathbb{N}.\ \Phi(x_i)) \Rightarrow \Phi(\bigsqcup_{i \in \mathbb{N}} x_i)}{\Phi(\mathrm{fix}(f))}$$

Building chain-closed sets

All the following are chain-closed:

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The "logical" view

Any formula written using:

- signature: continuous functions + constants
- relations: equality, inequality
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Given any set I, domains D, E, functions $(f_i)_{i \in I}, g \colon D \to E$, $e \in E$,

$$\Phi(x) := \forall y \in E, (\forall i \in I,\ f_i(x) \sqsubseteq y) \vee g(x) = e$$

is chain-closed.

Example: downset

Assume $f(d) \sqsubseteq d$, *i.e.* d is a pre-fixed point of the continuous $f : D \to D$. By Scott induction on $d\downarrow$, $\mathrm{fix}(f) \sqsubseteq d$.

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Proof!

Example: partial correctness

Let $w_\infty \colon \mathrm{State}_\perp \to \mathrm{State}_\perp$ be the denotation of

$$\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)$$

Recall that $w_\infty = \mathrm{fix}(F)$ where

$$F(w)(x,y) = \begin{cases} (x,y) & \text{if } x \leq 0 \\ w(x-1, x \cdot y) & \text{if } x > 0 \end{cases}$$

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Proof: by Scott induction!

PCF

Pcf

Syntax

Syntax of Pcf

Types: $$\tau ::= \texttt{nat} \mid \texttt{bool} \mid \tau \to \tau$$

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Terms:

$$\begin{aligned} t \quad ::= \quad & \texttt{0} \mid \texttt{succ}(t) \mid \texttt{pred}(t) \mid \\ & \texttt{true} \mid \texttt{false} \mid \texttt{zero?}(t) \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ t \\ & x \mid \texttt{fun}\ x{:}\tau.\ t \mid t\ t \mid \texttt{fix}(t) \end{aligned}$$

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 & & x \mid \texttt{fun}\ x{:}\tau.\ t \mid t\ t \mid \texttt{fix}(t)
\end{array}
$$

- λ-calculus + base types/functions + `fix`
- tiny ML (without references, ADTs, polymorphism...)

Variables: up to α-equivalence

Variables, substitutions and contexts

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- partial maps from variable to types
- finite lists $x_1\colon\tau_1, \ldots, x_n\colon\tau_n$

Typing for Pcf (I)

$\boxed{\Gamma \vdash t : \tau}$ The term t has type τ in context Γ

$$\textsc{Zero}\ \frac{}{\Gamma \vdash \mathtt{0} : \mathtt{nat}} \qquad \textsc{Succ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{succ}(t) : \mathtt{nat}} \qquad \textsc{Pred}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{pred}(t) : \mathtt{nat}}$$

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$$\textsc{True}\ \frac{}{\Gamma \vdash \mathtt{true} : \mathtt{bool}} \qquad \textsc{False}\ \frac{}{\Gamma \vdash \mathtt{false} : \mathtt{bool}} \qquad \textsc{IsZ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{zero?}(t) : \mathtt{bool}}$$

$$\textsc{If}\ \frac{\begin{array}{c}\Gamma \vdash b : \mathtt{bool} \\ \Gamma \vdash t : \tau \qquad \Gamma \vdash t' : \tau\end{array}}{\Gamma \vdash \mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t' : \tau}$$

Typing for Pcf (II)

$$\textsc{Var}\ \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \textsc{Fun}\ \frac{\Gamma, x{:}\sigma \vdash t : \tau}{\Gamma \vdash \mathtt{fun}\, x{:}\sigma.\, t : \sigma \to \tau} \qquad \textsc{App}\ \frac{\Gamma \vdash f : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash f\ u : \tau}$$

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$$\textsc{Pcf}_{\Gamma,\tau} \stackrel{\text{def}}{=} \{t \mid \Gamma \vdash t : \tau\} \qquad\qquad \textsc{Pcf}_{\tau} \stackrel{\text{def}}{=} \textsc{Pcf}_{\cdot,\tau}$$

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The **only** programs we care about!

Typing and substitution

If $\Gamma \vdash t : \tau$ and $\Gamma, x{:}\tau \vdash t' : \tau'$ both hold, then so does $\Gamma \vdash t'[t/x] : \tau'$.

Pcf

Operational semantics

Pcf values

Values:

$$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \underbrace{\mathtt{fun}\, x{:}\tau.\, t}_{\text{All functions } (< \mathtt{fun} >)}$$

PCF VALUES

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We will only evaluate closed term to values.

PCF evaluation

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v}$$

PCF evaluation

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PCF EVALUATION

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$$\textsc{IfT}\ \frac{b \Downarrow_{\texttt{bool}} \texttt{true} \qquad t_1 \Downarrow_\tau v}{\texttt{if}\ b\ \texttt{then}\ t_1\ \texttt{else}\ t_2 \Downarrow_\tau v} \qquad \textsc{IfF}\ \frac{b \Downarrow_{\texttt{bool}} \texttt{false} \qquad t_2 \Downarrow_\tau v}{\texttt{if}\ b\ \texttt{then}\ t_1\ \texttt{else}\ t_2 \Downarrow_\tau v}$$

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$$\mathtt{plus} \stackrel{\text{def}}{=} \mathtt{fun}\, x\colon \mathtt{nat}.\ \mathtt{fix}(\mathtt{fun}(p\colon \mathtt{nat} \to \mathtt{nat})(y\colon \mathtt{nat}).\ \mathtt{if}\ \mathtt{zero?}(y)\ \mathtt{then}\ x\ \mathtt{else}\ \mathtt{succ}(p\ \mathtt{pred}(y)))$$

$$\mathtt{plus}\ \underline{3}\ \underline{1} \Downarrow_{\mathtt{nat}} \underline{4}$$

