Where we're at

Domain theory ✓

Now is the time to ask questions!

Domain theory ✓

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Denotational semantics for PCF ←

Time to fire those Chekov's guns

PCF

Pcf

Terms and types

Types: $$\tau ::= \mathtt{nat} \mid \mathtt{bool} \mid \tau \to \tau$$

let rec f (x : nat) : nat =
. t.

//

fix(fun f : nat → nat .
fun x : nat .
t)

Types: $\tau ::= \mathtt{nat} \mid \mathtt{bool} \mid \tau \to \tau$

Terms:

$$
\begin{array}{rcl}
t & ::= & \mathtt{0} \mid \mathtt{succ}(t) \mid \mathtt{pred}(t) \mid \\
& & \mathtt{true} \mid \mathtt{false} \mid \mathtt{zero?}(t) \mid \mathtt{if}\ t\ \mathtt{then}\ t\ \mathtt{else}\ t \\
& & x \mid \mathtt{fun}\ x{:}\tau.\ t \mid t\ t \mid \mathtt{fix}(t)
\end{array}
$$

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\end{array}
$$

- λ-calculus + base types/functions + `fix`
- tiny ML (without references, ADTs, polymorphism...)

Variables: up to α-equivalence

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Substitution: $t[u/x]$

Variables, substitutions and contexts

Variables: up to α-equivalence

Substitution: $t[u/x]$

Contexts: $\cdot$ and $\Gamma, x\colon \tau$

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Contexts: $\cdot$ and $\Gamma, x\colon\tau$

- partial maps from variable to types
- finite lists $x_1\colon\tau_1, \dots, x_n\colon\tau_n$

Typing for Pcf (I)

$\boxed{\Gamma \vdash t : \tau}$ The term t has type τ in context Γ

$$\textsc{Zero}\ \frac{}{\Gamma \vdash \mathtt{0} : \mathtt{nat}} \qquad \textsc{Succ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{succ}(t) : \mathtt{nat}} \qquad \textsc{Pred}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{pred}(t) : \mathtt{nat}}$$

$\boxed{\Gamma \vdash t : \tau}$ The term t has type τ in context Γ

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$$\textsc{True}\ \frac{}{\Gamma \vdash \mathtt{true} : \mathtt{bool}} \qquad \textsc{False}\ \frac{}{\Gamma \vdash \mathtt{false} : \mathtt{bool}} \qquad \textsc{IsZ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{zero?}(t) : \mathtt{bool}}$$

$$\textsc{If}\ \frac{\begin{array}{c}\Gamma \vdash b : \mathtt{bool} \\ \Gamma \vdash t : \tau \qquad \Gamma \vdash t' : \tau\end{array}}{\Gamma \vdash \mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t' : \tau}$$

Typing for Pcf (II)

$$\textsc{Var}\ \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \textsc{Fun}\ \frac{\Gamma, x{:}\sigma \vdash t : \tau}{\Gamma \vdash \mathtt{fun}\, x{:}\sigma.\, t : \sigma \to \tau} \qquad \textsc{App}\ \frac{\Gamma \vdash f : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash f\ u : \tau}$$

$$\textsc{Fix}\ \frac{\Gamma \vdash f : \tau \to \tau}{\Gamma \vdash \mathtt{fix}(f) : \tau}$$

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$$\textsc{Fix}\ \frac{\Gamma \vdash f : \tau \to \tau}{\Gamma \vdash \mathtt{fix}(f) : \tau}$$

$$\textsc{Pcf}_{\Gamma,\tau} \stackrel{\text{def}}{=} \{t \mid \Gamma \vdash t : \tau\} \qquad \textsc{Pcf}_{\tau} \stackrel{\text{def}}{=} \textsc{Pcf}_{\cdot,\tau}$$

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The **only** programs we care about!

If $\Gamma \vdash t : \tau$ and $\Gamma, x{:}\tau \vdash t' : \tau'$ both hold, then so does $\Gamma \vdash t'[t/x] : \tau'$.

