Where we're at

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- partially ordered set = poset
- + lubs of all chains = cpo
- + least element = domain
- the least (pre)fixed point $\mathrm{fix}(f)$ of a monotone f is a fixed point:

$$f(\mathrm{fix}(f)) = \mathrm{fix}(f)$$

Least Fixed Points

Continuous functions

$$D \xrightarrow{f} E$$

$$\begin{tikzcd}
\bigsqcup d_n & \bigsqcup f(d_n) \arrow[rr, leftrightarrow, "?"] & & f(\bigsqcup(d_n)) \\
& & \vdots \arrow[ul] \arrow[ur] & \\
d_{n+1} \arrow[uu, dotted] & & f(d_{n+1}) \arrow[u] \arrow[uul] \arrow[uur] & \\
d_n \arrow[u] & & f(d_n) \arrow[u] & \\
d_1 \arrow[u, dotted] & & f(d_1) \arrow[u, dotted] & \\
d_0 \arrow[u] & & f(d_0) \arrow[u] &
\end{tikzcd}$$

$$
\begin{tikzcd}
D \arrow[r, "f"] & E
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bigsqcup d_n & \bigsqcup f(d_n) \arrow[rr] & & f(\bigsqcup(d_n)) \\
& & \vdots \arrow[ul] \arrow[ur] & \\
d_{n+1} \arrow[uu, dotted] & & f(d_{n+1}) \arrow[u] \arrow[uul] \arrow[uur] & \\
d_n \arrow[u] & & f(d_n) \arrow[u] & \\
d_1 \arrow[u, dotted] & & f(d_1) \arrow[u, dotted] & \\
d_0 \arrow[u] & & f(d_0) \arrow[u] &
\end{tikzcd}
$$

Given two cpos D and E, a function $f : D \to E$ is continuous if

- it is monotone, and
- it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, we have

$$f\Big(\bigsqcup_{n \geq 0} d_n\Big) = \bigsqcup_{n \geq 0} f(d_n)$$

Note: one direction is automatic.

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Note: one direction is automatic.

A function f is strict if $f(\perp_D) = \perp_E$.

All computable functions are continuous.

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Typical non-continuous function: "is a sequence the constant 0"? $(\mathbb{N} \rightharpoonup \mathbb{B}) \rightharpoonup \mathbb{B}$

$$\begin{array}{llllllll} 0 & 0 & \perp & \ldots & & & & \mapsto \perp \\ 0 & 0 & 0 & 0 & 1 & \ldots & & \mapsto 1 \\ \\ 0 & 0 & 0 & 0 & 0 & \overline{0} & & \mapsto 0 \end{array}$$

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$$
\begin{array}{llllllll}
0 & 0 & \perp & \ldots & & & & \mapsto \perp \\
0 & 0 & 0 & 0 & 1 & \ldots & & \mapsto 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & & \mapsto ? \\
 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \overline{0} & & \mapsto 0
\end{array}
$$

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$$
\begin{array}{cccccccccccl}
0 & 0 & \bot & \dots & & & & & & & \mapsto \bot \\
0 & 0 & 0 & 0 & 1 & \dots & & & & & \mapsto 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bot & \dots & \mapsto \bot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \mapsto ? \\
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Intuition: non-continuity ≈ "jump at infinity" ≈ non-computability

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$$
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bot & \ldots & \mapsto & \bot \\
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0 & 0 & 0 & 0 & 0 & \overline{0} & & & & & \mapsto & 0
\end{array}
$$

Intuition: non-continuity $\approx$ "jump at infinity" $\approx$ non-computability

Later in the course: show the thesis... by giving a denotational semantics.

Least Fixed Points

Kleene's fixed point theorem

Kleene's fixed point theorem

Let $f: D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

Let $f: D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

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It is thus also the least fixed point of f!

Let $f\colon D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

It is thus also the least fixed point of f!

$$\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq f^3(\bot) \sqsubseteq \cdots \sqsubseteq \bigsqcup_n f^n(\bot)$$

$$f(\bot) \sqsubseteq f(f(\bot))$$

$$f^2(\bot) \sqsubseteq f^3(\bot)$$

Claim: $\bigsqcup_n f^n(\bot)$ is a least pre-fixed point of f.

