Where we're at

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$. ✓

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$. ✓

Adequacy: for $\gamma = \mathtt{bool}$ or $\mathtt{nat}$, if $t \in \textsc{Pcf}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$. ✗

$$
\begin{aligned}
d \lhd_{\mathtt{nat}} t &\overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow t \Downarrow_{\mathtt{nat}} \underline{d}) \\
d \lhd_{\mathtt{bool}} t &\overset{\text{def}}{\Leftrightarrow} (d = \mathrm{true} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{true}) \\
&\qquad \wedge (d = \mathrm{false} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{false}) \\
d \lhd_{\tau \to \tau'} t &\overset{\text{def}}{\Leftrightarrow} \forall e \in [\![\tau]\!], u \in \textsc{Pcf}_{\tau} .(e \lhd_{\tau} u \Rightarrow d(e) \lhd_{\tau'} t\, u)
\end{aligned}
$$

Fundamental property

For any

- context Γ, type τ and term t such that $\Gamma \vdash t : \tau$
- environment ρ
- substitution σ
- such that $\rho \triangleleft_{\Gamma} \sigma$

we have $[\![t]\!]\,(\rho) \triangleleft_{\tau} t[\sigma]$.

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Implies adequacy when $\Gamma = \cdot$ and τ is a base type.

Adequacy

Proof of the fundamental property of formal approximation

Properties of formal approximation

1. The least element approximates any program: for any τ and $t \in \mathrm{PCF}_{\tau}$, $\perp_{[\![\tau]\!]} \lhd_{\tau} t$;

2. if $d' \sqsubseteq d$ and $d \lhd_{\tau} t$, then $d' \lhd_{\tau} t$;
3. the set $\{d \in [\![\tau]\!] \mid d \lhd_{\tau} t\}$ is chain-closed;

1. By induct° on the type τ

Already seen $\perp_{\mathbb{N}_\perp} \trianglelefteq_{nat} t$ for any $t \in PCF_{nat}$

$$\perp_{\mathbb{N}_\perp} = \perp_{[\![nat]\!]}$$

Similar for bool.

$$\perp_{[\![\sigma]\!] \to [\![\tau]\!]} \overset{?}{\trianglelefteq}_{\sigma \to \tau} t \quad (\overset{def}{\Leftrightarrow} \forall e \in [\![\sigma]\!], u \in PCF_\sigma, \underbrace{\perp_{[\![\sigma]\!] \to [\![\tau]\!]} e}_{\perp_{[\![\tau]\!]}} \trianglelefteq_\tau t\, u$$

By IH on τ, $\perp_{[\![\tau]\!]} \trianglelefteq_\tau t\, u$

So $\perp_{[\![\sigma]\!] \to [\![\tau]\!]} \trianglelefteq_{\sigma \to \tau} t$

2. By induction on τ

* for nat $\quad \underbrace{d \sqsubseteq d'}_{\text{2 possibilities}} \triangleleft_{nat} t$

- $d = d' \Rightarrow d \triangleleft_{nat} t$
- $d = \bot \Rightarrow \bot \triangleleft_{nat} t$

* bool is similar

* $d \sqsubseteq_{[\![\sigma]\!] \to [\![\tau]\!]} d' \triangleleft_{\sigma \to \tau} t$

Assume $e \in [\![\sigma]\!]$, $u \in PCF_\sigma$ s.t. $e \triangleleft_\sigma u$

$de \sqsubseteq_{[\![\tau]\!]} \underbrace{d'e \triangleleft_\tau tu}_{\text{bc } d' \triangleleft_{\sigma \to \tau} t}$ by IH on τ

$de \triangleleft_\tau tu$ And so $d \triangleleft_{\sigma \to \tau} t$

1. The least element approximates any program: for any τ and $t \in \text{PCF}_\tau$, $\bot_{[\![\tau]\!]} \lhd_\tau t$;
2. if $d' \sqsubseteq d$ and $d \lhd_\tau t$, then $d' \lhd_\tau t$;
3. the set $\{d \in [\![\tau]\!] \mid d \lhd_\tau t\}$ is chain-closed;
4. if $\forall v.\, t \Downarrow_\tau v \Rightarrow t' \Downarrow_\tau v$, and $d \lhd_\tau t$, then $d \lhd_\tau t'$.

