Denotational Semantics for Pcf

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Introducing denotational semantics

The aims of denotational semantics

- a mapping of PCF types τ to domains $[\![\tau]\!]$;
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$;
- denotation of open terms will be continuous functions.

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- denotation of open terms will be continuous functions.

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$.

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$.

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \mathrm{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$.

Adeqacy for function types?

$$v \stackrel{\text{def}}{=} \mathtt{fun}\, x{:}\,\mathtt{nat}.\,(\mathtt{fun}\, y{:}\,\mathtt{nat}.\, y)\,\mathtt{0} \quad \text{and} \quad v' \stackrel{\text{def}}{=} \mathtt{fun}\, x{:}\,\mathtt{nat}.\,\mathtt{0}.$$

The power of denotational semantics

Proof principle: to show

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau$$

it suffices to establish

$$[\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

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$$\begin{aligned} \mathcal{C}[t_1] \Downarrow_{\texttt{nat}} v &\Rightarrow [\![\mathcal{C}[t_1]]\!] = [\![v]\!] && \text{(soundness)} \\ &\Rightarrow [\![\mathcal{C}[t_2]]\!] = [\![v]\!] && \text{(compositionality on } [\![t_1]\!] = [\![t_2]\!]\text{)} \\ &\Rightarrow \mathcal{C}[t_2] \Downarrow_{\texttt{nat}} v && \text{(adequacy)} \end{aligned}$$

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\mathcal{C}[t_1] \Downarrow_{\mathtt{nat}} v &\Rightarrow [\![\mathcal{C}[t_1]]\!] = [\![v]\!] && \text{(soundness)} \\
&\Rightarrow [\![\mathcal{C}[t_2]]\!] = [\![v]\!] && \text{(compositionality on } [\![t_1]\!] = [\![t_2]\!]\text{)} \\
&\Rightarrow \mathcal{C}[t_2] \Downarrow_{\mathtt{nat}} v && \text{(adequacy)}
\end{aligned}$$

and symmetrically for $\mathcal{C}[t_2] \Downarrow_{\mathtt{nat}} v \Rightarrow \mathcal{C}[t_1] \Downarrow_{\mathtt{nat}} v$, and similarly for `bool`.

Proof principle: to show

$$t_1 \cong_{\text{ctx}} t_2 : \tau$$

it suffices to establish

$$[\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

Denotational equality is sound, but is it complete?
Does equality in the model imply contextual equivalence?

Proof principle: to show

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Denotational equality is **sound**, but is it **complete**?
Does equality in the model imply contextual equivalence?

Full abstraction.

Denotational Semantics for Pcf

Definition

Semantics of types

$$[\![\texttt{nat}]\!] \stackrel{\text{def}}{=} \mathbb{N}_\bot \qquad \text{(flat domain)}$$

$$[\![\texttt{bool}]\!] \stackrel{\text{def}}{=} \mathbb{B}_\bot \qquad \text{(flat domain)}$$

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!] \qquad \text{(function domain)}$$

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in \mathrm{dom}(\Gamma)} [\![\Gamma(x)]\!] \qquad \text{(environment)}$$

Semantics of contexts

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in \mathrm{dom}(\Gamma)} [\![\Gamma(x)]\!] \qquad \text{(environment)}$$

- $[\![\cdot]\!] = \mathbb{1}$ (one element set)
- $[\![x{:}\,\tau]\!] = (\{x\} \to [\![\tau]\!]) \cong [\![\tau]\!]$
- $[\![x_1{:}\,\tau_1, \dots, x_n{:}\,\tau_n]\!] \cong [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$