Evaluation (I)

$$\textsc{Fun}\ \frac{\texttt{plus} \Downarrow \texttt{plus} \qquad \texttt{plus}_{\underline{3}}\ \underline{1} \Downarrow \underline{4}}{\texttt{plus}\ \underline{3}\ \underline{1} \Downarrow_{\texttt{nat}} \underline{4}}$$

$$\begin{aligned}\texttt{plus}_x \quad &\stackrel{\text{def}}{=} \texttt{fix}(\texttt{fun}(p\colon \texttt{nat} \to \texttt{nat})(y\colon \texttt{nat}).\\ &\quad \texttt{if zero?}(y)\ \texttt{then}\ x\ \texttt{else succ}(p\ \texttt{pred}(y)))\end{aligned}$$

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$$\textsc{Fix}\ \dfrac{\textsc{Fun}\ \dfrac{\textsc{Val}\ \dfrac{}{(\texttt{fun}\ p\colon \texttt{nat} \to \texttt{nat}.\ \ldots) \Downarrow \ldots} \qquad \textsc{Val}\ \dfrac{}{(\texttt{fun}\ y\colon \texttt{nat}.\ldots)[p/\texttt{plus}_x] \Downarrow r_x}}{(\texttt{fun}(p\colon \texttt{nat} \to \texttt{nat})(y\colon \texttt{nat}).\ \ldots)\ \texttt{plus}_x \Downarrow r_x}}{\texttt{plus}_x \Downarrow \underbrace{\texttt{fun}\ y\colon \texttt{nat}.\ \texttt{if zero?}(y)\ \texttt{then}\ x\ \texttt{else succ}(\texttt{plus}_x\ \texttt{pred}(y))}_{r_x}}$$

$$
\textsc{Fun}\;\dfrac{
\mathtt{plus}_{\underline{3}} \Downarrow \underline{r_3}
\qquad
\textsc{IfF}\;\dfrac{
\textsc{ZeroS}\;\dfrac{\textsc{Val}\;\dfrac{}{\underline{1} \Downarrow \underline{1}}}{\mathtt{zero?}(\underline{1}) \Downarrow \mathtt{false}}
\qquad
\textsc{Succ}\;\dfrac{\dfrac{\dfrac{\textsc{ZeroZ}\;\dfrac{\textsc{Pred}\;\dfrac{\cdots}{\mathtt{pred}(\underline{1}) \Downarrow \mathtt{0}}}{\mathtt{zero?}(\mathtt{pred}(\underline{1})) \Downarrow \mathtt{true}}}{\cdots}}{\mathtt{plus}_{\underline{3}}\ \mathtt{pred}(\underline{1}) \Downarrow \underline{3}}}{\mathtt{succ}(\mathtt{plus}_{\underline{3}}\ \mathtt{pred}(\underline{1})) \Downarrow \underline{4}}
}{
\mathtt{if\ zero?}(\underline{1})\ \mathtt{then}\ \underline{3}\ \mathtt{else\ succ}(\mathtt{plus}_{\underline{3}}\ \mathtt{pred}(\underline{1})) \Downarrow \underline{4}
}
}{
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$$

Divergence

Divergence ($t \Uparrow_\tau$):

$$t : \tau \quad \wedge \quad \nexists v.\, t \Downarrow_\tau v$$

Divergence ($t \Uparrow_\tau$):

$$t : \tau \quad \wedge \quad \nexists v.\, t \Downarrow_\tau v$$

$$\Omega_\tau \stackrel{\text{def}}{=} \texttt{fix}(\texttt{fun}\, x{:}\tau.\, x)$$

$$\Omega_\tau \Uparrow_\tau \qquad \text{(diverges)}$$

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$$\dfrac{\dfrac{\mathtt{fun}\, x{:}\, \tau.\, x \Downarrow \mathtt{fun}\, x{:}\, \tau.\, x \qquad \overset{\mathcal{P}}{\mathtt{fix}(\mathtt{fun}\, x{:}\, \tau.\, x) \Downarrow v}}{(\mathtt{fun}\, x{:}\, \tau.\, x)\,(\mathtt{fix}(\mathtt{fun}\, x{:}\, \tau.\, x)) \Downarrow v}}{\mathtt{fix}(\mathtt{fun}\, x{:}\, \tau.\, x) \Downarrow v}$$

$$\textsc{Fun-CBN}\ \frac{t \Downarrow_{\sigma\to\tau} \mathsf{fun}\, x{:}\,\sigma.\, t' \qquad t'[u/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v}$$

$$\textsc{Fun-CBV}\ \frac{t \Downarrow_{\sigma\to\tau} \mathsf{fun}\, x{:}\,\sigma.\, t' \qquad u \Downarrow_\sigma v' \qquad t'[v'/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v}$$

Call-by-name and call-by-value

$$\textsc{Fun-CBN}\ \frac{t \Downarrow_{\sigma\to\tau} \mathtt{fun}\, x{:}\, \sigma.\, t' \qquad t'[u/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v}$$

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What does $(\mathtt{fun}\, x{:}\, \mathtt{nat}.\, \mathtt{0})\ \Omega_{\mathtt{nat}}$ denote?

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What does $(\mathtt{fun}\, x{:}\, \mathtt{nat}.\, \mathtt{0})\ \Omega_{\mathtt{nat}}$ denote?

In call-by-value, all functions are **strict**... but the least-fixed points of a strict function is **always $\bot$**!

