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- The denotation of recursive definitions (*e.g.* loops) are **fixed points**: some w such that $F(w) = w$;
- total functions are not a good fit, **partial** functions are better;
- there is some **"information" order** involved;
- we can compute fixed points by iterating F on the least element $\perp$ and taking a "**limit**".

Least Fixed Points

Least Fixed Points

Posets and monotone functions

PARTIALLY ORDERED SET

A partial order on a set D is a binary relation $\sqsubseteq$ that is

$$\begin{aligned} \text{reflexive: } & \forall d \in D.\ d \sqsubseteq d \\ \text{transitive: } & \forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d'' \\ \text{antisymmetric: } & \forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'. \end{aligned}$$

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antisymmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

$$\textsc{Refl}\ \frac{}{x \sqsubseteq x} \qquad \textsc{Trans}\ \frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z} \qquad \textsc{ASym}\ \frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

DOMAIN OF PARTIAL FUNCTIONS $X \rightharpoonup Y$

Underlying set: partial functions f with domain of definition $\mathbf{dom}(f) \subseteq X$ and taking values in Y;

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Refl : $f \sqsubseteq f$: $\mathrm{dom}(f) \subseteq \mathrm{dom}(f)$ and . . .

Trans : $f \sqsubseteq g \sqsubseteq h$:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[r, phantom, "\subseteq"] \arrow[rr, bend right, "\subseteq"'] & \mathrm{dom}(g) \arrow[r, phantom, "\subseteq"] & \mathrm{dom}(h) \end{tikzcd}$$

$x \in \mathrm{dom}(f)$ $\quad f(x) = g(x) = h(x)$

Asym : $f \sqsubseteq g \sqsubseteq f$ $\quad \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $\forall x \in \mathrm{dom}(f).$ $f(x) = g(x)$

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Proof!

MONOTONICITY

A function $f\colon D \to E$ between posets is monotone if

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$$\textsc{Mon}\ \frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)}$$

Least Fixed Points

Least elements and pre-fixed points

LEAST ELEMENT

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FIXED POINT

A fixed point for a function $f\colon D \to D$ is an element $d \in D$ satisfying $f(d) = d$.

(LEAST) PRE-FIXED POINT

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It is thus (uniquely) specified by the two properties:

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \text{LFP-LEAST}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

PROOFS WITH LEAST FIXED POINTS

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)}$$

The least pre-fixed point is a pre-fixed point.

PROOFS WITH LEAST FIXED POINTS

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \text{LFP-LEAST}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

To prove $\mathrm{fix}(f) \sqsubseteq d$, it is enough to show $f(d) \sqsubseteq d$.

PROOFS WITH LEAST FIXED POINTS

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad\qquad \text{LFP-LEAST}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

Application: least pre-fixed points of monotone functions are (least) fixed points.

$$\text{ASYM}\ \frac{\text{LFP-FIX}\ \dfrac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \dfrac{}{\mathrm{fix}(f) \sqsubseteq f(\mathrm{fix}(f))}}{f(\mathrm{fix}(f)) = \mathrm{fix}(f)}$$

PROOFS WITH LEAST FIXED POINTS

fixed point ⇒ prefixed point

least prefixed point ⇒ fixed point (least)

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad\qquad \text{LFP-LEAST}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

Application: least pre-fixed points of monotone functions are (least) fixed points.

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Least Fixed Points

Least upper bounds

LEAST UPPER BOUND OF A CHAIN

The **least upper bound** of countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$, written $\bigsqcup_{n \geq 0} d_n$, satisfies the two following properties:

$$\text{LUB-BOUND}\ \frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \qquad \text{LUB-LEAST}\ \frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x}$$

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- Other names: supremum, limit...
- Might write simply $\bigsqcup_n d_n$ or even $\bigsqcup d_n$
- Only lubs of chains – but can be generalized
- $\bigsqcup_{i \geq 0} d_i$ need not be one of the d_i – this is the interesting case!

PROPERTIES OF LUBS

Lubs are unique.

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$$\begin{array}{ccccc} e_0 \sqsubseteq & e_1 & \cdots & \sqsubseteq & \bigsqcup_n e_n \\ \rotatebox{90}{$\sqsubseteq$} & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} \\ d_0 & d_1 & \cdots & \sqsubseteq & \bigsqcup_n d_n \end{array}$$

Lubs are monotone: if for all $n \in \mathbb{N}$. $\underrightarrow{d_n} \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\text{Least}\ \dfrac{\text{Trans}\ \dfrac{\dfrac{}{d_n \sqsubseteq e_n} \qquad \dfrac{}{e_n \sqsubseteq \bigsqcup_n e_n}\ \text{Bound}}{\forall n.\ d_n \sqsubseteq \bigsqcup_n e_n}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

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$$\textsc{lub-mon}\ \frac{\forall i.\ d_i \sqsubseteq e_i}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

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For any d, $\bigsqcup_n d = d$.

