— *Solution notes* —

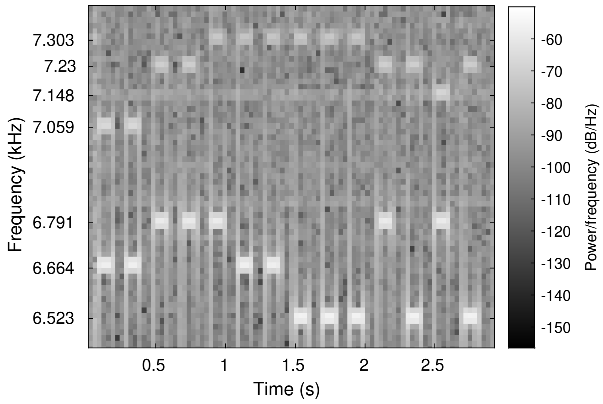
COMPUTER SCIENCE TRIPOS Part IB 75%, Part II 50% – 2020 – Paper 7

11 Digital Signal Processing (mgk25)

This question can only be attempted by Part II 50% candidates.

You are the new CTO of *Missampled Ltd*, a consulting company specializing in fixing digital-signal-recording accidents. These are the first customers seeking your help:

(*a*) A police officer has recorded a conversation between suspects on an analog phone line. The recording $\{x_n\}$ has sampling frequency $f_s = 16$ kHz. But the officer had accidentally activated a "scramble" switch on the recorder, and as a result the recording now sounds high-pitched and is unintelligible.



The manual of the recorder does not explain what the "scramble" switch does. Using a spectrogram (above), you spot at the start of the recording a sequence of 14 tone pairs.

spectral estimation

(*i*) What six computational steps are typically involved in producing such a spectrogram from a sequence of real-valued samples? [6 marks]

Answer:

- Split the input sequence of samples into blocks, typically each with a power-of-two length (e.g., 1024 samples) and 50% overlap between consecutive blocks
- Multiply each block with a window function (Hann, etc.) to reduce leakage and scalloping.
- Calculate the FFT of each block
- Discard the second half of the resulting block (the negative frequencies are redundant for real-valued input), and take the absolute value of each value in the first half of the FFT output
- Take the logarithm of each absolute value in that block (for dB scale)
- Make the resulting blocks columns in the spectrogram raster image

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spectral inversion, Exercise 14

(*ii*) The spectrogram reminds you of DTMF-encoded touch-tone digits, but the frequencies are clearly not the standard ones at 697, 770, 852, 941, 1209, 1336, and 1477 Hz. What appears to have happened to the frequencies in this recording, how can this transformation be explained as a simple time-domain operation on its samples, and how can you then restore it such that the officer can hear the original voices again? [6 marks]

Answer:

- Tones originally at frequency f appear to have been relocated to a new frequency of $f_s/2 - f = 8000\text{ Hz} - f$. In other words, the frequency spectrum appears to have been inverted, or equivalently the negative and positive frequencies have been swapped.

- The scrambling function appears to have multiplied the input signal with an 8000 Hz tone $\cos(2\pi \cdot t \cdot f_s/2) = \cos(\pi n) = -1^n$. That corresponds in the frequency domain to convolving the spectrum with $\frac{1}{2}\delta(f - f_s/2) + \frac{1}{2}\delta(f + f_s/2)$ and relocates the negative frequencies $[-8000, 0]$ Hz upwards by 8000 Hz and the positive frequencies $[0, 8000]$ Hz downwards by -8000 Hz, thereby, as these two frequency intervals mirror each other, inverting the amplitude spectrum.
- Simply negate every second sample by multiplying it with -1, i.e. $y_n = -1^n \cdot x_n$, to unscramble the signal.

[*Note:* The scrambling function could also have multiplied the signal with $-\cos(\pi n)$, but such a negation of the whole signal is inaudible for the human ear (in a mono channel).]

digital filters, z-transform

(*b*) A TV producer discovered that during the recording of a stage production, one of the microphones accidentally had activated the following digital FIR filter (where $\{x_n\}$ is the desired audio signal and $\{y_n\}$ is the available recorded sequence):

$$y_n = 0.4 \times \left(x_n + x_{n-1} + \frac{1}{2}x_{n-2}\right)$$

(*i*) What is the z-transform $H(z)$ of the impulse response of this filter? [2 marks]

Answer: This is a finite-impulse-response-filter of order two, with impulse response $0.4, 0.4, 0.2$, therefore

$$\frac{Y(z)}{X(z)} = H(z) = 0.4 \times \left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)$$

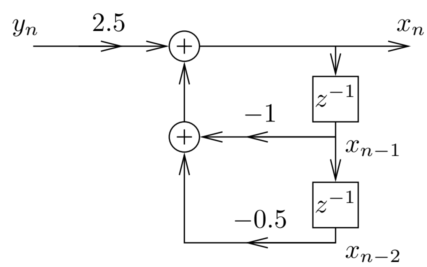
(*ii*) What is the z-transform of the impulse response of a filter G that, if applied to the recorded samples $\{y_n\}$, converts them back into the original waveform $\{x_n\}$? [2 marks]

Answer: Simply take the inverse of $H(z)$:

$$G(z) = \frac{X(z)}{Y(z)} = [H(z)]^{-1} = \frac{2.5}{1 + z^{-1} + \frac{1}{2}z^{-2}}$$

(*iii*) Draw a Direct Form I representation of G. [4 marks]

Answer:



COMPUTER SCIENCE TRIPOS Part II 50% – 2019 – Paper 7

11 Digital Signal Processing (mgk25)

This question can only be attempted by Part II 50% candidates.

(*a*) Name one advantage and one disadvantage of Finite-Impulse-Response (FIR) filters over Infinite-Impulse-Response (IIR) filters. [2 marks]

Answer: For example:

- Finite-impulse-response filters can easily be constructed with a symmetric impulse response, which results in a linear-phase filter, that is all frequencies experience the same delay.
- Infinite-impulse-response filters can often match a given filter specification at a much lower filter order, and therefore require fewer multiplications per sample.

(*b*) For each of the following discrete systems $\{y_n\} = T\{x_n\}$, either show that T is equivalent to a convolution operation, by providing an impulse response $\{h_n\}$ such that

$$y_n = \sum_{i=-\infty}^{\infty} h_i x_{n-i}$$

or explain why the system cannot be described through convolution.

(*i*) $y_n = \frac{1}{2}(x_{2n} + x_{2n+1})$ [2 marks]

Answer: A discrete system can be expressed as convolution if and only if it is linear and time invariant (LTI). This one is not time invariant: delaying the input by two samples will delay the output by just one sample.

(*ii*) $y_n = x_{n+4}$ [2 marks]

Answer: Non-causal LTI:

$$h_n = \begin{cases} 1, & n = -4 \\ 0, & n \neq -4 \end{cases}$$

(*iii*) $y_n = \frac{3}{2}x_{n-1} - \frac{1}{2}y_{n-2}$ [4 marks]

Answer: Causal LTI with impulse response $0, 1.5, 0, -0.75, 0, 0.375, 0, \ldots$, therefore

$$h_n = \begin{cases} 0, & n < 1 \lor n \text{ even} \\ \frac{3}{2}(-\frac{1}{2})^{\frac{n-1}{2}}, & n > 0 \land n \text{ odd} \end{cases}$$

(*c*) What is the z-transform of the impulse response of the system in Part (*b*)(*iii*)? [4 marks]

Answer:

$$\begin{aligned} y_n &= \tfrac{3}{2}x_{n-1} - \tfrac{1}{2}y_{n-2} \\ Y(z) &= \tfrac{3}{2}X(z) \cdot z^{-1} - \tfrac{1}{2}Y(z) \cdot z^{-2} \\ Y(z)(1 + \tfrac{1}{2}z^{-2}) &= \tfrac{3}{2}X(z) \cdot z^{-1} \\ H(z) = \frac{Y(z)}{X(z)} &= \frac{\tfrac{3}{2}z^{-1}}{1 + \tfrac{1}{2}z^{-2}} \end{aligned}$$

(*d*) Consider a digital filter where the z-transform of the impulse response is

$$H(z) = \frac{z^2 - 1}{z^2 + \frac{49}{64}}.$$

(*i*) Draw the location of poles and zeros of $H(z)$ in the z-plane. [2 marks]

Answer:

$$H(z) = \frac{z^2 - 1}{z^2 + \frac{49}{64}} = \frac{(z+1)(z-1)}{(z + \frac{7}{8}\mathrm{j})(z - \frac{7}{8}\mathrm{j})}$$

Therefore we have two zeros $H(\pm 1) = 0$ and two poles at $z = \pm\frac{7}{8}\mathrm{j}$, which is just inside the points $z = \pm\mathrm{j}$ on the unit circle.

(*ii*) What is this kind of filter called? [1 mark]

Answer: This is a *peak filter* or *peaking filter*. (It is intended to attenuate all but a very narrow band of frequencies, making it the opposite of a *notch filter*.)

(*iii*) A test signal $x(t) = \cos(2\pi f t)$ is sampled into $x_n = x(n/f_\mathrm{s})$, with rate $f_\mathrm{s} = 4$ kHz, and then passed through this filter. For what values of f will the root-mean-square level at the filter output be maximal? [3 marks]

Answer: The DTFT of a filter response at frequency f is $H(\mathrm{e}^{2\pi \mathrm{j} f/f_\mathrm{s}})$, and will be largest closest to the poles, which are here at $z = \pm\mathrm{j}$ and therefore $f = \pm\frac{1}{4}f_\mathrm{s} + mf_\mathrm{s} = (\pm 1 + 4m)$ kHz for all $m \in \mathbb{Z}$, i.e. $\ldots, -5, -3, -1, 1, 3, 5, \ldots$ kHz.

COMPUTER SCIENCE TRIPOS Part II – 2018 – Paper 8

5 Digital Signal Processing (MGK)

spectral estimation

Your friend Sam works on a physics experiment. This generates a voltage waveform $v(t)$ that is the sum of several signals:

- a sine wave $s(t) = A \cdot \sin(2\pi t f + \phi)$, the frequency f and phase ϕ of which are not known in advance, but f will be within 9.6 kHz $< f <$ 12.0 kHz;
- several other sine waves with frequencies below 8 kHz that Sam needs to ignore in her measurements;
- low levels of noise at all frequencies.