Small-step $t \rightsquigarrow_\tau u$:

$$\frac{}{(\mathsf{fun}\, x\colon \sigma.\, t)\; u \rightsquigarrow_\tau t[u/x]} \qquad \frac{t \rightsquigarrow_{\sigma \to \tau} t'}{t\; u \rightsquigarrow_\tau t'\; u} \qquad \ldots$$

Small-step $t \rightsquigarrow_\tau u$:

$$\frac{}{(\mathsf{fun}\, x\colon \sigma.\, t)\; u \rightsquigarrow_\tau t[u/x]} \qquad \frac{t \rightsquigarrow_{\sigma \to \tau} t'}{t\; u \rightsquigarrow_\tau t'\; u} \qquad \dots$$

We have $t \Downarrow_\tau v$ iff $t \rightsquigarrow_\tau^\star u$.

PCF is **Turing-complete**: for every partial recursive function ϕ, there is a PCF term $\underline{\phi} \in \mathrm{PCF}_{\mathtt{nat} \to \mathtt{nat}}$ such that for all $n \in \mathbb{N}$, if $\phi(n)$ is defined then $\underline{\phi}\ \underline{n} \Downarrow_{\mathtt{nat}} \underline{\phi(n)}$.

Turing-completeness

PCF is Turing-complete: for every partial recursive function ϕ, there is a PCF term $\underline{\phi} \in \text{PCF}_{\mathtt{nat} \to \mathtt{nat}}$ such that for all $n \in \mathbb{N}$, if $\phi(n)$ is defined then $\underline{\phi}\ \underline{n} \Downarrow_{\mathtt{nat}} \underline{\phi(n)}$.

(Later on: $\phi = \llbracket \underline{\phi} \rrbracket$).

Determinism

Evaluation in PCF is deterministic: if both $t \Downarrow_\tau v$ and $t \Downarrow_\tau v'$ hold, then $v = v'$.

Determinism

Evaluation in PCF is **deterministic**: if both $t \Downarrow_\tau v$ and $t \Downarrow_\tau v'$ hold, then $v = v'$.

By (rule) induction on evaluation $\Downarrow$:

$$P(t, \tau, v) \quad \stackrel{\text{def}}{=} \quad \forall v' \in \mathrm{PCF}_\tau \,.(t \Downarrow_\tau v' \Rightarrow v = v')$$

Intuition: there is always exactly one rule which applies.

Pcf

Contextual equivalence

Contextual equivalence – informal

Two phrases of a programming language are contextually equivalent if any occurrences of the first phrase in a complete program can be replaced by the second phrase without affecting the observable results of executing the program.

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Two phrases of a programming language are **contextually equivalent** if any occurrences of the first phrase in a **complete program** can be replaced by the second phrase without affecting the **observable results** of executing the program.

The intuitive notion of **program equivalence** for programmers.

Contextual equivalence – informal

Two phrases of a programming language are **contextually equivalent** if any occurrences of the first phrase in a **complete program** can be replaced by the second phrase without affecting the **observable results** of executing the program.

The intuitive notion of **program equivalence** for programmers.

But what's a complete program? What's an observable result?

Evaluation contexts

"Term with a hole":

$$\mathcal{C} ::= - \mid \texttt{succ}(\mathcal{C}) \mid \texttt{pred}(\mathcal{C}) \mid \texttt{zero?}(\mathcal{C}) \mid$$
$$\texttt{if}\ \mathcal{C}\ \texttt{then}\ t\ \texttt{else}\ t \mid \texttt{if}\ t\ \texttt{then}\ \mathcal{C}\ \texttt{else}\ t \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ \mathcal{C} \mid$$
$$\texttt{fun}\ x{:}\tau.\ \mathcal{C} \mid \mathcal{C}\ t \mid t\ \mathcal{C} \mid \texttt{fix}(\mathcal{C})$$

Evaluation contexts

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$$\begin{array}{rcl} \mathcal{C} & ::= & - \mid \texttt{succ}(\mathcal{C}) \mid \texttt{pred}(\mathcal{C}) \mid \texttt{zero?}(\mathcal{C}) \mid \\ & & \texttt{if}\ \mathcal{C}\ \texttt{then}\ t\ \texttt{else}\ t \mid \texttt{if}\ t\ \texttt{then}\ \mathcal{C}\ \texttt{else}\ t \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ \mathcal{C} \mid \\ & & \texttt{fun}\ x{:}\,\tau.\,\mathcal{C} \mid \mathcal{C}\ t \mid t\ \mathcal{C} \mid \texttt{fix}(\mathcal{C}) \end{array}$$

Typing extended to evaluation contexts: $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$.

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$$\texttt{fun}\ x{:}\tau.\ \mathcal{C} \mid \mathcal{C}\ t \mid t\ \mathcal{C} \mid \texttt{fix}(\mathcal{C})$$

Typing extended to evaluation contexts: $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$.

$$\frac{}{\Gamma \vdash_{\Gamma,\tau} - : \tau} \qquad \frac{\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau_1 \to \tau_2 \qquad \Gamma \vdash u : \tau_1}{\Gamma \vdash_{\Delta,\sigma} \mathcal{C}\ u : \tau_2} \qquad \dots$$

Given a type τ, a typing context Γ and terms $t, t' \in \mathrm{PCF}_{\Gamma,\tau}$, contextual equivalence, written $\Gamma \vdash t \cong_{\mathrm{ctx}} t' : \tau$ is defined to hold if for all evaluation contexts $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$, where γ is `nat` or `bool`, and for all values $v \in \mathrm{PCF}_{\gamma}$,

$$\mathcal{C}[t] \Downarrow_{\gamma} v \Leftrightarrow \mathcal{C}[t'] \Downarrow_{\gamma} v.$$

When Γ is the empty context, we simply write $t \cong_{\mathrm{ctx}} t' : \tau$ for $\cdot \vdash t \cong_{\mathrm{ctx}} t' : \tau$.

