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- Flat domains: "base" cases (\mathbb{N}_{\perp} , but also \mathbb{B}_{\perp}).
- Products of domains are domains, everything is componentwise.

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CONSTRUCTIONS ON DOMAINS FUNCTION DOMAINS

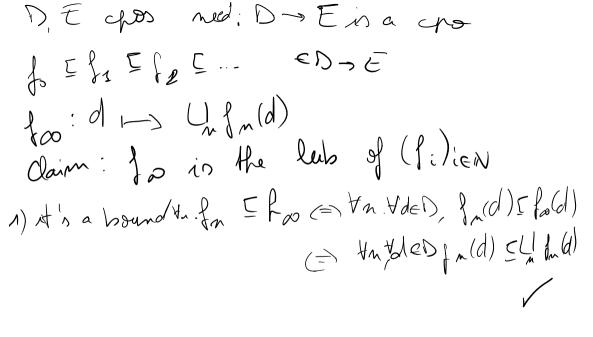
CPO/DOMAIN OF CONTINUOUS FUNCTIONS

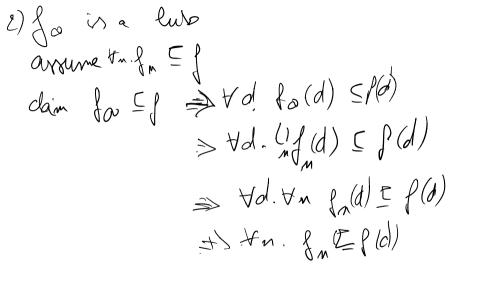
Given two cpos (D, \sqsubseteq_D) and (E, \sqsubseteq_E) , the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$${f:D \to E \mid \text{ is a } continuous function}}$$

equipped with the pointwise order:

$$f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D. \ f(d) \sqsubseteq_E f'(d).$$





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$$\frac{f \sqsubseteq_{D \to E} g \qquad x \sqsubseteq_{D} y}{f(x) \sqsubseteq_{E} g(y)}$$

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Argumentwise least elements and lubs:

$$\perp_{D \to E}(d) = \perp_{E} \qquad \left(\bigsqcup_{n \ge 0} f_n\right)(d) = \bigsqcup_{n \ge 0} f_n(d)$$

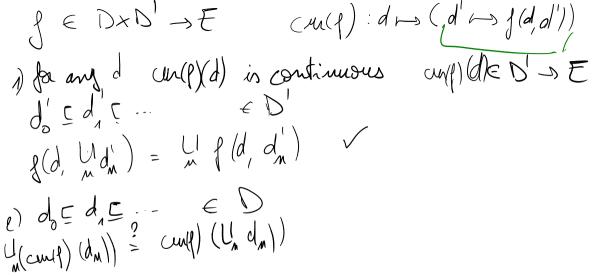
FUNCTION OPERATIONS ARE CONTINUOUS

Evaluation, currying $(f:(D'\times D)\to E)$ and composition

eval:
$$(D \to E) \times D \to E$$

 $(f,d) \mapsto f(d)$
cur $(f): D' \to (D \to E)$
 $d' \mapsto \lambda d \in D. f(d',d)$
 $\circ: ((E \to F) \times (D \to E)) \to (D \to F)$
 $(f,g) \mapsto \lambda d \in D. g(f(d))$

are all well-defined and continuous.



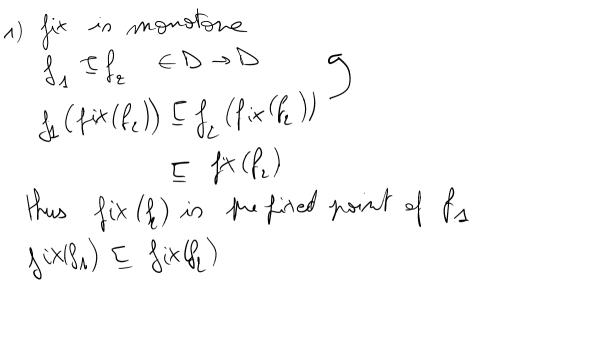
$$\begin{array}{l}
\text{LI}\left(d'\mapsto f(d_n,d')\right) = d'\mapsto \mathcal{L}\left\{d(d_n,d')\right\} \\
= d'\mapsto f(\mathcal{L}_nd_n,d') \\
= cun(f)(\mathcal{L}_nd_n)
\end{array}$$

Un well day = cure) (Unday)

CONTINUITY OF THE FIXED POINT OPERATOR

fix:
$$(D \to D) \to D$$

is continuous.



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$$f = f_1 = \dots \in D \rightarrow D$$

 $U(fix(f_n)) \stackrel{?}{=} fix(U_n f_n) = U(U_n f_n^m(L))$
 $U_m(U_n f_n^m(L))$ Liagonalisation

CONSTRUCTIONS ON DOMAINS BACK TO THE INTRODUCTION

THE SEMANTICS OF A WHILE LOOP

$$\llbracket \text{while } X > 0 \text{ do } (Y \coloneqq X * Y; X \coloneqq X - 1) \rrbracket$$

is a fixed point of the following $F: D \to D$, where D is (State \to State):

$$F(w)([X \mapsto x, Y \mapsto y]) = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \le 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0. \end{cases}$$

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$$F(\bot) = \bot$$

 $State_{\perp} \rightarrow State_{\perp}$ is a domain!