$x_i \mapsto d_i \in [\![\tau_i]\!]$

To every typing judgement

$$\Gamma \vdash t : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash t : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains. In other words,

$$[\![-]\!] : \textsc{Pcf}_{\Gamma,\tau} \to [\![\Gamma]\!] \to [\![\tau]\!]$$

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$$[\![-]\!] : \textsc{Pcf}_{\Gamma,\tau} \to [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\frac{\Gamma \vdash t : nat}{\Gamma \vdash succ(t) : nat}$$

$$[\![t]\!] \in [\![\Gamma]\!] \to [\![nat]\!]$$

$$[\![nat]\!] = \mathbb{N}_\bot$$

$$\Downarrow$$

$$[\![succ(t)]\!] \in [\![\Gamma]\!] \to \mathbb{N}_\bot$$

$$[\![succ(t)]\!] = succ([\![t]\!])$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$\begin{aligned} \text{succ} : \quad \mathbb{N} &\to \mathbb{N} \\ n &\mapsto n+1 \end{aligned}$$

$$\begin{aligned} \text{pred} : \quad \mathbb{N} &\rightharpoonup \mathbb{N} \\ n+1 &\mapsto n \\ 0 &\quad \text{undefined} \end{aligned}$$

$$\begin{aligned} \text{zero?} : \quad \mathbb{N} &\to \mathbb{B} \\ 0 &\mapsto \text{true} \\ n+1 &\mapsto \text{false} \end{aligned}$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$
\begin{aligned}
\mathrm{succ}_\bot : \quad \mathbb{N}_\bot &\to \mathbb{N}_\bot \\
n &\mapsto n+1 \\
\bot &\mapsto \bot
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{pred}_\bot : \quad \mathbb{N}_\bot &\to \mathbb{N}_\bot \\
n+1 &\mapsto n \\
0 &\mapsto \bot \\
\bot &\mapsto \bot
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{zero?}_\bot : \quad \mathbb{N}_\bot &\to \mathbb{B}_\bot \\
0 &\mapsto \mathrm{true} \\
n+1 &\mapsto \mathrm{false} \\
\bot &\mapsto \bot
\end{aligned}
$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$\overline{\Gamma \vdash 0 : \mathrm{nat}}$$

$$\rho \in \llbracket \Gamma \rrbracket$$

$$\llbracket \mathtt{0} \rrbracket(\rho) \stackrel{\text{def}}{=} 0 \in \mathbb{N}_\perp = \llbracket \mathrm{nat} \rrbracket$$

$$\llbracket \mathtt{true} \rrbracket(\rho) \stackrel{\text{def}}{=} \mathrm{true} \in \mathbb{B}_\perp$$

$$\llbracket \mathtt{false} \rrbracket(\rho) \stackrel{\text{def}}{=} \mathrm{false} \in \mathbb{B}_\perp$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$
\begin{aligned}
[\![\mathtt{0}]\!](\rho) &\stackrel{\text{def}}{=} 0 && \in \mathbb{N}_\perp \\
[\![\mathtt{true}]\!](\rho) &\stackrel{\text{def}}{=} \text{true} && \in \mathbb{B}_\perp \\
[\![\mathtt{false}]\!](\rho) &\stackrel{\text{def}}{=} \text{false} && \in \mathbb{B}_\perp
\end{aligned}
$$

$$
\frac{\Gamma \vdash t : nat}{\Gamma \vdash succ(t) : nat}
$$

$$
\begin{aligned}
[\![\mathtt{succ}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{succ}_\perp([\![t]\!](\rho)) && \in \mathbb{N}_\perp \\
[\![\mathtt{pred}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{pred}_\perp([\![t]\!](\rho)) && \in \mathbb{N}_\perp \\
[\![\mathtt{zero?}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{zero?}_\perp([\![t]\!](\rho)) && \in \mathbb{B}_\perp
\end{aligned}
$$

$$
[\![\mathtt{succ}(t)]\!] = \text{succ}_\perp \circ [\![t]\!]
$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$\frac{\Gamma \vdash b : \mathrm{bool} \quad \Gamma \vdash t : \tau \quad \Gamma \vdash t' : \tau}{\Gamma \vdash \mathrm{if}\ b\ \mathrm{then}\ t\ \mathrm{else}\ t' : \tau}$$