Pcf

Operational semantics

Values: $v ::= \underbrace{\texttt{0} \mid \texttt{succ}(v)}_{\underline{n}} \mid \texttt{true} \mid \texttt{false} \mid \underbrace{\texttt{fun}\, x{:}\tau.\, t}_{\text{All functions } (< \texttt{fun} >)}$

PCF values

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We will only evaluate closed term to values.

PCF evaluation

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v}$$

PCF evaluation

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v} \qquad \textsc{Succ}\ \frac{t \Downarrow_{\mathtt{nat}} v}{\mathtt{succ}(t) \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)} \qquad \textsc{Pred}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)}{\mathtt{pred}(t) \Downarrow_{\mathtt{nat}} v}$$

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$$\textsc{ZeroZ}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{0}}{\mathtt{zero?}(t) \Downarrow_{\mathtt{bool}} \mathtt{true}} \qquad \textsc{ZeroS}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)}{\mathtt{zero?}(t) \Downarrow_{\mathtt{bool}} \mathtt{false}}$$

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$$\textsc{IfT}\ \frac{b \Downarrow_{\mathtt{bool}} \mathtt{true} \qquad t_1 \Downarrow_\tau v}{\mathtt{if}\ b\ \mathtt{then}\ t_1\ \mathtt{else}\ t_2 \Downarrow_\tau v} \qquad \textsc{IfF}\ \frac{b \Downarrow_{\mathtt{bool}} \mathtt{false} \qquad t_2 \Downarrow_\tau v}{\mathtt{if}\ b\ \mathtt{then}\ t_1\ \mathtt{else}\ t_2 \Downarrow_\tau v}$$

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v} \qquad \textsc{Succ}\ \frac{t \Downarrow_{\mathtt{nat}} v}{\mathtt{succ}(t) \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)} \qquad \textsc{Pred}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)}{\mathtt{pred}(t) \Downarrow_{\mathtt{nat}} v}$$

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$$\textsc{Fun}\ \frac{t \Downarrow_{\sigma\to\tau} \mathtt{fun}\, x{:}\,\sigma.\, t' \qquad t'[u/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v} \qquad \textsc{Fix}\ \frac{t\ (\mathtt{fix}(t)) \Downarrow_\tau v}{\mathtt{fix}(t) \Downarrow_\tau v}$$

$$\begin{aligned}\texttt{plus} \;\stackrel{\text{def}}{=}\; & \texttt{fun}\, x\colon \texttt{nat}.\ \texttt{fix}(\texttt{fun}(p\colon \texttt{nat} \to \texttt{nat})(y\colon \texttt{nat}).\\ & \quad \texttt{if zero?}(y)\ \texttt{then}\ x\ \texttt{else}\ \texttt{succ}(p\ \texttt{pred}(y)))\end{aligned}$$

$$\texttt{plus}\ \underline{3}\ \underline{1} \Downarrow_{\texttt{nat}} \underline{4}$$

Evaluation (I)

$$\textsc{Fun}\ \frac{\texttt{plus} \Downarrow \texttt{plus} \qquad \texttt{plus}_{\underline{3}}\ \underline{1} \Downarrow \underline{4}}{\texttt{plus}\ \underline{3}\ \underline{1} \Downarrow_{\texttt{nat}} \underline{4}}$$

$$\begin{aligned}\texttt{plus}_x \quad \stackrel{\text{def}}{=} \ & \texttt{fix}(\texttt{fun}(p\colon \texttt{nat} \to \texttt{nat})(y\colon \texttt{nat}).\\ & \quad \texttt{if zero?}(y)\ \texttt{then}\ x\ \texttt{else succ}(p\ \texttt{pred}(y)))\end{aligned}$$