Sam needs to estimate the amplitude A of $s(t)$. She uses a USB audio recorder with a built-in 16 kHz anti-aliasing low-pass filter to digitize $v(t)$ at sampling frequency f_s = 48 kHz, recording s = 100 000 consecutive samples, resulting in real-valued samples $v_0, \ldots, v_{s-1}$. She implemented this algorithm to estimate A:

input $v_0, \ldots, v_{s-1}$

$b := 1000; \quad c := \left\lfloor \frac{s}{b} \right\rfloor$

$w_{k,l} := v_{kb+l}$ for all $0 \le k < c$, $0 \le l < b$

$$x_{k,n} := \sum_{m=0}^{b-1} w_{k,m} \cdot \mathrm{e}^{-2\pi \mathrm{j} \frac{nm}{b}} \qquad \text{for all } 0 \le k < c,\ 0 \le n < b$$

$$y_n := \left| \frac{1}{c} \cdot \sum_{k=0}^{c-1} x_{k,n} \right| \qquad \text{for all } 0 \le n < b$$

$z := \max\{y_{n_1}, \ldots, y_{n_2}\}$ with $n_1 = 200$, $n_2 = 220$

output z

decibel, root-mean square voltage

(*a*) Sam hopes that $A \approx z \cdot \alpha$ for some calibration constant α. She tries to determine α by connecting the USB audio recorder's input to a calibrated laboratory sine-wave generator set to output an amplitude of "60.0 dBμV". What amplitude A in volts will this test signal $A \cdot \sin(\ldots)$ have? [3 marks]

Answer: Since 0 dBμV means for a sine wave (AC signal) a root-mean square (rms) voltage of 1 μV, and 20 dB represents a 10-fold increase in voltage, 60 dBμV is a rms voltage of 1000 μV = 1 mV = 0.001 V. However, A is a peak voltage, and therefore a factor $\sqrt{2}$ higher than the rms voltage of a sine wave. Therefore $A = \sqrt{2}$ mV ≈ 0.001414 V.

DFT, scalloping, window, periodic averaging

(*b*) When Sam varies the test-signal frequency f in the range 9.6–12.0 kHz, she is disappointed that the output z varies greatly: for some f it even drops to zero!

Describe what Sam's algorithm tries to do, identify and explain *three* problems in it, and change *three* lines to make z more proportional to A across the expected range of f, and close to zero outside that range. [15 marks]

Answer: Sam's algorithm attempts to estimate an amplitude spectrogram of the input signal by splitting it into $c = 100$ blocks of $b = 1000$ samples each, then calculates the DFT of each block, averages the 100 results to reduce the impact of noise, and takes the absolute value to extract amplitudes. It then looks at the peak amplitude in a frequency range of interest to obtain a number proportional to A, ignoring sine waves and noise outside that range.

The three problems with this approach are:

(*i*) **Problem 1: use of a rectangular window function.** The DFT (line 4) calculates the spectrum of a periodic discrete sequence when fed with the samples of one period. If the period length f_s/f of the input signal does not match the length b of the DFT block, this results in effects known as *leakage* (signal energy spills to neighbouring frequency bins) and *scalloping* (the peak amplitude observed drops). Both are a consequence of the finite block size. Limiting the input to 1000 samples is equivalent to multiplication in the time domain with a 1000-sample wide rectangular pulse, which corresponds in the frequency domain to convolution with its Fourier transform: a sinc pulse.

Solution: apply a (better) window function. This replaces the abrupt end of the rectangular window with a gentler transition to zero (e.g., Hamming window), resulting in the frequency domain in a wider peak (less easy to miss when sampling the frequency domain) and less leakage. Change e.g.:

$$3: \quad w_{k,l} := v_{kb+l} \cdot \underline{\left[0.54 - 0.46 \times \cos\left(2\pi \tfrac{l}{b-1}\right)\right]} \qquad \text{for all } 0 \le k < c,\ 0 \le l < b$$

(*ii*) **Problem 2: averaging before taking the absolute value.** Since the DFT is a linear function, the averaging step in line 5 is equivalent to averaging before the DFT, which will attenuate all frequencies where the block spacing is not a multiple of the signal period, as sine waves of different phase (or equivalently: complex numbers of different angle) can cancel each other out.

Solution: take the absolute value before averaging This way, phase-related information is removed.

$$5: \quad y_n := \frac{1}{c} \cdot \sum_{k=0}^{c-1} \underline{|}x_{k,n}\underline{|} \qquad \text{for all } 0 \le n < b$$

(*iii*) **Problem 3: Incorrect range of frequency bins considered.** At a sampling frequency of f_s and a block length of b samples, the frequency spacing of the DFT output bins will be $f_s/b = 48\text{ kHz}/1000 = 48\text{ Hz}$. As a result, the frequencies 9.6 to 12.0 kHz correspond to bin numbers $n_1 = \frac{9600\text{ Hz}}{48\text{ Hz}} = 200$ to $\frac{12000\text{ Hz}}{48\text{ Hz}} = 250$. Therefore, change

$$6: \quad z := \max\{y_{n_1}, \ldots, y_{n_2}\} \qquad \text{with } n_1 = 200,\ n_2 = \underline{250}$$

[Bin ranges for slightly wider frequency ranges are equally acceptable, as long as they don't get near 8 kHz.]

FFT

(*c*) Suggest a small adjustment to b to accommodate a faster algorithm for one of the above steps. [2 marks]

Answer: Use $b := 1024$ and replace the DFT in line 4 with the real-valued FFT (which is fastest and easiest to implement with a power-of-two block length).

COMPUTER SCIENCE TRIPOS Part II – 2018 – Paper 9

6 Digital Signal Processing (MGK)

interpolation, sampling, aliasing

(*a*) When converting a digital audio signal from one sampling frequency to another, it is common practice to use a low-pass filter. What is the purpose of this low-pass filter, and what cut-off frequency should it have if the change of sampling frequency is

(*i*) from 12 kHz to 48 kHz;

(*ii*) from 48 kHz to 12 kHz. [4 marks]

Answer: In both cases the filter avoids aliasing and should have a cut-off frequency not higher than 6 kHz, half the lower of the two sampling frequencies.

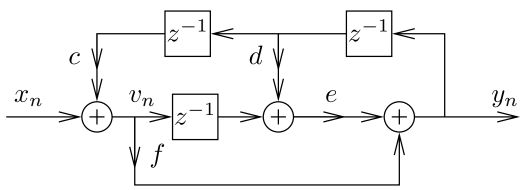
In more detail (can also be explained in aliasing diagrams):

We can resample a 12 kHz sequence at 48 kHz by inserting three zero samples after each 12 kHz sample, to quadruple the sampling frequency. However, at 12 kHz sampling frequency, the signal's spectrum (±6 kHz centered at 0 kHz) will have aliases centered at ±12 and ±24 kHz, and these will then appear as audible artifacts in the 48 kHz signal (which can represent frequencies up to ±24 kHz). Therefore, apply a 6 kHz low-pass filter to remove these aliases from the 48 kHz sequence.

Likewise, before reducing the sampling frequency from 48 kHz to 12 kHz by dropping all but every fourth sample, first apply a 6 kHz low-pass filter to remove frequencies that would otherwise alias at lower frequencies afterwards.

IIR filters

(*b*) You are working on the firmware of a quadcopter drone. Your colleague, through trial and error, found that the following recursive filter nicely avoids unwanted oscillations in the control system:



(*i*) What are the first three samples h_0, h_1, h_2 of the impulse response of this filter? [*Note:* All delay elements have been initialized to zero.] [6 marks]

Answer: Feed a single 1 through the filter diagram. One way to find h_k is look for all paths through the filter from input x_0 to output y_k that go through exactly k delay elements, multiply for each path all the factors encountered, then sum up all these products:

$$h_0 = f$$
$$h_1 = e + fde$$
$$h_2 = fcf + ede + fdede = cf^2 + de^2 + d^2e^2f$$

(One mark for each product path.)

Other, more labour intensive approaches include polynomial division of $H(z)$ or expanding the equations for y_n, both of which appear in part $(b)(ii)$.

(ii) What is the z-transform $H(z) = Y(z)/X(z)$ of the impulse response of this digital filter? [5 marks]

Answer:

$$
\begin{aligned}
y_n &= e \cdot v_{n-1} + de \cdot y_{n-1} + f \cdot v_n \\
v_n &= x_n + c \cdot y_{n-2} \\
Y(z) &= V(z) \cdot (f + ez^{-1}) + Y(z) \cdot dez^{-1} \\
V(z) &= X(z) + Y(z) \cdot cz^{-2} \\
Y(z) &= (X(z) + Y(z) \cdot cz^{-2}) \cdot (f + ez^{-1}) + Y(z) \cdot dez^{-1} \\
Y(z) &= X(z) \cdot (f + ez^{-1}) + Y(z) \cdot (cfz^{-2} + cez^{-3}) + Y(z) \cdot dez^{-1} \\
Y(z) &\cdot (1 - dez^{-1} - cfz^{-2} - cez^{-3}) = X(z) \cdot (f + ez^{-1}) \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{f + ez^{-1}}{1 - dez^{-1} - cfz^{-2} - cez^{-3}} = \frac{fz^3 + ez^2}{z^3 - dez^2 - cfz - ce}
\end{aligned}
$$

(iii) The software development kit of your flight controller can only implement digital filters of the form

$$y_n = \sum_{k=0}^{3} b_k \cdot x_{n-k} - \sum_{l=1}^{3} a_l \cdot y_{n-l}.$$

What coefficient values a_l and b_k $(0 \le k \le 3,\ 1 \le l \le 3)$ will implement the same impulse response as your colleague's filter? [5 marks]

Answer: The z-transform of the filter implemented by the library is

$$Y(z) = X(z) \cdot \sum_{k=0}^{3} b_k \cdot z^{-k} - \sum_{l=1}^{3} a_l \cdot z^{-l}$$

By matching this with equation

$$Y(z) = X(z) \cdot (f + ez^{-1}) + Y(z) \cdot (dez^{-1} + cfz^{-2} + cez^{-3})$$

from part $(b)(ii)$ we get

$$b_0 = f, b_1 = e, b_2 = b_3 = 0, a_1 = -de, a_2 = -cf, a_3 = -ce$$

COMPUTER SCIENCE TRIPOS Part II – 2017 – Paper 8

6 Digital Signal Processing (MGK)

sampling, IQ downconversion

A zoologist wants to record the echo-location sounds emitted by a bat. The species of bat to be recorded emits only sounds in the frequency range 40 kHz to 80 kHz and the microphone used includes an analog filter with that passband.

(*a*) Explain for each of the following sampling techniques how it can be used to convert a continuous ultrasonic microphone signal $x(t)$ into a discrete-time sequence $\{x_n\}$ and state for each technique the lowest sampling frequency f_s that enables the exact reconstruction of $x(t)$ from $\{x_n\}$:

(*i*) Passband sampling [3 marks]

Answer: Passband sampling unambiguously captures all frequencies in the range $k \cdot f_\mathrm{s}/2$ to $(k+1) \cdot f_\mathrm{s}/2$. With $k = 1$, the bat's signal can therefore be reconstructed from $x_n = x(n/f_\mathrm{s})$ without ambiguity, at $\underline{f_\mathrm{s} = 80 \text{ kHz}}$.