Given a type τ, a typing context Γ and terms $t, t' \in \textsc{Pcf}_{\Gamma,\tau}$, **contextual equivalence**, written $\Gamma \vdash t \cong_{\mathrm{ctx}} t' : \tau$ is defined to hold if for all evaluation contexts $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$, where γ is `nat` or `bool`, and for all values $v \in \textsc{Pcf}_{\gamma}$,

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When Γ is the empty context, we simply write $t \cong_{\mathrm{ctx}} t' : \tau$ for $\cdot \vdash t \cong_{\mathrm{ctx}} t' : \tau$.

Divergence is implicitly covered.

Denotational Semantics for Pcf

Denotational Semantics for Pcf

Introducing denotational semantics

The aims of denotational semantics

- a mapping of PCF types τ to domains $[\![\tau]\!]$;
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$;
- denotation of open terms will be continuous functions.

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- denotation of open terms will be continuous functions.

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$.

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$.

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \mathrm{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$.

$$v \stackrel{\text{def}}{=} \texttt{fun}\, x\colon \texttt{nat}.\, (\texttt{fun}\, y\colon \texttt{nat}.\, y)\, \texttt{0} \quad \text{and} \quad v' \stackrel{\text{def}}{=} \texttt{fun}\, x\colon \texttt{nat}.\, \texttt{0}.$$

Proof principle: to show

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau$$

it suffices to establish

$$[\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

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$$\begin{aligned}
\mathcal{C}[t_1] \Downarrow_{\mathtt{nat}} v &\Rightarrow [\![\mathcal{C}[t_1]]\!] = [\![v]\!] && \text{(soundness)}\\
&\Rightarrow [\![\mathcal{C}[t_2]]\!] = [\![v]\!] && \text{(compositionality on } [\![t_1]\!] = [\![t_2]\!]\text{)}\\
&\Rightarrow \mathcal{C}[t_2] \Downarrow_{\mathtt{nat}} v && \text{(adequacy)}
\end{aligned}$$

Proof principle: to show

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$$\begin{aligned} \mathcal{C}[t_1] \Downarrow_{\texttt{nat}} v &\Rightarrow [\![\mathcal{C}[t_1]]\!] = [\![v]\!] && \text{(soundness)} \\ &\Rightarrow [\![\mathcal{C}[t_2]]\!] = [\![v]\!] && \text{(compositionality on } [\![t_1]\!] = [\![t_2]\!]\text{)} \\ &\Rightarrow \mathcal{C}[t_2] \Downarrow_{\texttt{nat}} v && \text{(adequacy)} \end{aligned}$$

and symmetrically for $\mathcal{C}[t_2] \Downarrow_{\texttt{nat}} v \Rightarrow \mathcal{C}[t_1] \Downarrow_{\texttt{nat}} v$, and similarly for `bool`.

The power of denotational semantics

Proof principle: to show

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau$$

it suffices to establish

$$[\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

Denotational equality is **sound**, but is it **complete**?
Does equality in the model imply contextual equivalence?

Proof principle: to show

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau$$

it suffices to establish

$$[\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

Denotational equality is sound, but is it complete?
Does equality in the model imply contextual equivalence?

Full abstraction.

Denotational Semantics for Pcf

Definition

$$[\![\texttt{nat}]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp \qquad \text{(flat domain)}$$

$$[\![\texttt{bool}]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp \qquad \text{(flat domain)}$$

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!] \qquad \text{(function domain)}$$

Semantics of contexts

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in \mathrm{dom}(\Gamma)} [\![\Gamma(x)]\!] \qquad \text{(environment)}$$

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} [\![\Gamma(x)]\!] \qquad \text{(environment)}$$

- $[\![\cdot]\!] = \mathbb{1}$ (one element set)
- $[\![x : \tau]\!] = (\{x\} \to [\![\tau]\!]) \cong [\![\tau]\!]$
- $[\![x_1 : \tau_1, \ldots, x_n : \tau_n]\!] \cong [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$

Denotational semantics of Pcf

To every typing judgement

$$\Gamma \vdash t : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash t : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains. In other words,

$$[\![-]\!] : \textsc{Pcf}_{\Gamma,\tau} \to [\![\Gamma]\!] \to [\![\tau]\!]$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$\begin{aligned} \text{succ} : \quad \mathbb{N} &\to \mathbb{N} \\ n &\mapsto n+1 \end{aligned}$$

$$\begin{aligned} \text{pred} : \quad \mathbb{N} &\rightharpoonup \mathbb{N} \\ n+1 &\mapsto n \\ 0 &\quad \text{undefined} \end{aligned}$$

$$\begin{aligned} \text{zero?} : \quad \mathbb{N} &\to \mathbb{B} \\ 0 &\mapsto \text{true} \\ n+1 &\mapsto \text{false} \end{aligned}$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$\begin{aligned} \text{succ}_\bot : \quad \mathbb{N}_\bot &\to \mathbb{N}_\bot \\ n &\mapsto n+1 \\ \bot &\mapsto \bot \end{aligned}$$

$$\begin{aligned} \text{pred}_\bot : \quad \mathbb{N}_\bot &\to \mathbb{N}_\bot \\ n+1 &\mapsto n \\ 0 &\mapsto \bot \\ \bot &\mapsto \bot \end{aligned}$$

$$\begin{aligned} \text{zero?}_\bot : \quad \mathbb{N}_\bot &\to \mathbb{B}_\bot \\ 0 &\mapsto \text{true} \\ n+1 &\mapsto \text{false} \\ \bot &\mapsto \bot \end{aligned}$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$
\begin{aligned}
[\![\texttt{0}]\!](\rho) &\stackrel{\text{def}}{=} 0 && \in \mathbb{N}_\perp \\
[\![\texttt{true}]\!](\rho) &\stackrel{\text{def}}{=} \text{true} && \in \mathbb{B}_\perp \\
[\![\texttt{false}]\!](\rho) &\stackrel{\text{def}}{=} \text{false} && \in \mathbb{B}_\perp
\end{aligned}
$$

