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- We need to construct domains and continuous functions!

- Flat domains: "base" cases ($\mathbb{N}_\bot$, but also $\mathbb{B}_\bot$).
- Products of domains are domains, everything is componentwise.

Constructions on Domains

Function domains

Cpo/domain of continuous functions

Given two cpos $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the **function cpo** $(D \to E, \sqsubseteq)$ has underlying set

$$\{f : D \to E \mid \text{ is a } \textit{continuous} \text{ function}\}$$

equipped with the pointwise order:

$$f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D.\ f(d) \sqsubseteq_E f'(d).$$

D, E cpos need: D → E is a cpo

$f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots \quad \in D \to E$

$f_\infty : d \longmapsto \bigsqcup_n f_n(d)$

Claim: f_∞ is the lub of $(f_i)_{i \in \mathbb{N}}$

1) it's a bound $\forall n.\ f_n \sqsubseteq f_\infty \iff \forall n.\ \forall d \in D,\ f_n(d) \sqsubseteq f_\infty(d)$

$\iff \forall n, \forall d \in D\ f_n(d) \sqsubseteq \bigsqcup_n f_n(d)$

✓

2) f_∞ is a lub

assume $\forall n. f_n \sqsubseteq f$

claim $f_\infty \sqsubseteq f$ $\Rightarrow \forall d.\ f_\infty(d) \leq f(d)$

$\Rightarrow \forall d.\ \bigsqcup_n f_n(d) \sqsubseteq f(d)$

$\Rightarrow \forall d. \forall n\ f_n(d) \sqsubseteq f(d)$

$\Rightarrow \forall n.\ f_n \sqsubseteq f(d)$

CPO/DOMAIN OF CONTINUOUS FUNCTIONS

Given two cpos $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the **function cpo** $(D \to E, \sqsubseteq)$ has underlying set

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equipped with the pointwise order:

$$f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D.\ f(d) \sqsubseteq_E f'(d).$$

$$\frac{f \sqsubseteq_{D \to E} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq_E g(y)}$$

Cpo/domain of continuous functions

Given two cpos $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$\{f : D \to E \mid \text{ is a } \textit{continuous} \text{ function}\}$$

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Argumentwise least elements and lubs:

$$\bot_{D \to E}(d) = \bot_E \qquad \left(\bigsqcup_{n \geq 0} f_n\right)(d) = \bigsqcup_{n \geq 0} f_n(d)$$

Function operations are continuous

Evaluation, currying ($f : (D' \times D) \to E$) and composition

$$\begin{aligned} \text{eval} : \quad (D \to E) \times D &\to E \\ (f, d) &\mapsto f(d) \end{aligned}$$

$$\begin{aligned} \text{cur}(f) : \quad D' &\to (D \to E) \\ d' &\mapsto \lambda d \in D.\ f(d', d) \end{aligned}$$

$f(d', \cdot)$

$$\begin{aligned} \circ : \quad ((E \to F) \times (D \to E)) &\longrightarrow (D \to F) \\ (f, g) &\mapsto \lambda d \in D.\ g(f(d)) \end{aligned}$$

are all well-defined and continuous.

$f \in D \times D' \to E$ $\quad$ $\mathrm{cur}(f) : d \mapsto (d' \mapsto f(d, d'))$

1) for any d $\quad$ $\mathrm{cur}(f)(d)$ is continuous $\quad$ $\mathrm{cur}(f)(d) \in D' \to E$

$d'_0 \sqsubseteq d'_1 \sqsubseteq \dots \in D'$

$f(d, \bigsqcup_n d'_n) = \bigsqcup_n f(d, d'_n)$ ✓

2) $d_0 \sqsubseteq d_1 \sqsubseteq \dots \in D$

$\bigsqcup_n (\mathrm{cur}(f)(d_n)) \overset{?}{=} \mathrm{cur}(f)(\bigsqcup_n d_n)$

$$\bigsqcup_n \mathrm{cur}(f)(d_n) \stackrel{?}{=} \mathrm{cur}(f)\left(\bigsqcup_n d_n\right)$$

$$\bigsqcup_n \left(d' \mapsto f(d_n, d')\right) = d' \mapsto \bigsqcup_n \left(f(d_n, d')\right)$$

$$= d' \mapsto f\left(\bigsqcup_n d_n, d'\right)$$

$$= \mathrm{cur}(f)\left(\bigsqcup_n d_n\right)$$

Continuity of the fixed point operator

$$\text{fix}: \quad (D \to D) \quad \to \quad D$$

is continuous.