(*ii*) IQ downconversion [5 marks]

Answer: IQ downconversion unambiguously captures all frequencies in the range $f_\mathrm{c} - f_s/2$ to $f_\mathrm{c} + f_s/2$. It first shifts the centre frequency ($f_\mathrm{c} = 60$ kHz) to 0 by multiplying the incoming real-valued waveform with a complex phasor $\mathrm{e}^{-2\pi \mathrm{j} f_\mathrm{c} t}$, then applies to the result a low-pass filter with cut-off frequency $\pm f_s/2$ [impulse response: $f_\mathrm{s} \cdot \mathrm{sinc}(t f_\mathrm{s})$], to eliminate the negative frequencies, and finally samples its output at half the resulting single-sided bandwidth, that is at $\underline{f_\mathrm{s} = 40 \text{ kHz}}$. This way, the bat's signal $x(t)$ can be reconstructed from

$$x_n = f_\mathrm{s} \cdot \int x(s) \cdot \mathrm{e}^{-2\pi \mathrm{j} f_\mathrm{c} s} \cdot \mathrm{sinc}(n - s f_\mathrm{s}) \cdot \mathrm{d}s.$$

(*b*) Using a 32-bit floating-point data type, how many bytes per second are required to store each of the two resulting discrete sequences from part (*a*)? [2 marks]

Answer:

(*i*) 80 real-valued kilosamples per second at 4 bytes each: 320 kB/s

(*ii*) 40 complex-valued kilosamples per second at 8 bytes each: 320 kB/s

[*Note:* Complex valued samples require 32 bits for the real and 32 bits for the imaginary part.]

(*c*) Compare your answers to part (*b*) with the memory required for storing $x(t)$ sampled at the Nyquist rate of 160 kHz and explain the difference in terms of redundancy in the acquired spectrum. [2 marks]

Answer: Sampling at the Nyquist rate of 160 kHz requires 4 B × 160 kHz = 640 kB/s. This is twice the amount of information compared to the other techniques, because it can represent twice the bandwidth, from 0 to 80 kHz, including the unused (redundant) band 0 to 40 kHz.

DTFT

(*d*) If the sampling techniques from part (*a*) are applied to a test signal $x(t) = \cos(2\pi f t)$ with $f = 45$ kHz, what does the discrete-time Fourier transform of the resulting discrete sequence $\{x_n\}$ look like (over the normalized frequency range $-\pi < \dot{\omega} \leq \pi$) for each technique? [4 marks]

Answer: The 45 kHz test tone shows up in the Fourier spectrum as two Dirac deltas, at ± 45 kHz.

(*i*) After sampling at 80 kHz, there will be aliases at $(\pm 45 + k \times 80)$ kHz ($k \in \mathbb{Z}$), including the two at ± 35 kHz, which in normalized frequency corresponds to Dirac deltas at $\dot{w} = \pm\pi\frac{35\ \text{kHz}}{f_\text{s}/2} = \underline{\pm\frac{7}{8}\pi}$ (repeating every 2π).

(*ii*) After shifting by -60 kHz and filtering out the $(-45 - 60)$ kHz component, we end up with a single Dirac delta at $(45 - 60)$ kHz $= -15$ kHz, which in normalized frequency corresponds to a Dirac delta at $\dot{w} = \pi\frac{-15\ \text{kHz}}{f_\text{s}/2} = \underline{-\frac{3}{4}\pi}$ (repeating every 2π).

[Graphical answers are also acceptable, as are answers in kHz *if* they fall within the frequency interval $-f_\text{s}/2$ to $f_\text{s}/2$.]

(*e*) For both sampling techniques described in part (*a*), briefly outline the steps needed to reconstruct the original continuous waveform from the discrete sequence. [4 marks]

Answer:

Start in each case from $\hat{x}(t) = f_\text{s}^{-1} \cdot \sum_n x_n \cdot \delta(t - n/f_\text{s})$.

(*i*) Eliminate aliases from $\hat{x}(t)$ by applying a band-pass filter for the passband 40 to 80 kHz. This can be achieved by convolving with a modulated sinc function, using the impulse response of an $f_\text{s}/2 = 40$ kHz low-pass filter multiplied with an $f_\text{c} = 60$ kHz cosine.

Or as a formula (not essential for full marks):

$$x(t) = \sum_n x_n \cdot \text{sinc}(tf_\text{s} - n) \cdot \cos\big(2\pi f_\text{c}(t - n/f_\text{s})\big)$$

(*ii*) First apply a $f_\text{s}/2 = 20$ kHz low-pass filter (sinc interpolation), then multiply the result with a complex phasor rotating at $f_\text{c} = 60$ kHz to upconvert the signal, and finally recreate the missing negative frequencies by outputting only the real component (i.e., ignoring the imaginary component).

Or as a formula (not essential for full marks):

$$x(t) = 2\Re\Big[\text{e}^{2\pi \text{j} f_\text{c} t} \cdot \sum_n x_n \cdot \text{sinc}(tf_\text{s} - n)\Big]$$

COMPUTER SCIENCE TRIPOS Part II – 2017 – Paper 9

6 Digital Signal Processing (MGK)

IIR filters, z-transform

(*a*) A discrete-time LTI filter can be described through the locations of zeros and poles in the z-transform $H(z)$ of its impulse response. Consider an IIR filter of order 2 with $H(c_1) = H(c_2) = 0$ and $|H(z)| \to \infty$ for $z \to d_1$ and $z \to d_2$.

(*i*) What is the z-transform $H(z)$ of its impulse response? [2 marks]

Answer:

$$H(z) = \text{const.} \cdot \frac{(z - c_1) \cdot (z - c_2)}{(z - d_1) \cdot (z - d_2)}$$

(*ii*) What additional parameter (beyond c_1, c_2, d_1, d_2) is required to fully describe the impulse response of this filter? [1 mark]

Answer: A constant factor that scales the overall gain of the filter.

(*iii*) What is the magnitude of the discrete-time Fourier transform (DTFT) of the impulse response of this filter? [2 marks]

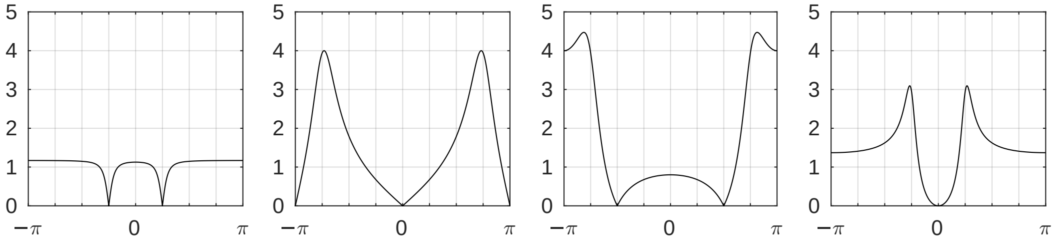
Answer:

$$|H(\mathrm{e}^{\mathrm{j}\dot{\omega}})| = |\text{const.}| \cdot \frac{|\mathrm{e}^{\mathrm{j}\dot{\omega}} - c_1| \cdot |\mathrm{e}^{\mathrm{j}\dot{\omega}} - c_2|}{|\mathrm{e}^{\mathrm{j}\dot{\omega}} - d_1| \cdot |\mathrm{e}^{\mathrm{j}\dot{\omega}} - d_2|}.$$

(*iv*) Under what condition on c_1, c_2, d_1, and d_2 is the impulse-response of this filter real-valued? [2 marks]

Answer: ($c_1 = c_2^*$ or $c_1, c_2 \in \mathbb{R}$) and ($d_1 = d_2^*$ or $d_1, d_2 \in \mathbb{R}$)
[*Note:* * denotes the complex conjugate]

(*b*) The following plots show the magnitude of the DTFT of the real-valued impulse response of four different IIR filters:



(*i*) (*ii*) (*iii*) (*iv*)

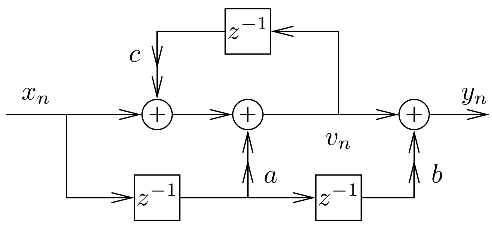
The z-transform of each impulse response has two zeros and two poles. Each zero or pole is at one of these 12 possible locations: $\mathrm{e}^{\pi \mathrm{j} k/4}$ with $k \in \{0, \ldots, 7\}$ or $0.6 \pm 0.6\mathrm{j}$ or $-0.5 \pm 0.5\mathrm{j}$.

For each filter, state the location of both zeros and both poles. Explain the reasoning behind your choice. [8 marks]

Answer: For a stable filter, the poles must be located inside the unit circle (not on it). The real-valued impulse response implies that zeros and poles of the corresponding z-transform are placed symmetrically across the real axis, therefore each zero and pole is on the real axis or is part of a conjugated pair (see part $(a)(iv)$). This means, each filter can either have the two poles at $0.6 \pm 0.6\mathrm{j}$ or the two at $-0.5 \pm 0.5\mathrm{j}$, depending on whether the DTFT magnitude shows a peak near $\pm\pi\mathrm{j}/4$ or $\pm 3\pi\mathrm{j}/4$. If the DTFT magnitude is zero at $\dot\omega$, this implies that the corresponding z-transform has a zero on the unit circle at $\mathrm{e}^{\mathrm{j}\dot\omega}$. A pole/zero pair close to each other contributes approximately unit gain for all frequencies except where $\mathrm{e}^{\mathrm{j}\dot\omega}$ is relatively much closer to one of them (see part $(a)(iii)$). These constraints leave only the following solution:

(i)	zeros at $\mathrm{e}^{\pm\pi\mathrm{j}/4}$,	poles at $0.6 \pm 0.6\mathrm{j}$
(ii)	zeros at ± 1,	poles at $-0.5 \pm 0.5\mathrm{j}$
(iii)	zeros at $\mathrm{e}^{\pm\pi\mathrm{j}/2} = \pm\mathrm{j}$,	poles at $-0.5 \pm 0.5\mathrm{j}$
(iv)	two zeros at 1,	poles at $0.6 \pm 0.6\mathrm{j}$

(c) What is the z-transform $H(z)$ of the impulse response of the following filter? [5 marks]