$$\textsc{Fun}\ \frac{\texttt{plus} \Downarrow \texttt{plus} \qquad \texttt{plus}_{\underline{3}}\ \underline{1} \Downarrow \underline{4}}{\texttt{plus}\ \underline{3}\ \underline{1} \Downarrow_{\texttt{nat}} \underline{4}}$$

$$\begin{aligned}\texttt{plus}_x \quad \stackrel{\text{def}}{=}\ & \texttt{fix}(\texttt{fun}(p\colon \texttt{nat} \to \texttt{nat})(y\colon \texttt{nat}).\\ & \quad \texttt{if zero?}(y)\ \texttt{then}\ x\ \texttt{else succ}(p\ \texttt{pred}(y)))\end{aligned}$$

$$\textsc{Fix}\ \frac{\textsc{Fun}\ \dfrac{\textsc{Val}\ \dfrac{}{(\texttt{fun}\ p\colon \texttt{nat} \to \texttt{nat}.\ \dots) \Downarrow r} \qquad \textsc{Val}\ \dfrac{}{(\texttt{fun}\ y\colon \texttt{nat}.\ \dots)[p/\texttt{plus}_x] \Downarrow r_x}}{(\texttt{fun}(p\colon \texttt{nat} \to \texttt{nat})(y\colon \texttt{nat}).\ \dots)\ \texttt{plus}_x \Downarrow r_x}}{\texttt{plus}_x \Downarrow \underbrace{\texttt{fun}\ y\colon \texttt{nat}.\ \texttt{if zero?}(y)\ \texttt{then}\ x\ \texttt{else succ}(\texttt{plus}_x\ \texttt{pred}(y))}_{r_x}}$$

Evaluation (II)

$$
\textsc{Fun}\;\dfrac{\mathtt{plus}_{\underline{3}} \Downarrow r_{\underline{3}} \qquad \textsc{IfF}\;\dfrac{\textsc{ZeroS}\;\dfrac{\textsc{Val}\;\dfrac{}{\underline{1} \Downarrow \underline{1}}}{\mathtt{zero?}(\underline{1}) \Downarrow \cancel{\mathtt{true}}\ \text{false}} \qquad \textsc{Succ}\;\dfrac{\dfrac{\dfrac{\textsc{ZeroZ}\;\dfrac{\textsc{Pred}\;\dfrac{\ldots}{\mathtt{pred}(\underline{1}) \Downarrow \mathtt{0}}}{\mathtt{zero?}(\mathtt{pred}(\underline{1})) \Downarrow \cancel{\mathtt{false}}\ \text{true}}}{\ldots}}{\mathtt{plus}_{\underline{3}}\ \mathtt{pred}(\underline{1}) \Downarrow \underline{3}}}{\mathtt{succ}(\mathtt{plus}_{\underline{3}}\ \mathtt{pred}(\underline{1})) \Downarrow \underline{4}}}{\mathtt{if}\ \mathtt{zero?}(\underline{1})\ \mathtt{then}\ \underline{3}\ \mathtt{else}\ \mathtt{succ}(\mathtt{plus}_{\underline{3}}\ \mathtt{pred}(\underline{1})) \Downarrow \underline{4}}}{\mathtt{plus}_{\underline{3}}\ \underline{1} \Downarrow_{\mathtt{nat}} \underline{4}}
$$

Divergence

Divergence ($t \Uparrow_\tau$):

$$t : \tau \quad \wedge \quad \nexists v.\, t \Downarrow_\tau v$$

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$$\Omega_\tau \stackrel{\text{def}}{=} \texttt{fix}(\texttt{fun}\, x{:}\,\tau.\, x)$$

$$\Omega_\tau \Uparrow_\tau \qquad \text{(diverges)}$$

Divergence ($t \Uparrow_\tau$):

$$t : \tau \quad \wedge \quad \nexists v.\, t \Downarrow_\tau v$$

$$\Omega_\tau \stackrel{\text{def}}{=} \mathtt{fix}(\mathtt{fun}\, x{:}\, \tau.\, x)$$