KLEENE'S FIXED POINT THEOREM

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$$w_{\infty} = \bigsqcup_{i \in \mathbb{N}} F^n(\bot)$$

is the least fixed point of F, and in particular a fixed point.

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We can compute explicitly

$$w_{\infty}[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x \ge 0 \end{cases}$$

And check this agrees with the operational semantics.



REASONING ON FIXED POINTS: SCOTT INDUCTION

Let D be a domain, $f: D \to D$ be a continuous function and $S \subseteq D$ be a subset of D. If the set S

- (i) contains ⊥,
- (ii) is chain-closed, i.e. the lub of any chain of elements of S is also in S,
- (iii) is stable for f, i.e. $f(S) \subseteq S$, $\forall x \in S$, $f(G) \in S$ then $fix(f) \in S$.

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$$\Phi(\bot) \qquad \Phi(x) \Rightarrow \Phi(f(x)) \qquad (\forall i \in \mathbb{N}. \ \Phi(x_i)) \Rightarrow \Phi(\bigsqcup_{i \in \mathbb{N}} x_i)$$
 Scottind
$$\frac{\Phi(\operatorname{fix}(f))}{}$$

Proof: fit(1) = 6(f(-1) ES by ii) it is enough to show to . f"(1) ES by induction on m: base if o(1) = 1 < 5 log i) IH: 1 1(1) CS

ind: { m+1 (+) = { ({ ({ (m)) } }

$$\{(x,y) \in D \times D \mid x \sqsubseteq y\} \ , \qquad d \downarrow^{\operatorname{def}} \{x \in D \mid x \sqsubseteq d\} \quad \text{and} \quad \{(x,y) \in D \times D \mid x = y\}$$

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$$f^{-1}S = \{x \in D \mid f(x) \in S\} \quad \text{if } S \subseteq E \text{ is chain-closed, and } f:D \to E \text{ is continuous}$$

$$\boxed{ \text{Def}} \left(f\left(y\right) \right) \text{ in Chain-Closed}$$

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$$\forall S \stackrel{\text{def}}{=} \{ y \in E \mid \forall x \in D. (x, y) \in S \} \subseteq E \quad \text{if } S \subseteq D \times E \text{ is}$$

THE "LOGICAL" VIEW

Any formula written using:

- signature: continuous functions + constants
- · relations: equality, inequality
- · logical connectives: conjuction, disjunction, universal quantification

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Given any set I, domains D, E, functions $(f_i)_{i \in I}$, $g: D \to E$, $e \in E$,

$$\Phi(x) := \forall y \in E, (\forall i \in I, f_i(x) \sqsubseteq y) \lor g(x) = e$$

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EXAMPLE: DOWNSET

Assume $f(d) \sqsubseteq d$, i.e. d is a pre-fixed point of the continuous $f: D \to D$. By Scott induction on $d \downarrow$, $\operatorname{fix}(f) \sqsubseteq d$.

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(a) $\downarrow \subseteq d$ (b) $\downarrow \subseteq d$ (c) $\downarrow \subseteq d$ (d) $\downarrow \subseteq d$ (e) $\downarrow \subseteq d$ (f) $\downarrow \subseteq d$ (f) $\downarrow \subseteq d$ (g) $\downarrow \subseteq$ iii) x Ed > f O) = f(d) Ed V Proof!

EXAMPLE: PARTIAL CORRECTNESS

Recall that $w_{\infty} = \operatorname{fix}(F)$ where

$$F(w)(x, y) = \begin{cases} (x, y) & \text{if } x \le 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$
$$F(w)(\bot) = \bot$$

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$$F(w)(\bot) = \bot \qquad \qquad \text{with } (x,y) \text{ if } x > 0$$

Claim:

$$\forall x. \ \forall y \ge 0. \ w_{\infty}(x,y) \ \downarrow \implies \ \pi_Y(w_{\infty}(x,y)) \ge 0$$

$$\overline{D}(\omega) := \forall \alpha. \forall \gamma \geq 0. \forall (\alpha, \gamma) \forall \beta = \forall \gamma (\omega(\alpha, \gamma)) \geq 0$$

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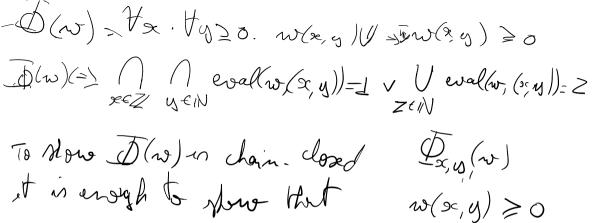
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Proof: by Scott induction!



 $w(9c,y) \geq 0$

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wo [
$$y \in State_1 \rightarrow State_2$$

assume $w \in (x, y) \ge 0$
med $(y, y) \ge 0$

((w; (x, y)) = w, (x, y)

 $\overline{\Phi}(1)$ V