$$
\begin{aligned}
[\![\mathtt{0}]\!]\,(\rho) &\stackrel{\text{def}}{=} 0 && \in \mathbb{N}_\perp \\
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[\![\mathtt{false}]\!]\,(\rho) &\stackrel{\text{def}}{=} \text{false} && \in \mathbb{B}_\perp
\end{aligned}
$$

$$
\begin{aligned}
[\![\mathtt{succ}(t)]\!]\,(\rho) &\stackrel{\text{def}}{=} \text{succ}_\perp([\![t]\!]\,(\rho)) && \in \mathbb{N}_\perp \\
[\![\mathtt{pred}(t)]\!]\,(\rho) &\stackrel{\text{def}}{=} \text{pred}_\perp([\![t]\!]\,(\rho)) && \in \mathbb{N}_\perp \\
[\![\mathtt{zero?}(t)]\!]\,(\rho) &\stackrel{\text{def}}{=} \text{zero?}_\perp([\![t]\!]\,(\rho)) && \in \mathbb{B}_\perp
\end{aligned}
$$

$$
[\![\mathtt{succ}(t)]\!] = \text{succ}_\perp \circ [\![t]\!]
$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$
\begin{aligned}
[\![\texttt{0}]\!]\,(\rho) &\overset{\text{def}}{=} 0 && \in \mathbb{N}_\perp \\
[\![\texttt{true}]\!]\,(\rho) &\overset{\text{def}}{=} \text{true} && \in \mathbb{B}_\perp \\
[\![\texttt{false}]\!]\,(\rho) &\overset{\text{def}}{=} \text{false} && \in \mathbb{B}_\perp
\end{aligned}
$$

$$
\begin{aligned}
[\![\texttt{succ}(t)]\!]\,(\rho) &\overset{\text{def}}{=} \text{succ}_\perp([\![t]\!]\,(\rho)) && \in \mathbb{N}_\perp \\
[\![\texttt{pred}(t)]\!]\,(\rho) &\overset{\text{def}}{=} \text{pred}_\perp([\![t]\!]\,(\rho)) && \in \mathbb{N}_\perp \\
[\![\texttt{zero?}(t)]\!]\,(\rho) &\overset{\text{def}}{=} \text{zero?}_\perp([\![t]\!]\,(\rho)) && \in \mathbb{B}_\perp
\end{aligned}
$$

$$
[\![\texttt{if}\ b\ \texttt{then}\ t\ \texttt{else}\ t']\!] \overset{\text{def}}{=} \text{if}([\![b]\!]\,(\rho), [\![t]\!]\,(\rho), [\![t']\!]\,(\rho)) \quad \in [\![\tau]\!]
$$

$$
[\![\texttt{if}\ b\ \texttt{then}\ t\ \texttt{else}\ t']\!] = \text{if} \circ \langle [\![b]\!]\,, \langle [\![t]\!]\,, [\![t']\!] \rangle\rangle
$$

$$[\![x]\!]\,(\rho) \stackrel{\text{def}}{=} \rho(x) \qquad \in [\![\Gamma(x)]\!]$$

$$[\![x]\!]\,(\rho) = \pi_x(\rho)$$

Denotation of the λ-calculus operations

$$\llbracket x \rrbracket (\rho) \stackrel{\text{def}}{=} \rho(x) \in \llbracket \Gamma(x) \rrbracket$$

$$\llbracket t_1\ t_2 \rrbracket (\rho) \stackrel{\text{def}}{=} (\llbracket t_1 \rrbracket (\rho))\ (\llbracket t_2 \rrbracket (\rho))$$

$$\llbracket t_1\ t_2 \rrbracket = \mathrm{eval} \circ \langle \llbracket t_1 \rrbracket , \llbracket t_2 \rrbracket \rangle$$

Denotation of the λ-calculus operations

$$\begin{aligned}
[\![x]\!]\,(\rho) &\stackrel{\text{def}}{=} \rho(x) && \in [\![\Gamma(x)]\!] \\
[\![t_1\ t_2]\!]\,(\rho) &\stackrel{\text{def}}{=} ([\![t_1]\!]\,(\rho))\ ([\![t_2]\!]\,(\rho)) \\
[\![\mathtt{fun}\,x{:}\,\tau.\,t]\!]\,(\rho) &\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!]\,.\ [\![t]\!]\,(\rho, d)
\end{aligned}$$

$$[\![\mathtt{fun}\,x{:}\,\tau.\,t]\!] = \mathrm{cur}([\![t]\!])$$

Denotation of fixed points

$$[\![\texttt{fix}\ f]\!]\,(\rho) \quad \stackrel{\text{def}}{=} \quad \mathrm{fix}([\![f]\!]\,(\rho))$$

Denotation of Pcf terms

For any Pcf term t such that $\Gamma \vdash t : \tau$, the object $[\![t]\!]$
is well-defined and a continuous function $[\![t]\!] : [\![\Gamma]\!] \to \tau$.