$$\Omega_\tau \Uparrow_\tau \qquad \text{(diverges)}$$

$$\dfrac{\dfrac{\mathtt{fun}\, x{:}\, \tau.\, x \Downarrow \mathtt{fun}\, x{:}\, \tau.\, x \qquad \overset{\mathcal{P}}{\mathtt{fix}(\mathtt{fun}\, x{:}\, \tau.\, x) \Downarrow v}}{(\mathtt{fun}\, x{:}\, \tau.\, x)\,(\mathtt{fix}(\mathtt{fun}\, x{:}\, \tau.\, x)) \Downarrow v}}{\mathtt{fix}(\mathtt{fun}\, x{:}\, \tau.\, x) \Downarrow v}$$

$$\textsc{Fun-CBN}\ \frac{t \Downarrow_{\sigma\to\tau} \mathsf{fun}\, x{:}\sigma.\, t' \qquad t'[u/x] \Downarrow_\tau v}{t\, u \Downarrow_\tau v}$$

$$\textsc{Fun-CBV}\ \frac{t \Downarrow_{\sigma\to\tau} \mathsf{fun}\, x{:}\sigma.\, t' \qquad \overset{u}{\cancel{t'}} \Downarrow_\sigma v' \qquad \overset{t'[v'/x]}{\cancel{v'[u/x]}} \Downarrow_\tau v}{t\, u \Downarrow_\tau v}$$

$$\textsc{Fun-CBN}\ \frac{t \Downarrow_{\sigma \to \tau} \mathtt{fun}\, x{:}\, \sigma.\, t' \qquad t'[u/x] \Downarrow_{\tau} v}{t\ u \Downarrow_{\tau} v}$$

$$\textsc{Fun-CBV}\ \frac{t \Downarrow_{\sigma \to \tau} \mathtt{fun}\, x{:}\, \sigma.\, t' \qquad t' \Downarrow_{\sigma} v' \qquad v'[u/x] \Downarrow_{\tau} v}{t\ u \Downarrow_{\tau} v}$$

What does $(\mathtt{fun}\, x{:}\, \mathtt{nat}.\, \mathtt{0})\ \Omega_{\mathtt{nat}}$ denote?

Call-by-name and call-by-value

$$\textsc{Fun-CBN}\ \frac{t \Downarrow_{\sigma\to\tau} \mathtt{fun}\, x{:}\,\sigma.\, t' \qquad t'[u/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v}$$

$$\textsc{Fun-CBV}\ \frac{t \Downarrow_{\sigma\to\tau} \mathtt{fun}\, x{:}\,\sigma.\, t' \qquad t' \Downarrow_\sigma v' \qquad v'[u/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v}$$

What does $(\mathtt{fun}\, x{:}\, \mathtt{nat}.\, \mathtt{0})\ \Omega_{\mathtt{nat}}$ denote?

In call-by-value, all functions are **strict**... but the least-fixed points of a strict function is **always $\perp$**!

Small-step semantic

Small-step $t \rightsquigarrow_\tau u$:

$$\frac{}{(\mathsf{fun}\, x\colon \sigma.\, t)\, u \rightsquigarrow_\tau t[u/x]} \qquad \frac{t \rightsquigarrow_{\sigma\to\tau} t'}{t\ u \rightsquigarrow_\tau t'\ u} \qquad \dots$$

Small-step $t \rightsquigarrow_\tau u$:

$$\frac{}{(\mathsf{fun}\, x\colon \sigma.\, t)\; u \rightsquigarrow_\tau t[u/x]} \qquad \frac{t \rightsquigarrow_{\sigma\to\tau} t'}{t\; u \rightsquigarrow_\tau t'\; u} \qquad \dots$$

We have $t \Downarrow_\tau v$ iff $t \rightsquigarrow^\star_\tau \cancel{u}\, v$.