4: By induction on τ

* nat: need $d \lhd_{nat} t' \overset{def}{\iff} (d \in \mathbb{N} \Rightarrow t' \Downarrow \underline{d})$

assumed $d \lhd_{nat} t \overset{def}{\iff} (d \in \mathbb{N} \Rightarrow t \Downarrow \underline{d})$

$+\ t \Downarrow \underline{d} \Rightarrow t' \Downarrow \underline{d}$

$\Rightarrow t' \Downarrow \underline{d}$

* bool: similar

* fun: assume $d \lhd_{\sigma \to \tau} t$

take $e \in [\![\sigma]\!]$, $u \in PCF_\sigma$ st. $e \lhd_\sigma u$

by assumption $de \lhd_\tau tu$

need $de \lhd_\tau t'u$

$$\frac{t' \Downarrow \mathrm{fun}\, x.b \quad b[u/x] \Downarrow v}{tu \Downarrow v} \Rightarrow \frac{t' \Downarrow \mathrm{fun}\, x.b \quad b[u/x] \Downarrow v}{t'u \Downarrow v}$$

$\Rightarrow$

IH on τ

$de \lhd_\tau t'u$

Thus $d \lhd_\tau t'$

Fundamental property

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- such that $\rho \lhd_\Gamma \sigma$

we have $[\![t]\!]\,(\rho) \lhd_\tau t[\sigma]$.

Fundamental property

For any

- context Γ, type τ and term t such that $\Gamma \vdash t : \tau$
- environment ρ
- substitution σ
- such that $\rho \lhd_\Gamma \sigma$

we have $[\![t]\!]\,(\rho) \lhd_\tau t[\sigma]$.

Proof! Induction on $\Gamma \vdash t : \tau$:

$$\forall \rho, \sigma.\, (\rho \lhd_\Gamma \sigma \;\Rightarrow [\![t]\!]\,(\rho) \lhd_\tau t[\sigma])$$

$\underline{0}$: $\quad 0 \Downarrow_{nat} 0 = \underline{\emptyset}$

$$[\![0]\!](e) = \emptyset \lhd_{nat} 0 \overset{df}{\iff} (\underbrace{\emptyset \in \mathbb{N} \Rightarrow 0 \Downarrow_{nat} \emptyset}_{\text{true}})$$

$\underline{\text{true}}, \underline{\text{false}}$: same

$\underline{\text{succ}}$: $\dfrac{t \Downarrow_{nat} v}{\text{succ}\, t \Downarrow_{nat} \text{succ}\, v}$

we need: $[\![\text{succ}\, t]\!](e) \lhd_{nat} (\text{succ}\, t)[\sigma]$

$\quad\quad \text{succ}_\perp [\![t]\!](e) \lhd_{nat} \text{succ}\,(t[\sigma])$

Enough to show $\boxed{d \lhd_{nat} t} \Rightarrow \boxed{\text{succ}_\perp d \lhd_{nat} \text{succ}\, t}$

need $\text{succ}_\perp d \in \mathbb{N} \Rightarrow \text{succ}\, t \Downarrow \underline{\text{succ}_\perp d} = \text{succ}\, \underline{d}$

IH $d \lhd_{nat} t \overset{def}{\iff} \underbrace{d \in N}_{\text{holds}} \Rightarrow t \Downarrow \underline{d}$ thus $\boxed{\cancel{succ\ t \Downarrow succ\ \underline{d}}}$

<u>pred</u>, <u>iszero</u>: similar

$$\text{if}: \quad \frac{b \Downarrow_{bool} \text{true} \qquad t_1 \Downarrow_\tau v}{\text{if } b \text{ then } t_1 \text{ else } t_2 \Downarrow_\tau v} \qquad [\![\text{if } b \text{ then } t_1 \text{ else } t_2]\!](e) = \text{if}_\perp \langle [\![b]\!](e), \langle [\![t_1]\!](e), [\![t_2]\!](e)\rangle\rangle$$

Enough $\boxed{d_b \lhd_{bool} b' \qquad d_1 \lhd_\tau t_1' \qquad d_2 \lhd_\tau t_2'}$