1) prefixed point

$$\begin{aligned} f\left(\bigsqcup_n f^n(\bot)\right) &= \bigsqcup_n f(f^n(\bot)) \\ &= \bigsqcup_n f^{n+1}(\bot) \\ &= \bigsqcup_n f^n(\bot) \end{aligned}$$

2) least prefixed points

Assume d st $f(d) \sqsubseteq d$

$\bot \sqsubseteq d$ $\quad\quad$ $\bot = f^0(\bot)$

$f(\bot) \sqsubseteq f(d) \sqsubseteq d$ $\quad\quad$ $f^n(\bot) \sqsubseteq d$

$f(f(\bot)) \sqsubseteq f(d) \sqsubseteq d$ $\quad\quad$ $f^{n+1}(\bot) = f(f^n(\bot)) \sqsubseteq f(d) \sqsubseteq d$

$\therefore$ by induction

$f^n(\bot) \sqsubseteq d$ for all n thus

$\sqcup_n f^n(\bot) \sqsubseteq d$

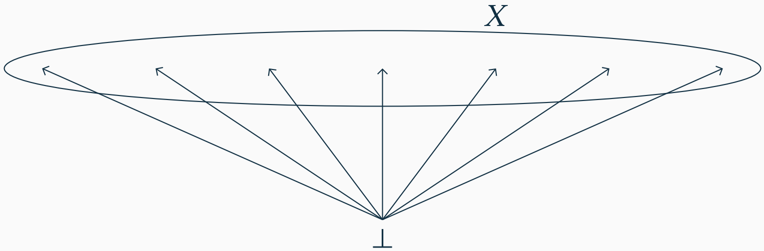
Constructions on Domains

Constructions on Domains

Flat domains

The flat domain on a set X is defined by:

- its underlying set $X \uplus \{\perp\}$;
- $x \sqsubseteq x'$ if either $x = \perp$ or $x = x'$.



Let $f : X \rightharpoonup Y$ be a partial function between two sets. Then

$$\begin{aligned} f_\perp : \quad X_\perp \quad &\to \quad Y_\perp \\ d \quad &\mapsto \begin{cases} f(d) & \text{if } d \in X \text{ and } f \text{ is defined at } d \\ \perp & \text{if } d \in X \text{ and } f \text{ is not defined at } d \\ \perp & \text{if } d = \perp \end{cases} \end{aligned}$$

defines a strict continuous function between the corresponding flat domains.

Constructions on Domains

Products of domains

Binary product

The product of two posets $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \land d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \land d_2 \sqsubseteq_2 d_2'$$

Binary product

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$$\text{POX}\ \frac{d_1 \sqsubseteq_1 d_1' \qquad d_2 \sqsubseteq_2 d_2'}{\overline{(d_1, d_2) \sqsubseteq (d_1', d_2')}}$$

Componentwise lubs and least elements

lubs of chains are computed componentwise:

$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}).$$

$$\begin{array}{c}
\bigsqcup_n d_{1,n} \quad \bigsqcup_n d_{2,n} \\
\vdots \\
\sqcup\!\!| \\
(d_{1,2}, d_{2,2}) \\
\sqcup\!\!| \\
(d_{1,1}, d_{2,1}) \\
\sqcup\!\!| \\
(d_{1,0}, d_{2,0}) \\
D_1 \qquad D_2
\end{array}$$

Proof: $(d_{1,n}, d_{2,n})$ is a chain $\Rightarrow (\bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n})$

is the lub of the chain

$$\dfrac{\dfrac{}{d_{1,n} \sqsubseteq \bigsqcup_k d_{1,k}}\,\text{Bound} \qquad \dfrac{}{d_{2,n} \sqsubseteq \bigsqcup_k d_{2,k}}\,\text{Bound}}{(d_{1,n}, d_{2,n}) \sqsubseteq (\bigsqcup_k d_{1,k}, \bigsqcup_k d_{2,k})}$$

$$\forall n.\ (d_{1,n}, d_{2,n}) \sqsubseteq (d_1, d_2)$$

then
$$\dfrac{\dfrac{\forall n.\ d_{1,n} \sqsubseteq d_1}{\bigsqcup_n d_{1,n} \sqsubseteq d_1}\,\text{least} \qquad \dfrac{\forall n.\ d_{2,n} \sqsubseteq d_2}{\bigsqcup_n d_{2,n} \sqsubseteq d_2}\,\text{least}}{(\bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n}) \sqsubseteq (d_1, d_2)}\,\text{POx}$$

lubs of chains are computed componentwise:

$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}).$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ have least elements, so does $(D_1 \times D_2, \sqsubseteq)$ with

$$\perp_{D_1\times D_2} = (\perp_{D_1}, \perp_{D_2})$$

Componentwise lubs and least elements

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Products of cpos (domains) are cpos (domains).