PCF is **Turing-complete**: for every partial recursive function ϕ, there is a PCF term $\underline{\phi} \in \mathrm{PCF}_{\mathtt{nat} \to \mathtt{nat}}$ such that for all $n \in \mathbb{N}$, if $\phi(n)$ is defined then $\underline{\phi}\ \underline{n} \Downarrow_{\mathtt{nat}} \underline{\phi(n)}$.

PCF is Turing-complete: for every partial recursive function ϕ, there is a PCF term $\underline{\phi} \in \mathrm{PCF}_{\mathtt{nat} \to \mathtt{nat}}$ such that for all $n \in \mathbb{N}$, if $\phi(n)$ is defined then $\underline{\phi}\ \underline{n} \Downarrow_{\mathtt{nat}} \underline{\phi(n)}$.

(Later on: $\phi = \llbracket \underline{\phi} \rrbracket$).

Evaluation in PCF is deterministic: if both $t \Downarrow_\tau v$ and $t \Downarrow_\tau v'$ hold, then $v = v'$.

Determinism

Evaluation in Pcf is **deterministic**: if both $t \Downarrow_\tau v$ and $t \Downarrow_\tau v'$ hold, then $v = v'$.

By (rule) induction on evaluation $\Downarrow$:

$$P(t, \tau, v) \quad \stackrel{\text{def}}{=} \quad \forall v' \in \mathrm{Pcf}_\tau\,.(t \Downarrow_\tau v' \Rightarrow v = v')$$

Intuition: there is always exactly one rule which applies.

PCF

Contextual equivalence

Contextual equivalence – informal

Two phrases of a programming language are **contextually equivalent** if any occurrences of the first phrase in a **complete program** can be replaced by the second phrase without affecting the **observable results** of executing the program.

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The intuitive notion of **program equivalence** for programmers.

Contextual equivalence – informal

Two phrases of a programming language are contextually equivalent if any occurrences of the first phrase in a complete program can be replaced by the second phrase without affecting the observable results of executing the program.

The intuitive notion of program equivalence for programmers.

But what's a complete program? What's an observable result?

Evaluation contexts

"Term with a hole":

$$\begin{aligned} \mathcal{C} \quad ::= \quad & - \mid \texttt{succ}(\mathcal{C}) \mid \texttt{pred}(\mathcal{C}) \mid \texttt{zero?}(\mathcal{C}) \mid \\ & \texttt{if}\ \mathcal{C}\ \texttt{then}\ t\ \texttt{else}\ t \mid \texttt{if}\ t\ \texttt{then}\ \mathcal{C}\ \texttt{else}\ t \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ \mathcal{C} \mid \\ & \texttt{fun}\ x{:}\tau.\ \mathcal{C} \mid \mathcal{C}\ t \mid t\ \mathcal{C} \mid \texttt{fix}(\mathcal{C}) \end{aligned}$$

"Term with a hole":

$$
\begin{array}{rcl}
\mathcal{C} & ::= & - \mid \mathtt{succ}(\mathcal{C}) \mid \mathtt{pred}(\mathcal{C}) \mid \mathtt{zero?}(\mathcal{C}) \mid \\
& & \mathtt{if}\ \mathcal{C}\ \mathtt{then}\ t\ \mathtt{else}\ t \mid \mathtt{if}\ t\ \mathtt{then}\ \mathcal{C}\ \mathtt{else}\ t \mid \mathtt{if}\ t\ \mathtt{then}\ t\ \mathtt{else}\ \mathcal{C} \mid \\
& & \mathtt{fun}\ x{:}\tau.\,\mathcal{C} \mid \mathcal{C}\ t \mid t\ \mathcal{C} \mid \mathtt{fix}(\mathcal{C})
\end{array}
$$

Typing extended to evaluation contexts: $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$.