PROPERTIES OF LUBS

$$d_0 \sqsubseteq d_1 \sqsubseteq \ldots d_N \sqsubseteq d_{N+1} \ldots \sqsubseteq \bigsqcup_n d_n$$

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For any chain and $N \in \mathbb{N}$, $\bigsqcup_n d_n = \bigsqcup_n d_{n+N}$.

$$\dfrac{\dfrac{\dfrac{}{\forall n.\, d_n \sqsubseteq d_{n+N}}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n d_{n+N}} \qquad \dfrac{\dfrac{}{\forall n.\, d_{n+N} \sqsubseteq \bigsqcup_n d_n}\,\text{Bound}}{\bigsqcup_n d_{n+N} \sqsubseteq \bigsqcup_n d_n}\,\text{least}}{\bigsqcup_n d_n = \bigsqcup_n d_{n+N}}\,\text{Asym}$$

PROPERTIES OF LUBS

Lubs are unique (if they exist).

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$ (if they exist).

For any d, $\bigsqcup_n d = d$ (and in particular it exists).

For any chain and $N \in \mathbb{N}$, $\bigsqcup_n d_n = \bigsqcup_n d_{n+N}$ (if any of the two exists).

DIAGONALISATION

Assume $d_{m,n} \in D$ $(m, n \geq 0)$ satisfies

$$m \leq m' \wedge n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}.$$

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Then, assuming they exist, the lubs form two chains

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots$$

and

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Moreover, again assuming the lubs of these chains exist,

$$\bigsqcup_{m\geq 0} \left(\bigsqcup_{n\geq 0} d_{m,n} \right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0} \left(\bigsqcup_{m\geq 0} d_{m,n} \right) .$$

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$\bigsqcup_m d_{m,0} \sqsubseteq \bigsqcup_m d_{m,1}$

$$\begin{array}{ccccccccc}
 & \rotatebox{90}{$\sqsubseteq$} & & \vdots & & & & & \\
 & d_{m,0} & & d_{m1} & & d_{k,k} & & & \\
 & \vdots & & \vdots & \rotatebox{45}{$\sqsubseteq$} & & & & \\
 & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & \\
 & d_{1,0} & & d_{1,1} & \cdots & \sqsubseteq d_{1,n} \sqsubseteq & \cdots & & \sqsubseteq \bigsqcup_n d_{1,n} \\
 & & \rotatebox{135}{$\sqsubseteq$} & \rotatebox{90}{$\sqsubseteq$} & & & & & \rotatebox{90}{$\sqsubseteq$} \\
 & \rotatebox{90}{$\sqsubseteq$} & & & & & & & \\
 & d_{0,0} & \sqsubseteq & d_{0,1} & \sqsubseteq \cdots & \sqsubseteq d_{0,n} \sqsubseteq & \cdots & & \sqsubseteq \bigsqcup_n d_{0,n}
\end{array}$$

Least Fixed Points

Complete partial orders and domains

CPOS AND DOMAINS

A **chain complete poset/cpo** is a poset $(D, \sqsubseteq)$ in which all chains have least upper bounds.

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A **domain** is a cpo with a least element $\bot$.

DOMAIN OF PARTIAL FUNCTIONS

Least element: $\bot$ is the totally undefined function.

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Lub of a chain: $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots$ has lub f such that

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \mathbf{dom}(f_n) \text{ for some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

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Beware: the definition of $\bigsqcup_{n \geq 0} f_n$ is unambiguous only if the f_i form a chain!

FINITE CPOS

$$e_i = d_0 \sqsubseteq d_0 \sqsubseteq \dots \sqsubseteq d_1 \sqsubseteq d_1 \dots \sqsubseteq \underset{\uparrow N}{d_n} \sqsubseteq d_n \dots$$

cpo

Finite posets are always ~~domains~~ – why?

$$\bigsqcup e_i = \bigsqcup e_{i+N} = \bigsqcup_i d_n = d_n$$

FINITE CPOS

cpo

Finite posets are always ~~domains~~ – why?

domain

Are they always ~~cpos~~?

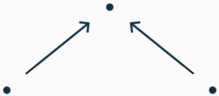
FINITE CPOS

cpo

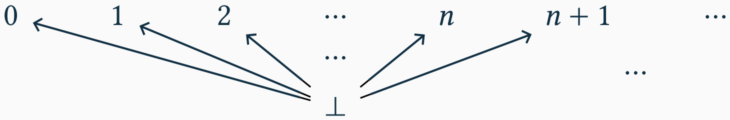
Finite posets are always ~~domains~~ – why?

domain

Are they always ~~cpos~~?



THE FLAT NATURAL NUMBERS $\mathbb{N}_\perp$



VERTICAL NATURAL NUMBERS

$$\begin{tikzcd}
\vdots \\
n+1 \arrow[u] \\
n \arrow[u] \\
1 \arrow[u, dotted] \\
0 \arrow[u]
\end{tikzcd}$$

No! (Why?)

Vertical natural numbers

$$\begin{tikzcd}
\omega \\
n+1 \arrow[u, dotted] \\
n \arrow[u] \\
1 \arrow[u, dotted] \\
0 \arrow[u]
\end{tikzcd}$$

Yes!

VERTICAL NATURAL NUMBERS

$$\begin{tikzcd}
\omega_1 & & \omega_2 \\
& \vdots \arrow[ul] \arrow[ur] & \\
& n+1 \arrow[u] \arrow[uul] \arrow[uur] & \\
& n \arrow[u] & \\
& 1 \arrow[u, dotted] & \\
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\end{tikzcd}$$

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