$$\begin{aligned}
[\![\mathtt{0}]\!](\rho) &\stackrel{\text{def}}{=} 0 && \in \mathbb{N}_\perp \\
[\![\mathtt{true}]\!](\rho) &\stackrel{\text{def}}{=} \mathrm{true} && \in \mathbb{B}_\perp \\
[\![\mathtt{false}]\!](\rho) &\stackrel{\text{def}}{=} \mathrm{false} && \in \mathbb{B}_\perp
\end{aligned}$$

$$\begin{aligned}
[\![\mathtt{succ}(t)]\!](\rho) &\stackrel{\text{def}}{=} \mathrm{succ}_\perp([\![t]\!](\rho)) && \in \mathbb{N}_\perp \\
[\![\mathtt{pred}(t)]\!](\rho) &\stackrel{\text{def}}{=} \mathrm{pred}_\perp([\![t]\!](\rho)) && \in \mathbb{N}_\perp \\
[\![\mathtt{zero?}(t)]\!](\rho) &\stackrel{\text{def}}{=} \mathrm{zero?}_\perp([\![t]\!](\rho)) && \in \mathbb{B}_\perp
\end{aligned}$$

$\mathrm{if} \in \mathbb{B}_\perp \times D \times D \to D$

$$[\![\mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t']\!] \stackrel{\text{def}}{=} \mathrm{if}([\![b]\!](\rho), [\![t]\!](\rho), [\![t']\!](\rho)) \in [\![\tau]\!]$$

$$[\![\mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t']\!] = \mathrm{if} \circ \langle [\![b]\!], \langle [\![t]\!], [\![t']\!] \rangle \rangle$$

Denotation of the λ-calculus operations

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}$$

$$[\![x]\!]\,(\rho) \stackrel{\text{def}}{=} \rho(x) \qquad \in [\![\Gamma(x)]\!]$$

$$[\![x]\!]\,(\rho) = \pi_x(\rho)$$

$$[\![x]\!] = \pi_x$$

Denotation of the λ-calculus operations

$$
\begin{aligned}
[\![x]\!]\,(\rho) &\overset{\text{def}}{=} \rho(x) && \in [\![\Gamma(x)]\!] \\
[\![t_1\ t_2]\!]\,(\rho) &\overset{\text{def}}{=} ([\![t_1]\!]\,(\rho))\ ([\![t_2]\!]\,(\rho))
\end{aligned}
$$

$$
[\![t_1\ t_2]\!] = \mathrm{eval} \circ \langle [\![t_1]\!]\,, [\![t_2]\!] \rangle
$$

Denotation of the λ-calculus operations

$$\frac{\Gamma, x:\tau \vdash t : \sigma}{\Gamma \vdash \mathsf{fun}\, x:\tau.\, t : \tau \to \sigma}$$

$$\llbracket x \rrbracket\,(\rho) \stackrel{\text{def}}{=} \rho(x) \qquad \in \llbracket \Gamma(x) \rrbracket$$

$$\llbracket t_1\ t_2 \rrbracket\,(\rho) \stackrel{\text{def}}{=} (\llbracket t_1 \rrbracket\,(\rho))\ (\llbracket t_2 \rrbracket\,(\rho))$$

$$\llbracket \mathsf{fun}\, x{:}\, \tau.\, t \rrbracket\,(\rho) \stackrel{\text{def}}{=} \lambda d \in \llbracket \tau \rrbracket\,.\ \llbracket t \rrbracket\,(\rho, d)$$

$$\llbracket t \rrbracket \in \llbracket \Gamma, x:\tau \rrbracket \to \llbracket \sigma \rrbracket$$

$$\cong$$

$$(\llbracket \Gamma \rrbracket \times \llbracket \tau \rrbracket) \to \llbracket \sigma \rrbracket \qquad \llbracket \mathsf{fun}\, x{:}\, \tau.\, t \rrbracket = \mathrm{cur}(\llbracket t \rrbracket)$$

$$\downarrow \mathrm{cur}$$

$$\llbracket \Gamma \rrbracket \to (\llbracket \tau \rrbracket \to \llbracket \sigma \rrbracket)$$

$$= \llbracket \tau \to \sigma \rrbracket$$

Denotation of fixed points

$$[\![\texttt{fix}\ f]\!]\,(\rho) \quad \stackrel{\text{def}}{=} \quad \mathrm{fix}([\![f]\!]\,(\rho))$$