"Term with a hole":

$$\begin{array}{rcl} \mathcal{C} & ::= & - \mid \mathtt{succ}(\mathcal{C}) \mid \mathtt{pred}(\mathcal{C}) \mid \mathtt{zero?}(\mathcal{C}) \mid \\ & & \mathtt{if}\ \mathcal{C}\ \mathtt{then}\ t\ \mathtt{else}\ t \mid \mathtt{if}\ t\ \mathtt{then}\ \mathcal{C}\ \mathtt{else}\ t \mid \mathtt{if}\ t\ \mathtt{then}\ t\ \mathtt{else}\ \mathcal{C} \mid \\ & & \mathtt{fun}\ x{:}\tau.\,\mathcal{C} \mid \mathcal{C}\ t \mid t\ \mathcal{C} \mid \mathtt{fix}(\mathcal{C}) \end{array}$$

Typing extended to evaluation contexts: $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$.

$$\frac{}{\Gamma \vdash_{\Gamma,\tau} - : \tau} \qquad \frac{\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau_1 \to \tau_2 \qquad \Gamma \vdash u : \tau_1}{\Gamma \vdash_{\Delta,\sigma} \mathcal{C}\ u : \tau_2} \qquad \dots$$

Given a type τ, a typing context Γ and terms $t, t' \in \textsc{Pcf}_{\Gamma,\tau}$, contextual equivalence, written $\Gamma \vdash t \cong_{\text{ctx}} t' : \tau$ is defined to hold if for all evaluation contexts $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$, where γ is `nat` or `bool`, and for all values $v \in \textsc{Pcf}_{\gamma}$,

$$\mathcal{C}[t] \Downarrow_{\gamma} v \Leftrightarrow \mathcal{C}[t'] \Downarrow_{\gamma} v.$$

When Γ is the empty context, we simply write $t \cong_{\text{ctx}} t' : \tau$ for $\cdot \vdash t \cong_{\text{ctx}} t' : \tau$.

Given a type τ, a typing context Γ and terms $t, t' \in \text{PCF}_{\Gamma,\tau}$, **contextual equivalence**, written $\Gamma \vdash t \cong_{\text{ctx}} t' : \tau$ is defined to hold if for all evaluation contexts $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$, where γ is `nat` or `bool`, and for all values $v \in \text{PCF}_\gamma$,

$t = x$

$C = \text{fun } x : \text{nat} . -$ $\qquad t' = x + \underline{0}$

$$\mathcal{C}[t] \Downarrow_\gamma v \Leftrightarrow \mathcal{C}[t'] \Downarrow_\gamma v.$$

When Γ is the empty context, we simply write $t \cong_{\text{ctx}} t' : \tau$ for $\cdot \vdash t \cong_{\text{ctx}} t' : \tau$.

$C[t] \Downarrow \underline{1}$

$C[t'] \Downarrow \underline{1}$

Divergence is implicitly covered.

$C = (\text{fun } x : \text{nat} . -)\ \underline{1}$

Pcf

Introducing denotational semantics

- a mapping of PCF types τ to domains $[\![\tau]\!]$;
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$;
- denotation of open terms will be continuous functions.

$$\Gamma \vdash t : \tau$$

$$[\![\Gamma]\!] \to [\![\tau]\!]$$

- a mapping of PCF types τ to domains $[\![\tau]\!]$;
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$;
- denotation of open terms will be continuous functions.

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$.

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$.

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \mathrm{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$.

$[\![t]\!] = \mathrm{true}$

$\mathbb{B}_\perp$

Adeqacy for function types?

$$\llbracket (\mathsf{fun}\, y : \mathsf{nat}.\, y)\, \emptyset \rrbracket = 0$$

$$\Downarrow \underline{0} \qquad\qquad \llbracket \underline{0} \rrbracket$$

$$v \stackrel{\text{def}}{=} \mathtt{fun}\, x\colon \mathtt{nat}.\, (\mathtt{fun}\, y\colon \mathtt{nat}.\, y)\, \mathtt{0} \quad \text{and} \quad v' \stackrel{\text{def}}{=} \mathtt{fun}\, x\colon \mathtt{nat}.\, \mathtt{0}.$$