1) fix is monotone

$f_1 \sqsubseteq f_2 \in D \to D$

$f_1(fix(f_2)) \sqsubseteq f_2(fix(f_2))$

$\sqsubseteq fix(f_2)$

Thus $fix(f_2)$ is the fixed point of f_1

$fix(f_1) \sqsubseteq fix(f_2)$

2) $f_0 \sqsubseteq f_1 \sqsubseteq \dots \quad \in D \to D$

$$\bigsqcup_n (\mathrm{fix}(f_n)) \overset{?}{=} \mathrm{fix}\left(\bigsqcup_n f_n\right) = \bigsqcup_m \left(\bigsqcup_n f_n^{\,m}(\bot)\right)$$

$$\bigsqcup_n \left(\bigsqcup_m (f_n)^m(\bot)\right)$$

Diagonalisation

Constructions on Domains

Back to the introduction

The semantics of a while loop

$$[\![\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\mathbf{State} \rightharpoonup \mathbf{State})$:

$$F(w)([X \mapsto x, Y \mapsto y]) \quad = \quad \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0. \end{cases}$$

The semantics of a while loop

$$[\![\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\text{State}_\perp \to \text{State}_\perp)$:

$$F(w)([X \mapsto x, Y \mapsto y]) = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x-1, Y \mapsto x \cdot y]) & \text{if } x > 0. \end{cases}$$

$$F(\perp) = \perp$$

$\text{State}_\perp \to \text{State}_\perp$ is a domain!

Kleene's fixed point theorem

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We can compute explicitly

$$w_\infty[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x \geq 0 \end{cases}$$

And check this agrees with the operational semantics.

Scott Induction

Let D be a domain, $f : D \to D$ be a continuous function and $S \subseteq D$ be a subset of D. If the set S

(i) contains $\bot$,

(ii) is chain-closed, *i.e.* the lub of any chain of elements of S is also in S,

(iii) is stable for f, *i.e.* $f(S) \subseteq S$, $\forall x \in S,\ f(x) \in S$

then $\mathrm{fix}(f) \in S$.

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$$\textsc{ScottInd}\ \frac{\Phi(\bot) \qquad \Phi(x) \Rightarrow \Phi(f(x)) \qquad (\forall i \in \mathbb{N}.\ \Phi(x_i)) \Rightarrow \Phi(\bigsqcup_{i\in\mathbb{N}} x_i)}{\Phi(\mathrm{fix}(f))}$$

Proof: $fix(f) = \sqcup \{ f^n(\bot) \} \overset{?}{\in} S$

by ii) it is enough to show $\forall n.\ f^n(\bot) \in S$

by induction on n:

<u>base</u>: $f^0(\bot) = \bot \in S$ by i)

<u>ind</u>: $f^{n+1}(\bot) = f(f^n(\bot))$ IH: $f^n(\bot) \in S$

$\in S$ by iii)

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$x_i \sqsubseteq y_i \qquad \bigsqcup x_i \sqsubseteq \bigsqcup y_i$

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$$f^{-1}S = \{x \in D \mid f(x) \in S\} \qquad \text{if } S \subseteq E \text{ is chain-closed, and } f\colon D \to E \text{ is continuous}$$

$\Phi(x)$ chain closed $\quad \Phi(f(x))$ is chain-closed

Building chain-closed sets

All the following are chain-closed:

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$$S \cup T \qquad \text{and} \qquad \bigcap_{i \in I} S_i \qquad \text{if } S, T \text{ and } S_i \text{ are}$$

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$f^{-1}S = \{x \in D \mid f(x) \in S\}$ if $S \subseteq E$ is chain-closed, and $f : D \to E$ is continuous

$S \cup T$ and $\bigcap_{i \in I} S_i$ if S, T and S_i are

$\forall S \stackrel{\text{def}}{=} \{y \in E \mid \forall x \in D.\ (x, y) \in S\} \subseteq E$ if $S \subseteq D \times E$ is

The "logical" view

Any formula written using:

- signature: continuous functions + constants
- relations: equality, inequality
- logical connectives: conjuction, disjunction, universal quantification

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Given any set I, domains D, E, functions $(f_i)_{i \in I}$, $g \colon D \to E$, $e \in E$,

$$\Phi(x) := \forall y \in E, (\forall i \in I, f_i(x) \sqsubseteq y) \vee g(x) = e$$

is chain-closed.

Example: downset

Assume $f(d) \sqsubseteq d$, *i.e.* d is a pre-fixed point of the continuous $f : D \to D$. By Scott induction on $d \downarrow$, $\mathrm{fix}(f) \sqsubseteq d$.

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$\Phi(x) = x \sqsubseteq d$

i) $\bot \sqsubseteq d$ ✓

ii) $d_0 \sqsubseteq \dots d_i \sqsubseteq d \Rightarrow \bigsqcup_n d_n \sqsubseteq d$ ✓

Proof!

iii) $x \sqsubseteq d \Rightarrow f(x) \sqsubseteq f(d) \sqsubseteq d$ ✓

$\Phi(\mathrm{fix}\, f)$ $\quad$ $\mathrm{fix}(f) \sqsubseteq d$

Let $w_\infty \colon \mathrm{State}_\perp \to \mathrm{State}_\perp$ be the denotation of

$$\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)$$

$[X \mapsto x \quad Y \mapsto y]$

$\mathrm{State} = \mathbb{N} \times \mathbb{N}$

Recall that $w_\infty = \mathrm{fix}(F)$ where

$$F(w)(x, y) = \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

$$F(w)(\perp) = \perp$$

Example: partial correctness

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$$F(w)(\perp) = \perp$$

$$w(x,y)\Downarrow \iff w(x,y) \neq \perp$$

Claim:

$$\forall x.\, \forall y \geq 0.\, w_\infty(x,y)\Downarrow \implies \pi_Y\left(w_\infty(x,y)\right) \geq 0$$

$$\Phi(w) := \forall x. \forall y \geq 0.\ w(x,y)\Downarrow \Rightarrow \pi_Y(w(x,y)) \geq 0$$

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Claim:

$$\forall x.\ \forall y \geq 0.\ w_\infty(x, y) \Downarrow \implies \pi_Y\left(w_\infty(x, y)\right) \geq 0$$

Proof: by Scott induction!

$$\Phi(w) = \forall x . \forall y \geq 0 . \; w(x, y) \Downarrow \Rightarrow w(x, y) \geq 0$$

$$\Phi(w) \iff \bigcap_{x \in \mathbb{Z}} \bigcap_{y \in \mathbb{N}} eval(w, (x, y)) = \bot \vee \bigcup_{z \in \mathbb{N}} eval(w, (x, y)) = z$$

To show $\Phi(w)$ is chain-closed it is enough to show that $\Phi_{x,y}(w)$ — $w(x, y) \geq 0$ — is chain-closed

$w_0 \sqsubseteq w_1 \sqsubseteq \dots$ $\in State_\perp \to State_\perp$

assume $w_i(x, y) \geq 0$

need $(\bigsqcup_i w_i)(x, y) \overset{?}{\geq} 0$

$$\bigsqcup_i (w_i(x, y)) = w_? (x, y)$$

$\mathbb{Z}_\perp$

$\underline{\bar{\Phi}(t)}$ ✓