$[\![\Gamma \vdash t : \tau]\!]$

For any Pcf term t such that $\Gamma \vdash t : \tau$, the object $[\![t]\!]$ is well-defined and a continuous function $[\![t]\!] : [\![\Gamma]\!] \to \tau$.

$\Gamma \leq \Delta \longrightarrow wk : [\![\Delta]\!] \to [\![\Gamma]\!]$

$[\![\Gamma \vdash t : \tau]\!] \circ wk_{\Delta,\Gamma} = [\![\Delta \vdash t : \tau]\!]$

For any Pcf term t such that $\Gamma \vdash t : \tau$, the object $[\![t]\!]$ is well-defined and a continuous function $[\![t]\!] : [\![\Gamma]\!] \to \tau$.

$\mathbb{1}_\bot$

If $t \in \text{Pcf}_\tau$: $[\![t]\!] \in [\![\cdot]\!] \to [\![\tau]\!] = \mathbb{1} \to [\![\tau]\!] \cong [\![\tau]\!]$

$\bot \stackrel{?}{=} (\bot, 0)$

Denotational Semantics for Pcf

Compositionality

Compositionality

Suppose $t, u \in \textsc{Pcf}_{\Delta,\sigma}$, such that

$$[\![t]\!] = [\![u]\!] : [\![\Delta]\!] \to [\![\sigma]\!]$$

Suppose moreover that $\mathcal{C}[-]$ is a $\textsc{Pcf}$ context such that $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$. Then

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[u]]\!] : [\![\Gamma]\!] \to [\![\tau]\!] \,.$$

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}]\!]([\![t]\!]) = [\![\mathcal{C}]\!]([\![u]\!]) = [\![\mathcal{C}[u]]\!]$$

A denotation for evaluation contexts

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

A denotation for evaluation contexts

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$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\begin{aligned} [\![-]\!]\,(d) &= d \\ [\![\mathcal{C}\ t]\!]\,(d)(\rho) &= ([\![\mathcal{C}]\!]\,(d)(\rho))([\![t]\!]\,(\rho)) \\ &\vdots \end{aligned}$$

A denotation for evaluation contexts

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

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If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$ and $\Delta \vdash t : \sigma$, then

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}]\!]\,([\![t]\!])$$

Assume

$$\left.\begin{array}{r} \Gamma \vdash u : \sigma \\ \Gamma, x{:}\sigma \vdash t : \tau \end{array}\right\} \Gamma \vdash t[u/x] : \tau$$

$$[\![\Gamma, x:\sigma]\!] = [\![\Gamma]\!] \times [\![\sigma]\!]$$

Then for all $\rho \in [\![\Gamma]\!]$

$$[\![t[u/x]]\!]\,(\rho) = [\![t]\!]\,(\rho[x \mapsto [\![u]\!]\,(\rho)]).$$

In particular when $\Gamma = \cdot$, $[\![t]\!] : [\![\sigma]\!] \to [\![\tau]\!]$ and

$[\![u]\!] \in [\![\sigma]\!]$

$$[\![t[u/x]]\!] = [\![t]\!]\,([\![u]\!])$$

$$[\![u]\!] \in [\![\Gamma]\!] \to [\![\sigma]\!]$$

$$\rho \in [\![\Gamma]\!] \quad [\![u]\!](\rho) \in [\![\sigma]\!]$$