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If $t \in \text{Pcf}_\tau$: $[\![t]\!] \in [\![\cdot]\!] \to [\![\tau]\!] = \mathbb{1} \to [\![\tau]\!] \cong [\![\tau]\!]$

Denotational Semantics for Pcf

Compositionality

Suppose $t, u \in \textsc{Pcf}_{\Delta,\sigma}$, such that

$$[\![t]\!] = [\![u]\!] : [\![\Delta]\!] \to [\![\sigma]\!]$$

Suppose moreover that $\mathcal{C}[-]$ is a $\textsc{Pcf}$ context such that $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$. Then

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[u]]\!] : [\![\Gamma]\!] \to [\![\tau]\!] .$$

A denotation for evaluation contexts

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

A denotation for evaluation contexts

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\begin{aligned} [\![-]\!]\,(d) &= d \\ [\![\mathcal{C}\ t]\!]\,(d)(\rho) &= ([\![\mathcal{C}]\!]\,(d)(\rho))([\![t]\!]\,(\rho)) \\ &\vdots \end{aligned}$$

A denotation for evaluation contexts

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\begin{aligned}
[\![-]\!]\,(d) &= d \\
[\![\mathcal{C}\;t]\!]\,(d)(\rho) &= ([\![\mathcal{C}]\!]\,(d)(\rho))([\![t]\!]\,(\rho)) \\
&\vdots
\end{aligned}$$

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$ and $\Delta \vdash t : \sigma$, then

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}]\!]\,([\![t]\!])$$

Assume

$$\Gamma \vdash u : \sigma$$
$$\Gamma, x{:}\sigma \vdash t : \tau$$

Then for all $\rho \in [\![\Gamma]\!]$

$$[\![t[u/x]]\!]\,(\rho) = [\![t]\!]\,(\rho[x \mapsto [\![u]\!]\,(\rho)]).$$

In particular when $\Gamma = \cdot$, $[\![t]\!] : [\![\sigma]\!] \to [\![\tau]\!]$ and

$$[\![t[u/x]]\!] = [\![t]\!]\,([\![u]\!])$$

Denotational Semantics for Pcf

Soundness

Soundness

For all PCF types τ and all closed terms $t, v \in \text{PCF}_\tau$ with v a value, if $t \Downarrow_\tau v$ is derivable, then

$$[\![t]\!] = [\![v]\!] \in [\![\tau]\!]$$

Divergence

If $t \in \textsc{Pcf}_{\mathtt{nat}}$ and $[\![t]\!] = \bot$, then $t \Uparrow_{\mathtt{nat}}$.

Adequacy

Reminder: Adequacy

For any closed Pcf term t and value v of ground type $\gamma \in \{\mathtt{nat}, \mathtt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_\gamma v$$

Reminder: Adequacy

For any closed PCF term t and value v of ground type $\gamma \in \{\mathtt{nat}, \mathtt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_\gamma v$$

Adequacy does not hold at function types or for open terms

Reminder: Adequacy

For any **closed** PCF term t and value v of **ground** type $\gamma \in \{\mathtt{nat}, \mathtt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_\gamma v$$

Adequacy does **not** hold at function types or for open terms

$$[\![\mathtt{fun}\, x\colon\tau.\,(\mathtt{fun}\, y\colon\tau.\, y)\, x]\!] \quad = \quad [\![\mathtt{fun}\, x\colon\tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\mathtt{fun}\, x\colon\tau.\,(\mathtt{fun}\, y\colon\tau.\, y)\, x \not\Downarrow_{\tau\to\tau} \mathtt{fun}\, x\colon\tau.\, x$$

Reminder: Adequacy

For any **closed** PCF term t and value v of **ground** type $\gamma \in \{\texttt{nat}, \texttt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_\gamma v$$

Adequacy does **not** hold at function types or for open terms

More serious:

$$\begin{aligned} & [\![\texttt{fun}\, x\colon \texttt{nat}.\, (\texttt{if zero?}(f\ x)\ \texttt{then true else true})]\!] \\ \stackrel{?}{=}\ & [\![\texttt{fun}\, x\colon \texttt{nat}.\, \texttt{true}]\!] \end{aligned}$$

Adequacy

Formal approximation relation

Proof idea: introduce a relation R such that

1. if $t \in \textsc{Pcf}_{\texttt{nat}}$, $n \in \mathbb{N}$, and $R(n, t)$, then $t \Downarrow_\gamma \underline{n}$ (same for booleans);
2. for any well-typed term t, $R(\llbracket t \rrbracket, t)$.

Proof idea: introduce a relation R such that

1. if $t \in \textsc{Pcf}_{\mathtt{nat}}$, $n \in \mathbb{N}$, and $R(n, t)$, then $t \Downarrow_\gamma \underline{n}$ (same for booleans);
2. for any well-typed term t, $R(\llbracket t \rrbracket, t)$.

But at non-base types, adequacy does not hold.

Proof idea: introduce a relation R such that

1. if $t \in \text{PCF}_{\texttt{nat}}$, $n \in \mathbb{N}$, and $R(n, t)$, then $t \Downarrow_\gamma \underline{n}$ (same for booleans);
2. for any well-typed term t, $R(\llbracket t \rrbracket, t)$.

But at non-base types, adequacy does not hold.

We must define a family of relations, tailored for each type: formal approximation

$$\triangleleft_\tau \subseteq \llbracket \tau \rrbracket \times \text{PCF}_\tau$$

Proof idea: introduce a relation R such that

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A **logical relation**.

$$d \lhd_{\texttt{nat}} t \overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow t \Downarrow_{\texttt{nat}} \underline{d})$$

$$\begin{aligned} d \lhd_{\texttt{bool}} t \overset{\text{def}}{\Leftrightarrow} & (d = \text{true} \Rightarrow t \Downarrow_{\texttt{bool}} \texttt{true}) \\ & \wedge (d = \text{false} \Rightarrow t \Downarrow_{\texttt{bool}} \texttt{false}) \end{aligned}$$

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Exactly what we need to get 1.

Formal approximation at base types

$$d \lhd_{\texttt{nat}} t \overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow t \Downarrow_{\texttt{nat}} \underline{d})$$

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Exactly what we need to get 1.

Note though that $\bot \lhd_{\texttt{nat}} t$ for any $t \in \mathrm{PCF}_{\texttt{nat}}$.

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Assume $\llbracket u \rrbracket \lhd_\tau u$ and $\llbracket t \rrbracket \lhd_{\tau \to \tau'} t$, how do we get $\llbracket t\ u \rrbracket = \llbracket t \rrbracket (\llbracket u \rrbracket) \lhd_\tau t\ u$?