Answer: First describe the filter through finite-difference equations:

$$y_n = v_n + b x_{n-2}$$
$$v_n = x_n + c v_{n-1} + a x_{n-1}$$

Then convert these into equations for the z-transforms of the sequences $\{v_n\}$, $\{x_n\}$, $\{y_n\}$:

$$Y(z) = V(z) + b z^{-2} X(z)$$
$$V(z) = X(z) + c z^{-1} V(z) + a z^{-1} X(z)$$

Finally, solve for $H(z) = \dfrac{Y(z)}{X(z)}$:

$$\begin{aligned}
V(z) - c z^{-1} V(z) &= X(z) + a z^{-1} X(z) \\
V(z)(1 - c z^{-1}) &= X(z)(1 + a z^{-1}) \\
V(z) &= X(z) \frac{1 + a z^{-1}}{1 - c z^{-1}} \\
Y(z) &= X(z) \frac{1 + a z^{-1}}{1 - c z^{-1}} + b z^{-2} X(z) \\
H(z) = \frac{Y(z)}{X(z)} &= \frac{1 + a z^{-1}}{1 - c z^{-1}} + b z^{-2} = \frac{1 + a z^{-1} + (1 - c z^{-1}) b z^{-2}}{1 - c z^{-1}} \\
&= \frac{1 + a z^{-1} + b z^{-2} - b c z^{-3}}{1 - c z^{-1}}
\end{aligned}$$

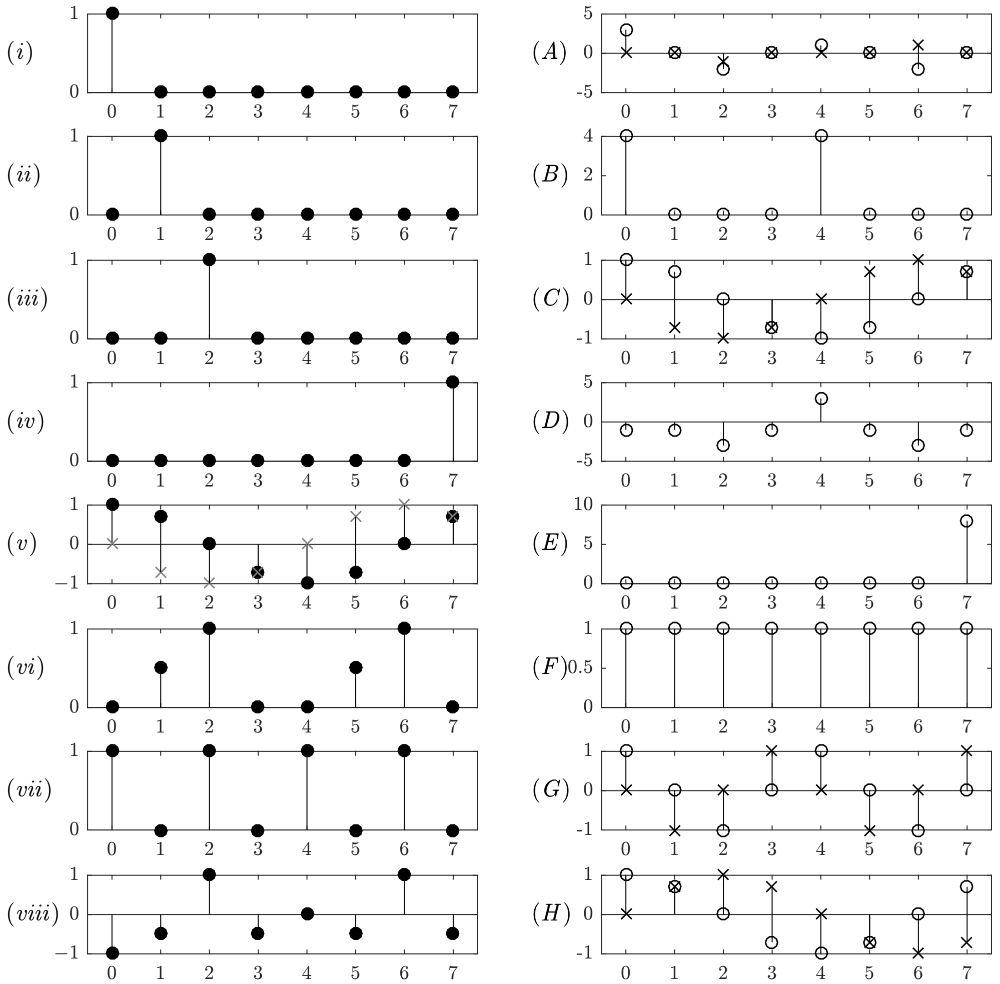
COMPUTER SCIENCE TRIPOS Part II – 2016 – Paper 8

6 Digital Signal Processing (MGK)

DFT properties

(*a*) Figures (*i*)–(*viii*) show eight different input vectors $x \in \mathbb{C}^8$. For each, identify one of figures (*A*)–(*H*) that shows the DFT output $X \in \mathbb{C}^8$ with $X_k = \sum_{n=0}^{7} x_n \cdot \mathrm{e}^{-2\pi \mathrm{j} kn/8}$.

Briefly explain each choice. Real components are shown as circles. For non-real vectors, the imaginary components are shown in addition as crosses. [8 marks]



Answer:

(*i*) F – pulse at 0 ⇒ flat spectrum

(*ii*) C – delaying (*i*) by one sample ⇒ multiply F with a phasor at negative block frequency

(*iii*) G – (*i*) delayed by two samples ⇒ multiply F with a phasor at $-2 \times$ block frequency

(*iv*) H – (*ii*) with negative time ⇒ complex conjugate of C

(*v*) E – complex phasor with negative block frequency ⇒ pulse at $-1 = 7$

(*vi*) A – periodic in time domain at twice the block frequency ⇒ only multiples of twice the block frequency are non-zero

(*vii*) B – DC + half the sampling frequency ⇒ peaks at 0 and 8/2=4 (*viii*) D – symmetric in time domain ($x_n = x_{8-n}$) $\Rightarrow$ real-valued in frequency domain

(*b*) Are these statements true or false? Explain your answers. [3 marks each]

z-transform

(*i*) The system $y_n = x_n + y_{n-1}$ has an impulse response with z-transform $\frac{1}{1+z}$.

Answer: False. The z-transform of $y_n = x_n + y_{n-1}$ is $Y(z) = X(z) + z^{-1}Y(z)$, therefore $Y(z) - z^{-1}Y(z) = Y(z)(1 - z^{-1}) = X(z)$ and therefore the z-transform of its impulse response is $Y(z)/X(Z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$.

IQ sampling or pass-band sampling

(*ii*) A continuous signal can *only* be reconstructed after sampling if the sampling frequency is larger than twice the highest frequency in the signal.

Answer: False. The stated requirement is sufficient, but not necessary. For example, after IQ sampling, any signal can be reconstructed if the sampling frequency was higher than its bandwidth, which is the difference between its highest and lowest frequency.

convolution, power spectrum

(*iii*) Convolution of a signal with a triangular window function causes its power spectrum to be multiplied with a sinc^3 function.

Answer: False. A triangular window can be created by convolving two equal rectangular windows (which have a sinc-shaped Fourier transform), therefore the Fourier transform of a triangular window is a sinc^2 function. Convolution with a triangular window corresponds to multiplying the Fourier transform with this sinc^2 function, and this is in the power spectrum ($|\cdot|^2$ of the Fourier transform) equivalent to multiplications with a sinc^4 function.

z-transform, impulse response, convolution

(*iv*) To convert the z-transform $H(z)$ of the impulse response of any LTI filter into the z-transform of its step response, divide $H(z)$ by $1 - z^{-1}$.

Answer: True. The unit impulse sequence $\ldots, 0, 0, 0, 1, 0, 0, 0, \ldots$ has the z-transform $z^0 = 1$. It can be converted into the unit-step sequence $\ldots, 0, 0, 0, 1, 1, 1, 1, \ldots$ by feeding it through the accumulator system $y_n = x_n + y_{n-1}$, which has a z-transform of $\frac{1}{1-z^{-1}}$ (see part $(b)(i)$). Since convolution is associative, we can apply this accumulator system also after any LTI filter, to its impulse response, and obtain this way its response to the unit-step sequence. In the z-domain, this is equivalent to multiplication with $\frac{1}{1-z^{-1}}$.

COMPUTER SCIENCE TRIPOS Part II – 2016 – Paper 9

6 Digital Signal Processing (MGK)

Fourier transform

(*a*) Let δ be the Dirac delta function and $T, b > 0$ be time intervals. Give the Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\, \mathrm{e}^{-2\pi \mathrm{j} f t}\, \mathrm{d}t$$

of the following two functions:

Dirac comb

(*i*) $x(t) = c_T(t)$, where $c_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ [3 marks]

Answer: $X(f) = T^{-1} \cdot \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$

[Two marks for recalling the formula as given in the lecture on Dirac combs, and one mark for adjusting the scale factor T correctly.]

rect and sinc, properties of the Fourier transform

(*ii*) $x(t) = r_b(t)$, where $r_b(t) = \begin{cases} 1 & \text{if } |t| < b \\ \frac{1}{2} & \text{if } |t| = b \\ 0 & \text{otherwise} \end{cases}$ [5 marks]

Answer: $X(f) = 2b\,\mathrm{sinc}(2bf)$.

One possible way to find the answer: Recall that if

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

then $\mathcal{F}\{\mathrm{rect}(t)\} = \mathrm{sinc}(f) = \frac{\sin \pi f}{\pi f}$.

Also recall the time-scaling property of the Fourier transform, namely that if $X(f) = \int_{-\infty}^{\infty} x(t)\, \mathrm{e}^{-2\pi \mathrm{j} f t}\, \mathrm{d}t$, then $\int_{-\infty}^{\infty} x(at)\, \mathrm{e}^{-2\pi \mathrm{j} f t}\, \mathrm{d}t = X(f/a)/|a|$.

Then, observing that $x(t) = \mathrm{rect}(t/2b)$, substitute $a := (2b)^{-1}$ to obtain

$$X(f) = 2b\,\mathrm{sinc}(2bf) = 2b\frac{\sin 2\pi bf}{2\pi bf} = \frac{\sin 2\pi bf}{\pi f}.$$

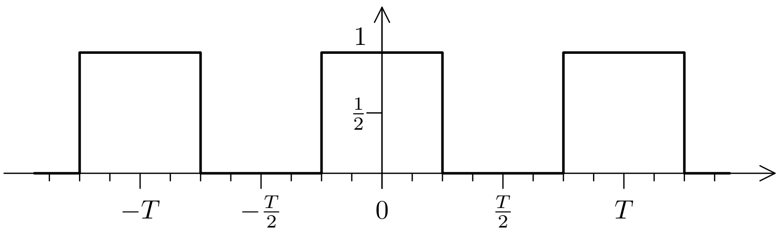
[Two marks each for recalling the required formulæ given in the lectures on the rect and sinc functions and the properties of the Fourier transform, and one mark for substituting the scale factor b correctly.]