$\Rightarrow \boxed{\text{if}_\perp \langle d_b, \langle t_1', t_2' \rangle\rangle \lhd_\tau \text{if } b' \text{ then } t_1' \text{ else } t_2'}$

three case depending on d_b

- $d_b = \perp$ then $\text{if}_\perp \langle \perp, \langle \dots \rangle\rangle = \perp \overset{?}{,} \lhd_\tau \text{if } b' \text{ then } t_1' \text{ else } t_2'$ ✓
- $d_b = \text{true}$ then $\text{if}_\perp \langle \text{true}, \langle d_1, d_2 \rangle\rangle = d_1 \lhd_\tau \text{if } b' \text{ then } t_1' \text{ else } t_2'$

$\text{true} \lhd_{bool} b' \Rightarrow b' \Downarrow \text{true}$ thus $t_1' \Downarrow v \Rightarrow \text{if } b' \text{ then } t_1' \text{ else } t_2' \Downarrow v$

thus by 4. $d_1 \lhd_\tau \text{if } b' \text{ then } t_1' \text{ else } t_2'$ ✓

Var: $\Gamma(x) = \tau$

Need: $[\![x]\!](e) \triangleleft_\tau x[\sigma]$

$$\underbrace{e(x) \triangleleft_\tau \sigma(x)}$$

hypothesis $e \triangleleft_\Gamma \sigma$

App: $\boxed{d_1 \triangleleft_{\sigma\to\tau} t_1 \quad d_2 \triangleleft_\sigma t_2} \Rightarrow \boxed{d_1 d_2 \triangleleft_\tau t_1 t_2}$

by def of $\triangleleft_{\sigma\to\tau}$

Fun: $[\![\mathrm{fun}\ x:\tau'.t]\!](e) = (\lambda d \in [\![\tau']\!].[\![t]\!](e[x\mapsto d]))$

$(\mathrm{fun}\ x:\tau'.t)[\sigma] = \mathrm{fun}\ x:\tau'.t[\sigma]$

Need $[\![\mathrm{fun}\ x:\tau'.t]\!](e) \lhd_{\sigma\to\tau} (\mathrm{fun}\ x:\tau'.t)[\sigma]$

Take $d \in [\![\tau']\!]$, $u \in PCF_{\tau'}$, need

$[\![\mathrm{fun}\ x:\tau'.t]\!](e)(d) \lhd_{\tau} (\mathrm{fun}\ x:\tau'.t)[\sigma]\ u$

$\overset{u}{\ }[\![t]\!](e[x\mapsto d])$ $\quad (\mathrm{fun}\ x:\tau'.t[\sigma])\ u$

$$\frac{\mathrm{fun}\ldots \Downarrow \mathrm{fun}\ x:\tau'.\ t[\sigma] \qquad t[\sigma][u/x] \Downarrow v}{(\mathrm{fun}\ x:\tau'.t[\sigma])\ u \Downarrow v}$$

$= t[\sigma[x\mapsto u]]$

and $e[x\mapsto d] \lhd_{\Gamma, x:\tau'} \sigma[x\mapsto u]$

Illam t: $[\![t]\!](e[x \mapsto d]) \lhd_\tau t[\sigma[x \mapsto u]]$

4. $[\![t]\!](e[x \mapsto d]) \lhd_\tau (\mathrm{fun}\ x : \tau'. t[\sigma])\ u \Leftarrow$

<u>Fix</u>: Assume $d \lhd_\tau f$ need $\mathrm{fix}\ d \lhd_\tau \mathrm{fix}\ f$

Scott ind. $\{ e \mid e \lhd_\tau \mathrm{fix}\ f \}$

contains $\bot$ 1.

down-closed 3.

Assume $e \lhd \mathrm{fix}\ f$

$d\,e \lhd f(\mathrm{fix}\ f)$

is enough

$$\frac{f(\mathrm{fix}\ f) \to v}{\mathrm{fix}\ f \to v}$$

$d \triangleleft_{i \to i} f \qquad e \triangleleft \mathrm{fix}\, f \qquad d(e) \triangleleft f(\mathrm{fix}\, f)$

$\triangleleft\ \mathrm{fix}\, f$

$\{ e \mid e \triangleleft \mathrm{fix}\, f \}$ stable by d

$\mathrm{fix}\, d^{\varphi} \triangleleft\ \mathrm{fix}\, f$