Define

$$d \lhd_{\tau \to \tau'} t \overset{\text{def}}{\Leftrightarrow} \forall e \in \llbracket \tau \rrbracket, u \in \text{PCF}_\tau .(e \lhd_\tau u \Rightarrow d(e) \lhd_{\tau'} t\ u)$$

$$\textsc{Abs}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathsf{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

To prove Item 2, we need to talk about **open** terms.

$$\text{Abs}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathsf{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

To prove Item 2, we need to talk about **open** terms.

$$[\![t]\!]\,([\![u]\!]) = [\![(t[u/x])]\!]$$ Semantic application ≈ syntactic substitution

Formal approximation for open terms

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To prove Item 2, we need to talk about open terms.

$$[\![t]\!]\left([\![u]\!]\right) = [\![(t[u/x])]\!] \qquad \text{Semantic application} \approx \text{syntactic substitution}$$

Parallel substitution: maps each $x \in \mathrm{dom}(\Gamma)$ to $\sigma(x) \in \textsc{Pcf}_{\Gamma(x)}$.

$$\rho \lhd_\Gamma \sigma \overset{\text{def}}{\Leftrightarrow} \forall x \in \mathrm{dom}(\Gamma), \rho(x) \lhd_{\Gamma(x)} \sigma(x)$$

The fundamental theorem

For any

- context Γ and type τ
- term t such that $\Gamma \vdash t : \tau$
- environment ρ
- substitution σ
- such that $\rho \lhd_\Gamma \sigma$

we have

$$[\![t]\!]\,(\rho) \lhd_\tau t[\sigma].$$

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Corollary: if $\cdot \vdash t : \tau$,

$$[\![t]\!] \lhd_\tau t.$$

Adequacy

Proof of the fundamental property of formal approximation

Properties of formal approximation

1. The least element approximates any program: for any τ and $t \in \textsc{Pcf}_\tau$, $\bot_{[\![\tau]\!]} \lhd_\tau t$;
2. if $d' \sqsubseteq d$ and $d \lhd_\tau t$, then $d' \lhd_\tau t$;
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3. the set $\{d \in [\![\tau]\!] \mid d \lhd_\tau t\}$ is chain-closed;
4. if $\forall v.\, t \Downarrow_\tau v \Rightarrow t' \Downarrow_\tau v$, and $d \lhd_\tau t$, then $d \lhd_\tau t'$.

Fundamental property

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Proof! Induction on $\Gamma \vdash t : \tau$:

$$\forall \rho, \sigma.\, (\rho \lhd_\Gamma \sigma \;\Rightarrow [\![t]\!]\,(\rho) \lhd_\tau t[\sigma])$$

Adequacy

Extensionality

Contextual preorder is the one-sided version of contextual equivalence: $\Gamma \vdash t \leq_{\mathrm{ctx}} t' : \tau$ if for all $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$ and for all values v,

$$\mathcal{C}[t] \Downarrow_\gamma v \Rightarrow \mathcal{C}[t'] \Downarrow_\gamma v.$$

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$$\Gamma \vdash t \cong_{\text{ctx}} t' : \tau \Leftrightarrow (\Gamma \vdash t \leq_{\text{ctx}} t' : \tau \wedge \Gamma \vdash t' \leq_{\text{ctx}} t : \tau)$$

Let τ be a type, and assume $t_1, t_2 \in \textsc{Pcf}_\tau$ are such that $t_1 \leq_{\mathrm{ctx}} t_2 : \tau$. Then

$$d \lhd_\tau t_1 \Rightarrow d \lhd_\tau t_2.$$

Lemma: application contexts

To characterise contextual preorder between closed terms, applicative contexts are enough.

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Let t_1, t_2 be closed terms of type τ. Then $t_1 \leq_{\mathrm{ctx}} t_2 : \tau$ if and only if, for every term $f : \tau \to \texttt{bool}$,

$$f\ t_1 \Downarrow_{\texttt{bool}} \texttt{true} \Rightarrow f\ t_2 \Downarrow_{\texttt{bool}} \texttt{true}.$$

Contextual preorder and formal approximation

Formal approximation corresponds to the contextual preorder.

Formal approximation **corresponds to** the contextual preorder.

For all PCF types τ and all closed terms $t_1, t_2 \in \mathrm{PCF}_\tau$

$$t_1 \leq_{\mathrm{ctx}} t_2 : \tau \Leftrightarrow [\![t_1]\!] \lhd_\tau t_2.$$

Extensionality properties of contextual preorder

For $\gamma = \texttt{bool}$ or $\texttt{nat}$, $t_1 \leq_{\text{ctx}} t_2 : \gamma$ holds if and only if

$$\forall v.\, (t_1 \Downarrow_\gamma v \Rightarrow t_2 \Downarrow_\gamma v).$$

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At a function type $\tau \to \tau'$, $t_1 \leq_{\text{ctx}} t_2 : \tau \to \tau'$ holds if and only if

$$\forall t \in \textsc{Pcf}_\tau\, .\, (t_1\, t \leq_{\text{ctx}} t_2\, t : \tau').$$

Full abstraction

Full abstraction

Failure of full abstraction

Full abstraction

A denotational model is fully abstract if

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau \Rightarrow [\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

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A form of completeness of semantic equivalence wrt. program equivalence.

The domain model of PCF is not fully abstract.