Alternatively, solve the integral directly:

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} r_b(t)\, \mathrm{e}^{-2\pi \mathrm{j} f t}\, \mathrm{d}t = \int_{-b}^{b} \mathrm{e}^{-2\pi \mathrm{j} f t}\, \mathrm{d}t = \frac{1}{-2\pi \mathrm{j} f}\left[\mathrm{e}^{-2\pi \mathrm{j} f t}\right]_{-b}^{b} \\ &= \frac{1}{-2\pi \mathrm{j} f}\left(\mathrm{e}^{-2\pi \mathrm{j} f b} - \mathrm{e}^{2\pi \mathrm{j} f b}\right) = \frac{1}{\pi f}\frac{1}{2\mathrm{j}}\left(\mathrm{e}^{2\pi \mathrm{j} f b} - \mathrm{e}^{-2\pi \mathrm{j} f b}\right) \\ &= \frac{1}{\pi f}\sin 2\pi f b \end{aligned}$$

convolution theorem, rect and sinc

(b) Consider this periodic, binary, square-wave clock signal $p(t)$, with period T, duty cycle 0.5 and maximum amplitude 1:



Show that its Fourier transform is

$$P(f) = \frac{1}{2}\delta(f) + \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{2k+1}{T}\right) \cdot \frac{(-1)^k}{k+\frac{1}{2}}.$$

Hint: Use the answers from part (a). [8 marks]

Answer: Construct the square-wave signal $p(t)$ by convolving a Dirac comb of period T with a rectangular window of width $T/2$. In the notation of part (a):

$$p = c_T * r_{T/4}$$

Then apply the convolution theorem and the results of part (a):

$$P(f) = C_T(f) \cdot R_{T/4}(f) = T^{-1} \cdot \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \cdot \frac{T}{2} \operatorname{sinc} \frac{Tf}{2}$$

$$= \frac{1}{2} \cdot \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \cdot \operatorname{sinc} \frac{Tf}{2}$$

Since $\delta(f - n/T) \neq 0$ only if $Tf = n$, substitute $Tf := n$:

$$= \frac{1}{2} \cdot \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \cdot \operatorname{sinc} \frac{n}{2}$$

$$= \frac{1}{2} \cdot \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \cdot \frac{\sin \pi n/2}{\pi n/2}$$

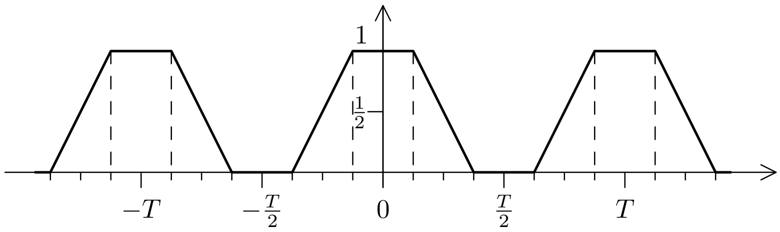
Since for all *even* integers n we have $\sin \pi\frac{n}{2} = 0$ and therefore (except at $n = 0$) $\operatorname{sinc} \pi\frac{n}{2} = 0$, and for all *odd* integers n we have $\sin \pi\frac{n}{2} = (-1)^{(n-1)/2}$, substitute $2k+1 = n$ or $k = (n-1)/2$, and take care of $n = 0$ separately:

$$= \frac{1}{2} \cdot \delta\left(f - \frac{0}{T}\right) \cdot \operatorname{sinc} \frac{0}{2} + \frac{1}{2} \cdot \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{2k+1}{T}\right) \cdot \frac{(-1)^k}{\pi(2k+1)/2}$$

$$= \frac{1}{2}\delta(f) + \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{2k+1}{T}\right) \cdot \frac{(-1)^k}{k+\frac{1}{2}}$$

[Two marks for correct application of convolution theorem, two marks for substituting Tf, two marks for substituting n, and two marks for dealing with $n = 0$.]

(c) Real-world digital signals need some time to transition between low and high. What is the Fourier transform of the periodic, trapezoid-wave clock signal $q(t)$, shown below, with period T and transition time $T/4$?



[4 marks]

Answer: We can obtain $q(t)$, which has a high/low transition time of $T/4$, by convolving $p(t)$ from part (b) with a rectangular window of width $T/4$:

$$q = p * r_{T/8}$$

Applying the convolution theorem:

$$\begin{aligned} Q(f) &= P(f) \cdot R_{T/8}(f) = P(f) \cdot \frac{T}{4} \operatorname{sinc}\left(\frac{T}{4} f\right) \\ &= \frac{1}{2}\delta(f) + \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{2k+1}{T}\right) \cdot \frac{(-1)^k}{k+\frac{1}{2}} \cdot \frac{T}{4} \operatorname{sinc}\left(\frac{T}{4} \cdot \frac{2k+1}{T}\right) \end{aligned}$$

[Two marks for basic idea and two marks for correct execution.]

COMPUTER SCIENCE TRIPOS Part II – 2015 – Paper 8

4 Digital Signal Processing (MGK)

Discrete-time Fourier transform (DTFT)

The discrete-time Fourier transform (DTFT) of a discrete sequence $\{x_n\}$ can be defined as

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{\infty} x_n \cdot \mathrm{e}^{-\mathrm{j}\omega n}$$

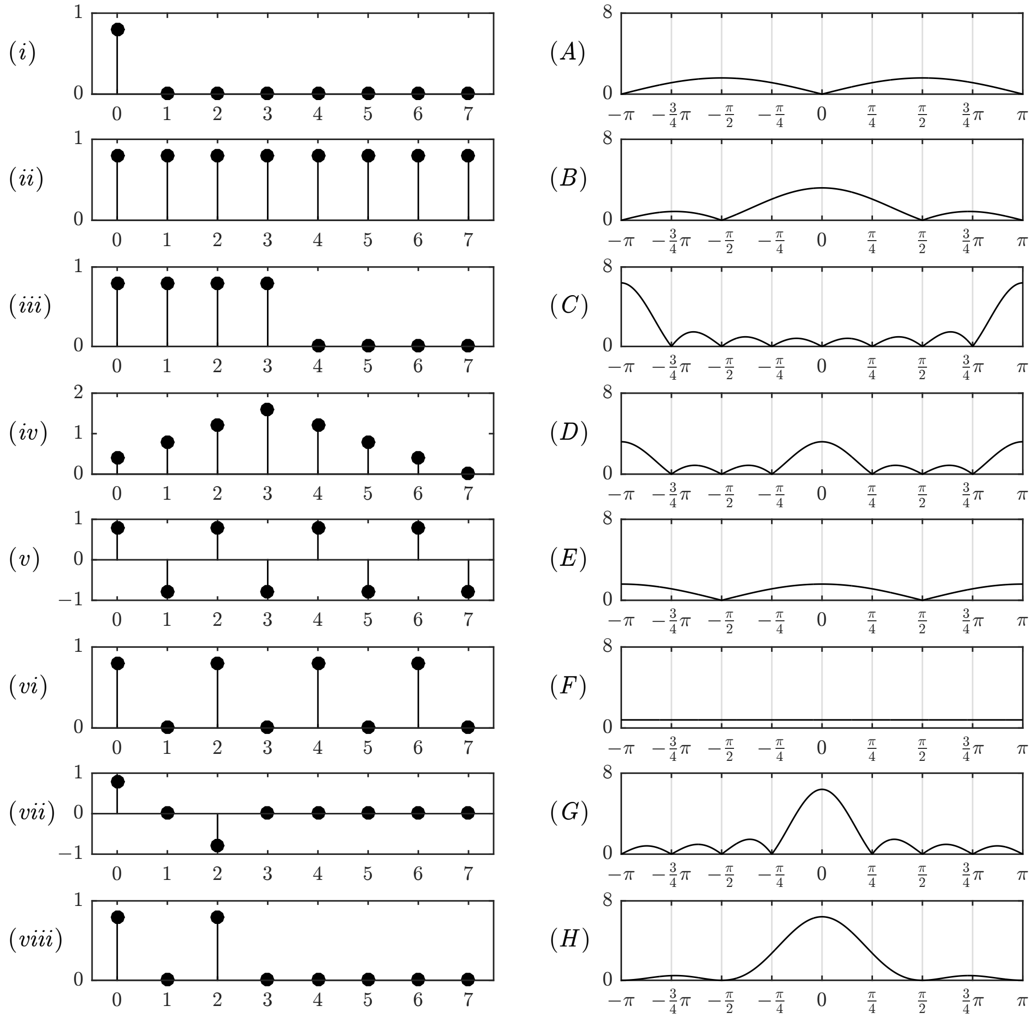
(*a*) If $\{x_n\}$ was the result of sampling a signal at sampling rate f_s and we want to know its DTFT at frequency f, what will be the corresponding value for ω? [2 marks]

Answer: $\omega = 2\pi \frac{f}{f_\mathrm{s}}$

(*b*) If $\{x_n\}$ has only real values and we know the value of $X(\mathrm{e}^{\mathrm{j}\pi/4})$, what is the value of $X(\mathrm{e}^{\mathrm{j}\pi\times 3.75})$? [2 marks]

Answer: $X(\mathrm{e}^{\mathrm{j}\pi\times 3.75}) = X(\mathrm{e}^{\mathrm{j}(3.75\pi-4\pi)}) = X(\mathrm{e}^{\mathrm{j}(-0.25\pi)}) = X(\mathrm{e}^{\mathrm{j}\pi/4})^*$
[Recall that the DTFT is 2π periodic with $X(\mathrm{e}^{\mathrm{j}\omega}) = X(\mathrm{e}^{\mathrm{j}(\omega+2\pi k)})$ for any $k \in \mathbb{Z}$ and that, for real-valued sequences, the DTFT at negative frequencies is the complex conjugate of the DTFT at the corresponding positive frequency, that is $X(\mathrm{e}^{-\mathrm{j}\omega}) = X(\mathrm{e}^{\mathrm{j}\omega})^*$.]