A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e').$$

Functions of two arguments

A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:

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$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e').$$

Moreover, it is continuous if and only if it preserves lubs in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m\,,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$
$$f(d\,,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

$\Rightarrow f(\bigsqcup_n (d_n, e_n)) = \bigsqcup_n f(d_n, e_n)$

$$\bigsqcup_m \bigsqcup_n f(d_m, e_n) = \bigsqcup_n \bigsqcup_m f(d_m, e_n) = \bigsqcup_n f(d_n, e_n)$$

$$f\left(\bigsqcup_n (d_n, e_n)\right) = f\left(\bigsqcup_m \bigsqcup_n (d_m, e_n)\right)$$

$$= f\left(\bigsqcup_m d_m, \bigsqcup_n e_n\right)$$

$$= \bigsqcup_m f\left(d_m, \bigsqcup_n e_n\right)$$

$$= \bigsqcup_m \bigsqcup_n f(d_m, e_n)$$

$$= \bigsqcup_n f(d_n, e_n)$$

$$\textsc{mon}\times\ \frac{f \text{ monotone} \qquad x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')}$$

$$f\left(\bigsqcup_m x_m, \bigsqcup_n y_n\right) = \bigsqcup_m \bigsqcup_n f(x_m, y_n) = \bigsqcup_k f(x_k, y_k)$$

Let D_1 and D_2 be cpos. The projections

$$\begin{array}{rrcl} \pi_1 : & D_1 \times D_2 & \to & D_1 \\ & (d_1, d_2) & \mapsto & d_1 \end{array} \qquad \begin{array}{rrcl} \pi_2 : & D_1 \times D_2 & \to & D_2 \\ & (d_1, d_2) & \mapsto & d_2 \end{array}$$

are continuous functions.

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are continuous functions.

If $f_1 : D \to D_1$ and $f_2 : D \to D_2$ are continuous functions from a cpo D, then the pairing function

$$\begin{array}{rccl} \langle f_1, f_2 \rangle : & D & \to & D_1 \times D_2 \\ & d & \mapsto & (f_1(d), f_2(d)) \end{array}$$

is continuous.

The conditional function

$$\begin{aligned} \mathrm{if} : \quad \mathbb{B}_\perp \times (D \times D) &\to D \\ (x, d) &\mapsto \begin{cases} \pi_1(d) & \text{if } x = \mathrm{true} \\ \pi_2(d) & \text{if } x = \mathrm{false} \\ \perp_D & \text{if } x = \perp \end{cases} \end{aligned}$$

is continuous.

Assume (b_n, d_n) a chain in $\mathbb{B}_\perp \times (D \times D)$

chain in a flat domain : ultimately constant

- b_n is ult. $\perp$ $\Rightarrow$ $if(b_n, d_n)$ is ultimately $\perp_D$

  $\bigsqcup(b_n, d_n) = (\bigsqcup_n b_n, \bigsqcup_n d_n) = (\perp, \bigsqcup_n d_n)$

  $if(\bigsqcup_n (b_n, d_n)) = if(\perp, \bigsqcup_n d_n) = \perp_D$

- b_n is ult. true $\Rightarrow$ $if(b_n, d_n)$ is $\pi_1(d_n)$ and so

  $\bigsqcup(if(b_n, d_n)) = \bigsqcup_n \pi_1(d_n)$

  $\bigsqcup_n (b_n, d_n) = \bigsqcup_n \pi_1(d_n)$

  $if(\bigsqcup_n (b_n, d_n)) = if(true, \bigsqcup_n d_n)$

  $= \pi_1(\bigsqcup_n d_n) = \bigsqcup_n \pi_1(d_n)$

- b_n is ult. false $\Rightarrow$ ...

General product

Given a set I, suppose that for each $i \in I$ we are given a set X_i. The (cartesian) **product** of the X_i is

$$\prod_{i \in I} X_i$$

Two ways to see it:

- tuples: $(\dots, x_i, \dots)_{i \in I}$ such that $x_i \in X_i$;

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- heterogeneous functions: p defined on I such that $p(i) \in X_i$.

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Special case: $\prod_{i \in \mathbb{B}} D_i$ corresponds to $D_{\mathrm{true}} \times D_{\mathrm{false}}$.

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Projections (for any $i \in I$):

$$\pi_i : \left(\prod_{i \in I} X_i\right) \to X_i$$

Given a set I, suppose that for each $i \in I$ we are given a cpo $(D_i, \sqsubseteq_i)$. The **product** of this whole family of cpos has

- underlying set equal to $\prod_{i \in I} D_i$;

General product of domains

Given a set I, suppose that for each $i \in I$ we are given a cpo $(D_i, \sqsubseteq_i)$. The product of this whole family of cpos has

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- componentwise order

$$p \sqsubseteq p' \stackrel{\text{def}}{\Leftrightarrow} \forall i \in I .\, p_i \sqsubseteq_i p'_i .$$

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I-indexed products of cpos (domains) are cpos (domains), and projections are continuous.