The *parallel or* function $\mathbf{por} : \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is defined as given by the following table:

por	true	false	$\perp$
true	true	true	true
false	true	false	$\perp$
$\perp$	true	$\perp$	$\perp$

Left sequential or

The (left) sequential or function $\mathbf{or} : \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is defined as

$$\mathbf{or} \stackrel{\text{def}}{=} [\![\texttt{fun}\, x\colon \texttt{bool}.\ \texttt{fun}\, y\colon \texttt{bool}.\ \texttt{if}\ x\ \texttt{then}\ \texttt{true}\ \texttt{else}\ y]\!]$$

It is given by the following table:

or	true	false	$\perp$
true	true	true	true
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$\perp$	$\perp$	$\perp$	$\perp$

Parallel vs sequential or

por	true	false	⊥
true	true	true	true
false	true	false	⊥
⊥	true	⊥	⊥

or	true	false	⊥
true	true	true	true
false	true	false	⊥
⊥	⊥	⊥	⊥

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or	true	false	⊥
true	true	true	true
false	true	false	⊥
⊥	⊥	⊥	⊥

or is sequential, but por is not.

Undefinability or parallel or

There is **no** closed Pcf term

$$t : \texttt{bool} \to \texttt{bool} \to \texttt{bool}$$

satisfying

$$[\![t]\!] = \mathrm{por} : \mathbb{B}_\perp \to \mathbb{B}_\perp \to \mathbb{B}_\perp \ .$$

Failure of full abstraction

The denotational model of PCF in domains and continuous functions is not fully abstract.

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For well-chosen $T_{\texttt{true}}$ and $T_{\texttt{false}}$,

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Idea:

- for all $f \in PCF_{\mathtt{bool}\to\mathtt{bool}\to\mathtt{bool}}$, ensure $T_b\ f \Uparrow_{\mathtt{bool}}$...
- but $[\![T_b]\!]\ (\mathrm{por}) = [\![b]\!]$.

Example of full abstraction failure

$$
\begin{aligned}
T_b \stackrel{\text{def}}{=}\ & \texttt{fun}\ f\colon \texttt{bool} \to (\texttt{bool} \to \texttt{bool}).\\
& \quad \texttt{if}(f\ \texttt{true}\ \Omega_{\texttt{bool}})\ \texttt{then}\\
& \quad\quad \texttt{if}\ (f\ \Omega_{\texttt{bool}}\ \texttt{true})\ \texttt{then}\\
& \quad\quad\quad \texttt{if}\ (f\ \texttt{false}\ \texttt{false})\ \texttt{then}\ \Omega_{\texttt{bool}}\ \texttt{else}\ b\\
& \quad\quad \texttt{else}\ \Omega_{\texttt{bool}}\\
& \quad \texttt{else}\ \Omega_{\texttt{bool}}
\end{aligned}
$$

Full abstraction

Beyond full abstraction failure

Interpreting full abstraction failure

- Pcf is not expressive enough to present the model?
- The model does not adequately capture Pcf?
- Contexts are too weak: they do not distinguish enough programs?

$\boxed{\Gamma \vdash t : \tau}$

$$\ldots \qquad \textsc{Por}\ \frac{\Gamma \vdash t_1 : \tau \qquad \Gamma \vdash t_2 : \tau}{\Gamma \vdash \mathtt{por}(t_1, t_2) : \tau}$$

$\boxed{t \Downarrow_\tau v}$

$$\textsc{PorL}\ \frac{t_1 \Downarrow_{\mathtt{bool}} \mathtt{true}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{true}} \qquad \textsc{PorR}\ \frac{t_2 \Downarrow_{\mathtt{bool}} \mathtt{true}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{true}}$$

$$\textsc{PorF}\ \frac{t_1 \Downarrow_{\mathtt{bool}} \mathtt{false} \qquad t_2 \Downarrow_{\mathtt{bool}} \mathtt{false}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{false}}$$

If we extend the semantics of Pcf to Pcf+`por` with

$$[\![\texttt{por}]\!] = \mathrm{por}$$

the resulting denotational semantics is fully abstract.

Full abstraction for Pcf+`por`

If we extend the semantics of Pcf to Pcf+`por` with

$$[\![\texttt{por}]\!] = \mathrm{por}$$

the resulting denotational semantics is fully abstract…

but is Pcf+`por` still a reasonable model of programming language?

Fully abstract semantics

Fully abstract semantics for Pcf

- first step: dI-domains & stable functions $\rightarrow$ no $\mathbf{por}$ any more, but still not fully abstract...
- only proper answers in the late 90s (!): logical relations and game semantics

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Real languages have effects

- If you add effects (references, control flow...) to a language, contexts become *much more* expressive.
- Full abstraction becomes different: somewhat easier... but is contextual equivalence still a reasonable idea?

Where to go from here?

Towards full abstraction

Source of a very rich literature:

- linear logic
- logical relations
- game semantics
- bisimulations techniques
- ...

Categorical semantics

Separate

- the structure needed to interpret a language (generic)
- how to construct this structure in particular examples (specific)

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Interpret:

- a type τ as an object in a category;
- a term $\Gamma \vdash t : \tau$ as a morphism/arrow $[\![t]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$.

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Example: λ-calculus → cartesian closed categories

OCaml's ADT:

```
type 'a tree =
  | Leaf
  | Node of 'a * 'a tree * 'a tree
```

It is a fixed point equation! We can use domain theory to solve it.

Beyond pure languages

Effects: control flow (errors), mutability/state, input-output...

An important aspect of programming languages!

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Modelled as a monad T (example: $T(A) \stackrel{\text{def}}{=} (A \times \mathrm{State})^{\mathrm{State}}$)

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An important aspect of programming languages!

Modelled as a monad T (example: $T(A) \stackrel{\text{def}}{=} (A \times \mathrm{State})^{\mathrm{State}}$)

Denotation of a computation: $[\![\Gamma]\!] \to T([\![\tau]\!])$

More semantics

Easter: axiomatic semantic (Hoare Logic and Model Checking)

Easter: **axiomatic semantic** (Hoare Logic and Model Checking)

In the end, the most interesting aspects of semantics is in the **interaction** between different approaches.