(*c*) Each of the eight plots (*i*)–(*viii*) below shows real-valued samples $x_0, \ldots, x_7$ from a discrete sequence $\{x_n\}$, with $x_n = 0$ for $n < 0$ or $n > 7$. For each of these eight sequences, identify which of the eight plots (*A*)–(*H*) shows the magnitude $|X(\mathrm{e}^{\mathrm{j}\omega})|$ of the corresponding discrete-time Fourier transform. [8 × 2 marks]



Answer:

(*i*) (F) – unit impulse has flat spectrum

(*ii*) (G) – sinc with first zero at $f_s/8$

(*iii*) (B) – sinc with first zero at $f_s/4$

(*iv*) (H) – (*iii*) convolved with itself $\Rightarrow$ sinc^2 with first zero at $f_s/4$

(*v*) (C) – frequency inversion of (G)

(*vi*) (D) – like (*v*), but half amplitude and DC offset

(*vii*) (A) – like (*iii*) but modulated to $f_s/4$

(*viii*) (E) – like (*vi*) but with shorter window

COMPUTER SCIENCE TRIPOS Part II – 2015 – Paper 9

5 Digital Signal Processing (MGK)

sampling

(*a*) You have been asked to design a long-wave radio receiver that can simultaneously monitor two radio signals at 60 ± 10 kHz and 100 ± 10 kHz, that is two stations with 20 kHz bandwidth each. Analog filters in your antenna amplifier suppress all signals outside those two bands.

What is the lowest possible sampling frequency that you can use for a *single* time-domain discrete sequence that records signals from these two bands simultaneously and unambiguously if you use

(*i*) base-band sampling; [2 marks]

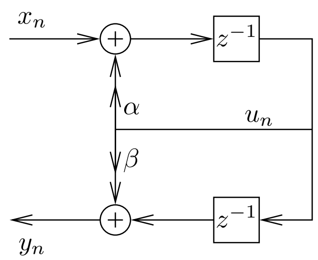
Answer: The lowest possible sampling frequency to avoid aliasing is the Nyquist limit of twice the highest frequency appearing in the signal, that is $2 \times (100 + 10) = 220$ kHz.

(*ii*) IQ sampling. [2 marks]

Answer: If we down-convert the input signal by its centre frequency of $(60 - 10 + 100 + 10)/2 = 80$ kHz, we end up with a complex-valued signal stretching from $60 - 10 - 80 = -30$ kHz to $100 + 10 - 80 = +30$ kHz, for which the Nyquist limit is $2 \times 30 = 60$ kHz.

z-transform, IIR filters

(*b*) Consider the following digital filter with two multipliers:



(*i*) State the equations that define the elements of the output sequence $\{y_n\}$ and the intermediate sequence $\{u_n\}$ in terms of other values from $\{x_n\}$, $\{y_n\}$ or $\{u_n\}$. [4 marks]

Answer:

$$y_n = u_{n-1} + \beta \cdot u_n$$
$$u_n = x_{n-1} + \alpha \cdot u_{n-1}$$

(*ii*) Convert these equations into equivalent equations for the z-transforms $X(z)$, $Y(z)$ and $U(z)$ of these three discrete sequences, and then solve for $U(z)$ and $Y(z)$. [4 marks]

Answer:

$$
\begin{aligned}
Y(z) &= U(z)\cdot z^{-1} + \beta\cdot U(z) = U(z)\cdot(z^{-1}+\beta)\\
U(z) &= X(z)\cdot z^{-1} + \alpha\cdot U(z)\cdot z^{-1}
\end{aligned}
$$

Solving for $U(z)$:

$$
\begin{aligned}
U(z) - \alpha\cdot U(z)\cdot z^{-1} &= X(z)\cdot z^{-1}\\
\Longrightarrow \qquad U(z) &= X(z)\cdot\frac{z^{-1}}{1-\alpha z^{-1}}
\end{aligned}
$$

(*iii*) What is the z-transform $H(z)$ of the impulse response of this digital filter? [4 marks]

Answer: Any linear time-invariant system where the z-transform of the input and output sequences are $X(z)$ and $Y(z)$, respectively, has an impulse-response sequence with z-transform $H(z) = Y(z)/X(z)$. Therefore:

$$
\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= \frac{U(z)\cdot(z^{-1}+\beta)}{X(z)} = \frac{X(z)\cdot\frac{z^{-1}}{1-\alpha z^{-1}}\cdot(z^{-1}+\beta)}{X(z)}\\
&= \frac{z^{-1}}{1-\alpha z^{-1}}\cdot(z^{-1}+\beta) = \frac{\beta z^{-1}+z^{-2}}{1-\alpha z^{-1}}
\end{aligned}
$$

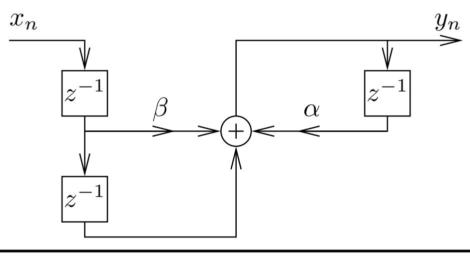
(*iv*) Draw the block diagram of an equivalent Direct Form I filter. [4 marks]

Answer: Students who prefer to work from basic principles may want to go from the z-transform back to the constant-coefficient difference equation:

$$
\begin{aligned}
Y(z) &= X(z)\cdot\frac{\beta z^{-1}+z^{-2}}{1-\alpha z^{-1}}\\
Y(z)\cdot(1-\alpha z^{-1}) &= X(z)\cdot(\beta z^{-1}+z^{-2})\\
Y(z) - \alpha Y(z) z^{-1} &= \beta X(z) z^{-1} + X(z) z^{-2}\\
y_n - \alpha y_{n-1} &= \beta x_{n-1} + x_{n-2}\\
y_n &= \alpha y_{n-1} + \beta x_{n-1} + x_{n-2}
\end{aligned}
$$

Students who prefer to remember the patterns shown in the lecture may prefer to recognize $H(z) = \frac{\beta z^{-1}+z^{-2}}{1-\alpha z^{-1}} = \frac{b_0+b_1z^{-1}+b_2z^{-2}}{a_0+a_1z^{-1}} \Longrightarrow b_0 = 0, b_1 = \beta, b_2 = 1, a_0 = 1, a_1 = -\alpha$ and then fill these coefficients into the Direct Form I diagram from the lecture.

Either approach should lead to something equivalent to:



COMPUTER SCIENCE TRIPOS Part II – 2014 – Paper 8

6 Digital Signal Processing (MGK)

IIR filters, z-transform

(*a*) Consider a causal, order-2 digital filter with real-valued infinite impulse response sequence $h_0, h_1, h_2, \dots$

(*i*) What is the z-transform $H(z)$ of this filter's impulse response? [2 marks]

Answer: $H(z) = \sum_{n=0}^{\infty} h_n z^{-n}$

(*ii*) Express $H(z)$ in terms of the locations c_1, c_2 of its two zeros and the locations d_1, d_2 of its two poles in $\mathbb{C}$. [4 marks]

Answer: $H(z) = C \cdot \frac{(z - c_1)(z - c_2)}{(z - d_1)(z - d_2)}$, where $C \in \mathbb{R}$ is a constant.

Can also be written as $H(z) = C \cdot \frac{(1 - c_1 z^{-1})(1 - c_2 z^{-1})}{(1 - d_1 z^{-1})(1 - d_2 z^{-1})}$.

(*iii*) Give a necessary condition for c_1, c_2, d_1, d_2 to ensure that $\{h_n\}$ has only real values. [4 marks]

Answer: For h_n to have only values in $\mathbb{R}$, we need to ensure in the z-domain that $H(z) = H(z^*)$, i.e. that the z-domain is symmetric with respect to the real axis. This requires for each zero and pole a complex-conjugate partner, that is $c_1 = c_2^*$ and $d_1 = d_2^*$.

(*iv*) If we operate that filter at sampling frequency f_s, what will its amplitude gain at frequency f be? [2 marks]

Answer: $\left|H(e^{2\pi j f/f_s})\right| = \left|\sum_{n=0}^{\infty} h_n e^{-2\pi j n f/f_s}\right| = C \cdot \frac{\left|e^{2\pi j f/f_s} - c_1\right| \cdot \left|e^{2\pi j f/f_s} - c_2\right|}{\left|e^{2\pi j f/f_s} - d_1\right| \cdot \left|e^{2\pi j f/f_s} - d_2\right|}$

[*Note:* Any of the above is sufficient for full marks.]

(*b*) A *notch filter* aims to suppress a single frequency f_c. One way of designing an order-2 notch filter, as in part (*a*), involves placing the zeros directly onto the unit circle, and the poles right next to them inside the unit circle, at distance $0 < \alpha < 1$ from 0:

$$c_1 = e^{j\omega}, \quad d_1 = \alpha \cdot c_1, \quad c_2 = e^{-j\omega}, \quad d_2 = \alpha \cdot c_2, \quad \text{with} \quad \omega = 2\pi f_c/f_s$$

(*i*) What is the z-transform of the impulse response of the resulting filter, written as a fraction of two polynomials of z^{-1}? [4 marks]

Answer:

$$
\begin{aligned}
H(z) = \frac{(z-c_1)(z-c_2)}{(z-d_1)(z-d_2)} &= \frac{(z-\mathrm{e}^{\mathrm{j}\omega})(z-\mathrm{e}^{-\mathrm{j}\omega})}{(z-\alpha\cdot\mathrm{e}^{\mathrm{j}\omega})(z-\alpha\cdot\mathrm{e}^{-\mathrm{j}\omega})} \\
&= \frac{z^2-(\mathrm{e}^{\mathrm{j}\omega}+\mathrm{e}^{-\mathrm{j}\omega})z+\mathrm{e}^{\mathrm{j}\omega}\mathrm{e}^{-\mathrm{j}\omega}}{z^2-\alpha(\mathrm{e}^{\mathrm{j}\omega}+\mathrm{e}^{-\mathrm{j}\omega})z+\alpha^2\mathrm{e}^{\mathrm{j}\omega}\mathrm{e}^{-\mathrm{j}\omega}} \\
&= \frac{z^2-2\cos(\omega)z+1}{z^2-\alpha 2\cos(\omega)z+\alpha^2} = \frac{1-2\cos(\omega)z^{-1}+z^{-2}}{1-\alpha 2\cos(\omega)z^{-1}+\alpha^2 z^{-2}}
\end{aligned}
$$

(*ii*) The *OxyMax* is a medical device designed in the United States. It processes a heart-beat signal with a sampling rate of $f_\mathrm{s} = 600$ Hz. It contains the following C function, which implements a notch filter, as in part (*b*)(*i*), to suppress in the input signal interference from the North American power grid at $f_\mathrm{c} = 60$ Hz:

```
double mains_notch(double sample) {
  static double x[4], y[4];
  static int n = 0;
  x[n&3] = sample;
  y[n&3] = sample + x[(n-1)&3] * b1 + x[(n-2)&3]
                  - y[(n-1)&3] * a1 - y[(n-2)&3] * a2;
  return y[n++&3];
}
```

The U.S. version initializes the constants used with `b1` $= -2\cos(\pi/5)$, `a1` $=$ `b1` $\times\, 0.9$ and `a2` $= 0.81$. What changed constant(s) will instead suppress the power-grid frequency at $f_\mathrm{c} = 50$ Hz for the European version?

[4 marks]

Answer: The C function implements an order-2 IIR filter

$$y_n = x_n + b_1 x_{n-1} + x_{n-2} - a_1 y_{n-1} - a_2 y_{n-2}$$

the impulse-response of which has a z-transform of the form

$$H(z) = \frac{1+b_1 z^{-1}+z^{-2}}{1+a_1 z^{-1}+a_2 z^{-2}}$$

That matches the result of part (*b*)(*i*) with

$$b_1 = -2\cos(\omega), \quad a_1 = -\alpha\cdot 2\cos(\omega), \quad a_2 = \alpha^2$$

The above *C* code implements these coefficients with $\alpha = 0.9$ and $b_1 = -2\cos(\omega) = -2\cos(2\pi f_\mathrm{c}/f_\mathrm{s}) = -2\cos(2\pi\times 60\text{ Hz}/600\text{ Hz}) = -2\cos(\pi/5)$. If we change from $f_\mathrm{c} = 60$ Hz to $f_\mathrm{c} = 50$ Hz, all we need to do is to change `b1` from $-2\cos(\pi/\underline{5})$ to

$$b_1 = -2\cos(2\pi f_\mathrm{c}/f_\mathrm{s}) = -2\cos(2\pi\times 50\text{ Hz}/600\text{ Hz}) = -2\cos(\pi/\underline{6})$$

(and `a1` accordingly).

— Solution notes —

COMPUTER SCIENCE TRIPOS Part II – 2014 – Paper 9

5 Digital Signal Processing (MGK)

A discrete sequence $\{x_n\}$ can be converted into a continuous representation

$$\hat{x}(t) = t_s \cdot \sum_{n=-\infty}^{\infty} \delta(t - n \cdot t_s) \cdot x_n,$$

where t_s is the sampling period.

Dirac's delta

(*a*) State two characteristic properties of Dirac's δ function. [2 marks]

Answer: Any two of

- $x \neq 0 \Rightarrow \delta(x) = 0$
- $\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$ or $\int_{a}^{b} \delta(x)\,\mathrm{d}x = 1$ for any $a < 0$ and $b > 0$
- $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,\mathrm{d}x = f(0)$

Sampling, aliasing

(*b*) Describe briefly how this representation helps to explain aliasing. [4 marks]

Answer:

- multiplying a continuous function $x(t)$ with a Dirac comb

$$s(t) = t_s \cdot \sum_{n=-\infty}^{\infty} \delta(t - n \cdot t_s)$$

  represents sampling, namely the discarding of any information about $x(t)$ for locations other than $t = t_s \cdot n$ where $n \in \mathbb{Z}$
- the Fourier transform of a Dirac comb $s(t)$ with period t_s is another Dirac comb with period t_s^{-1}:

$$S(f) = \sum_{n=-\infty}^{\infty} \delta(f - n/t_s)$$

- multiplying in the time domain corresponds to convolution in the frequency domain
- therefore: sampling in the time domain ($\hat{x}(t) = x(t) \cdot s(t)$) causes the spectrum $X(f)$ to be repeated every t_s^{-1} in

$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} X(f - n/t_s)$$

Interpolation, convolution

(*c*) Define three functions $h(t)$, such that convolving $\hat{x}(t)$ with $h(t)$ results in

(*i*) the output of an idealized analog-to-digital converter that holds the output voltage of each sample x_n for the time interval from $t = n \cdot t_s$ until the next sample x_{n+1} arrives at time $t = (n+1) \cdot t_s$; [4 marks]

Answer: Impulse response of an idealized ADC:

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t < t_\mathrm{s} \\ 0, & t \geq t_\mathrm{s} \end{cases}$$

(A properly labeled plot is equally acceptable.)

(*ii*) linear interpolation of $\{x_n\}$; [4 marks]

Answer: Impulse response of a linear interpolator:

$$h(t) = \begin{cases} 0, & t < -t_\mathrm{s} \\ 1 - |t/t_\mathrm{s}|, & -t_\mathrm{s} \leq t \leq t_\mathrm{s} \\ 0, & t > t_\mathrm{s} \end{cases}$$

(A properly labeled plot is equally acceptable.)

Band-pass sampling

(*iii*) reconstruction of a signal $x(t)$ that was sampled as $x_n = x(n \cdot t_\mathrm{s})$, assuming that the Fourier transform of $x(t)$ is zero at any frequency f with $|f|^{-1} \leq t_\mathrm{s}$ or $|f|^{-1} \geq 2t_\mathrm{s}$. [6 marks]

Answer: We can rewrite the bandwidth constraint $t_\mathrm{s} < |f|^{-1} < 2t_\mathrm{s}$ given for $x(t)$ as

$$\frac{1}{2} f_\mathrm{s} < |f| < f_\mathrm{s}$$

where $f_\mathrm{s} = 1/t_\mathrm{s}$ is the sampling frequency. Therefore, the reconstruction filter that we need to apply has to be a band-pass filter with center frequency $\frac{3}{4} f_\mathrm{s}$ and (single-sided) bandwidth $f_\mathrm{s}/2$.

Recall that an ideal low-pass filter with cut-off frequency f_c has impulse response $2f_\mathrm{c}\,\mathrm{sinc}(2f_\mathrm{c}t)$.

Solution 1: We can build the required band-pass filter by amplitude modulating a suitably scaled sinc low-pass filter with cut-off frequency $f_\mathrm{s}/2$:

$$h(t) = f_\mathrm{s}\,\mathrm{sinc}(tf_\mathrm{s}/2) \cdot \cos\left(\frac{3}{2}\pi t f_\mathrm{s}\right) = f_\mathrm{s} \frac{\sin \pi t f_\mathrm{s}/2}{\pi t f_\mathrm{s}/2} \cdot \cos\left(2\pi t \frac{3}{4} f_\mathrm{s}\right)$$

Solution 2: We can also construct the same band-pass filter by subtracting a lowpass filter with cut-off frequency $f_\mathrm{s}/2$ from one with cut-off frequency f_s (resulting in the same impulse response as above):

$$h(t) = 2f_\mathrm{s}\,\mathrm{sinc}(2f_\mathrm{s}t) - f_\mathrm{s}\,\mathrm{sinc}(f_\mathrm{s}t) = 2f_\mathrm{s} \frac{\sin 2\pi t f_\mathrm{s}}{2\pi t f_\mathrm{s}} - f_\mathrm{s} \frac{\sin \pi t f_\mathrm{s}}{\pi t f_\mathrm{s}}$$

COMPUTER SCIENCE TRIPOS Part II – 2013 – Paper 8

6 Digital Signal Processing (MGK)

IIR filters, z-transform

Consider the discrete system

$$y_n = \sum_{i=0}^{\infty} x_{n-2i} \cdot \left(-\frac{1}{2}\right)^i$$

(*a*) Write down the first 8 samples of the impulse response of this filter. [2 marks]

Answer: $1, 0, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{8}, 0, \ldots$

(*b*) Provide the finite-difference equation of an equivalent recursive filter that can be implemented with not more than two delay elements. [4 marks]

Answer: $y_n = x_n - \frac{1}{2}y_{n-2}$ or $y_n + \frac{1}{2}y_{n-2} = x_n$

(*c*) What is the z-transform $H(z)$ of the impulse response of this filter? [4 marks]

Answer: $H(z) = \dfrac{1}{1 + \frac{1}{2}z^{-2}}$

Route 1: Recall from the lecture that a filter described by the finite-difference equation

$$\sum_{l=0}^{k} a_l \cdot y_{n-l} = \sum_{l=0}^{m} b_l \cdot x_{n-l} \tag{1}$$

has an impulse response with a z-transform of

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_k z^{-k}} \tag{2}$$

Taking from Part (*b*) the equation $y_n + \frac{1}{2}y_{n-2} = x_n$, we see that this is equation (1) with $a_0 = 1, a_2 = \frac{1}{2}$, $b_0 = 1$, and all other coefficients zero. This turns equation (2) into the answer:

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-2}}$$

Route 2: Recalling that $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, we can also directly calculate the z-transform of the impulse response $1, 0, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{8}, 0, \ldots$ as

$$H(z) = z^0 - \frac{1}{2}z^{-2} + \frac{1}{4}z^{-4} - \cdots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{-2n} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}z^{-2}\right)^n = \frac{1}{1 + \frac{1}{2}z^{-2}}.$$

(*d*) Where are the zeros and poles of $H(z)$? [6 marks]

Answer: $H(z) = \dfrac{1}{1 + \frac{1}{2}z^{-2}} = \dfrac{z^2}{z^2 + \frac{1}{2}} = \dfrac{z^2}{\left(z + \mathrm{j}\sqrt{\frac{1}{2}}\right)\left(z - \mathrm{j}\sqrt{\frac{1}{2}}\right)}$

Poles at $z = \pm\mathrm{j}\sqrt{\frac{1}{2}}$ and two zeros at $z = 0$.

(*e*) We now operate this discrete system at sampling frequency $f_\mathrm{s} = 1$ MHz and feed it with input $x_n = \cos(2\pi f n/f_\mathrm{s})$. For which f (with $0 \leq f \leq f_\mathrm{s}/2$) will the peak amplitude of the output sequence $\{y_n\}$ be largest, and how large will it be? [4 marks]

Answer: We are looking for a maximum in the discrete-time Fourier transform of the impulse response of this discrete system. The discrete-time Fourier transform is found in the z plane on the unit circle ($z = \mathrm{e}^{2\pi \mathrm{j} f/f_\mathrm{s}}$). As the poles are located here on the j axis, the largest $|H(z)|$ values will also be on the j axis, at $z = \pm\mathrm{j}$. This corresponds to a positive frequency of $f = f_\mathrm{s}/4 = 250$ kHz.

Since $H(\pm\mathrm{j}) = \dfrac{-1}{-1+\frac{1}{2}} = 2$ we have at that frequency $y_n = 2x_n$, with peak amplitude 2 (= 6 dB gain).

COMPUTER SCIENCE TRIPOS Part II – 2013 – Paper 9

5 Digital Signal Processing (MGK)

FIR low-pass filters

(*a*) Consider a digital filter with impulse response

$$h_i = 2\alpha \cdot \frac{\sin[2\pi(i-n/2)\alpha]}{2\pi(i-n/2)\alpha} \cdot w_i \quad \text{where} \quad w_i = \begin{cases} 1, & 0 \le i \le n \\ 0, & \text{otherwise} \end{cases}.$$

(*i*) What type of filter is this? [4 marks]

Answer: causal finite-impulse response (FIR) low-pass filter of order n based on a rectangular window

(*ii*) How are the sampling rate f_s at which this filter is operated and its −6 dB cut-off frequency f_c related to parameter α? [2 marks]

Answer: $\alpha = f_c/f_s$

FIR band-pass filters

(*b*) In an open-source audio-effect library, you find a C routine for processing a recorded voice to sound like it came over an analog phone line:

```
#include <math.h>
#define N 512
#define PI 3.14159265358979323846
void phone_effect(double *x, double *y, int m)
{
  double w, p, f, g, h[N+1];
  int i, k;
  for (i = 0; i <= N; i++) {
    w = 0.54 - 0.46 * cos(2*PI*i/N);
    p = 2 * PI * (i-N/2) / 10;
    f = w * ((p == 0) ? 1 : sin(p)/p) / 5;
    p = 2 * PI * (i-N/2) / 100;
    g = w * ((p == 0) ? 1 : sin(p)/p) / 50;
    h[i] = f - g;
  }
  for (i = 0; i < m; i++) {
    y[i] = 0;
    for (k = 0; k <= N && k <= i; k++)
      y[i] += x[i - k] * h[k];
  }
}
```

The input array x and the output array y each hold m samples of an audio recording (mono) at sampling frequency $f_s = 32$ kHz.

(*i*) Explain in detail what operation is implemented here (e.g., type of filter, order, cut-off frequency) and how it has been constructed. [8 marks]

Answer: This function convolves an input sequence $\{x_i\}$ with an impulse response sequence $\{h_i\}$ to produce an output sequence $\{y_i\}$:

$$y_i = \sum_{k=0}^{n} x_{i-k} \cdot h_k \qquad \text{or} \qquad \{y_i\} = \{x_i\} * \{h_i\}$$

The impulse response $h_i = f_i - g_i$ is constructed as the difference of two window-based FIR low-pass filters

$$f_i = 2f_\mathrm{h}/f_\mathrm{s} \cdot \frac{\sin[2\pi(i-n/2)f_\mathrm{h}/f_\mathrm{s}]}{2\pi(i-n/2)f_\mathrm{h}/f_\mathrm{s}} \cdot w_i, \quad g_i = 2f_\mathrm{l}/f_\mathrm{s} \cdot \frac{\sin[2\pi(i-n/2)f_\mathrm{l}/f_\mathrm{s}]}{2\pi(i-n/2)f_\mathrm{l}/f_\mathrm{s}} \cdot w_i$$

with cut-off frequencies $f_\mathrm{h} = f_\mathrm{s}/10 = 3.2$ kHz (for $\{f_i\}$) and $f_\mathrm{l} = f_\mathrm{s}/100 = 320$ Hz (for $\{g_i\}$). As a result, the output sequence

$$\{y_i\} = \{x_i\} * \{h_i\} = \{x_i\} * (\{f_i\} - \{g_i\}) = \{x_i\} * \{f_i\} - \{x_i\} * \{g_i\}$$

first removes frequencies above f_h by convolution with $\{f_i\}$, and then removes frequencies below f_l, by subtracting the result of convolution with $\{g_i\}$. Overall, what is implemented here is a window-based FIR band-pass filter of order $n = 512$, with a pass-band of 320 to 3200 Hz.

(*ii*) You want to use this algorithm on audio recordings with a sampling rate of 48 kHz. What do you have to change in the source code to ensure that the audible effect remains the same? [6 marks]

Answer: If we want to multiply f_s with 1.5 while keeping f_c the same, then we have to divide $\alpha = f_\mathrm{c}/f_\mathrm{s}$ with 1.5. So we need to replace constants in four lines:

```
p = 2 * PI * (i-N/2) / 15;
f = w * ((p == 0) ? 1 : sin(p)/p) / 7.5;
p = 2 * PI * (i-N/2) / 150;
g = w * ((p == 0) ? 1 : sin(p)/p) / 75;
```

COMPUTER SCIENCE TRIPOS Part II – 2012 – Paper 8

6 Digital Signal Processing (MGK)

BBC Radio Cambridgeshire broadcasts a radio signal on a carrier frequency of 1026 kHz with a (double-sided) bandwidth of 10 kHz. You connect a long wire (antenna) via an amplifier and bandpass filter (0.5–2.0 MHz) to an analog-to-digital converter (ADC) with a sampling frequency of 5 MHz.

discrete Fourier transform

(*a*) You record $n = 500$ consecutive samples $(x_0, x_1, \ldots, x_{499})$ from the analog-to-digital converter output and calculate the Discrete Fourier Transform (DFT)

$$X_k = \sum_{i=0}^{n-1} x_i \cdot \mathrm{e}^{-2\pi \mathrm{j} \frac{ik}{n}}$$

(*i*) For which index value(s) k do you expect $|X_k|$ to best indicate the received signal strength of this radio station? [4 marks]

Answer: With a sampling rate of 5000 kHz and block length of 500 samples, the base frequency associated with X_1 (one period = block length) is 5000 kHz/500 = 10 kHz. Therefore X_{102} and X_{103} represent 1020 kHz and 1030 kHz, respectively, and both are closest to 1026 ± 5 kHz. (Same for $X_{500-102} = X_{398}$ and $X_{500-103} = X_{397}$, see (*a*)(*iii*).)

spectral estimation, windowing

(*ii*) What preprocessing step would improve this indication? [4 marks]

Answer: Multiplying the block $(x_0, x_1, \ldots, x_{499})$ with a windowing function (e.g., Hamming: $x'_i = x_i \cdot \left[0.54 - 0.46 \times \cos\left(2\pi \frac{i}{499}\right)\right]$) would reduce leakage and scalloping effects of signals whose frequencies fall between the exact integer multiples of the base frequency of the DFT block, as is the case here with the *BBC Radio Cambridgeshire* carrier.

(*iii*) What redundancy do you expect to find in the DFT output vector X, considering that the input signal is real-valued? [2 marks]

Answer: $X_i^* = X_{500-i}$ for all $0 < i < 500$, $\quad X_0, X_{250} \in \mathbb{R}$.

fast Fourier transform

(*iv*) Explain a technique that exploits this redundancy to calculate this real-valued DFT more efficiently. (You can assume that an FFT implementation is already available and that $n = 512$ is used instead.) [4 marks]

Answer: Given two real-valued vectors x' and x'', we can calculate their DFTs simultaneously by forming $x_i = x'_i + \mathrm{j} \cdot x''$ and calculating the DFT of that as X. We then extract $X'_i = (X_i + X^*_{n-i})/2$ and $X''_i = (X_i - X^*_{n-i})/(2\mathrm{j})$. We apply this technique to combine the two real-valued 256-element FFTs that are recursively requested after the first round into a single complex-valued one.

IQ downconversion

(*b*) You want to record the output of this radio station for later analysis, but you do not yet know how it was modulated. How can you convert the ADC output sequence $\{x_i\}$ such that the resulting sequence encodes efficiently what is happening in the frequency range 1021–1031 kHz, with as low a sample rate as possible? [4 marks]

Answer: IQ downconversion

(*i*) Shift the frequency spectrum by the centre frequency of the signal of interest, that is multiply the time-domain signal with a 1026 kHz phasor: $y_i = x_i \cdot \mathrm{e}^{2\pi \mathrm{j} i \frac{1026\ \mathrm{kHz}}{5000\ \mathrm{kHz}}}$

(*ii*) Apply a 5 kHz lowpass filter

(*iii*) Sample again at 10 kHz sampling frequency, that is discard all but every 500-th sample.

---

modulation

(*c*) You finally learn that the signal recorded in (*b*) was an amplitude-modulated positive mono audio signal. How can you demodulate it? [2 marks]

Answer: Simply take the absolute value of each IQ sample.

COMPUTER SCIENCE TRIPOS Part II – 2012 – Paper 9

5 Digital Signal Processing (MGK)

z-transform

(*a*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

(*i*) The z-transform of a sequence shows on the unit circle its discrete-time cosine transform. [1 mark]

Answer: The z-transform of a sequence shows on the unit circle its discrete-time *Fourier* transform.

(*ii*) Delaying a sequence by two samples corresponds in the z-domain to multiplication with z^2. [1 mark]

Answer: Delaying a sequence by two samples corresponds in the z-domain to multiplication with z^{-2}.

infinite impulse response (IIR) filters, z-transform

(*b*) Consider a causal digital IIR filter of order 2, operated at a sampling frequency of 48 kHz, where the impulse response $\{h_n\}$ has (for $n > 2$) the shape of a sine wave of frequency 8 kHz (amplitude and phase do not matter).

(*i*) Where in the z domain can you place two zeros and two poles to achieve such an impulse response $\{h_n\}$ in the time domain? [4 marks]

Answer: A filter with that impulse response has two poles on the unit circle, at mutually complex-conjugate positions (to keep the impulse response real), and at the angle associated with the required frequency (e.g., 0° for 0 Hz, 180° for half the sampling frequency). So place both zeros at $z = 0$, and the poles at $z = \mathrm{e}^{\pm 2\pi \mathrm{j} \frac{8\ \mathrm{kHz}}{48\ \mathrm{kHz}}} = \mathrm{e}^{\pm\pi\mathrm{j}/3} = \frac{1}{2} \pm \mathrm{j} \cdot \sin\frac{\pi}{3} = (1 \pm \mathrm{j}\cdot\sqrt{3})/2$.

(*ii*) Write down the z transform of $\{h_n\}$ as a rational function (with those zeros and poles). [6 marks]

Answer: $\dfrac{z^2}{(z - \mathrm{e}^{-\pi\mathrm{j}/3})(z - \mathrm{e}^{+\pi\mathrm{j}/3})} = \dfrac{z^2}{z^2 - z \cdot (\mathrm{e}^{-\pi\mathrm{j}/3} + \mathrm{e}^{-\pi\mathrm{j}/3}) + 1} = \dfrac{z^2}{z^2 - z + 1}$

(*iii*) Provide the constant-coefficient difference equation that describes the time-domain behaviour of that filter. [4 marks]

Answer: If $\frac{z^2}{z^2 - z + 1} = \frac{1}{1 - z^{-1} + z^{-2}}$ is the z transform of the digital filter, then

$$y_n = x_n + y_{n-1} - y_{n-2},$$

is its constant-coefficient difference equation, where $\{y_n\} = \{x_n\} * \{h_n\}$.

(*iv*) How can you use such a filter design to digitally generate an 8 kHz sinewave sampled at 48 kHz with very little computational effort? [4 marks]

Answer: Just feed in a single impulse into the filter, and it will then oscillate as desired:

```
y0 = 1
y1 = 0
y2 = 0
for ever:
  y2 = y1
  y1 = y0
  y0 = y1 - y2
  output y
```

This needs just one subtraction per sample!

COMPUTER SCIENCE TRIPOS Part II – 2011 – Paper 8

6 Digital Signal Processing (MGK)

Fourier transform

(a) What can you say about the Fourier transform $X(f)$ if

(i) $x(t)$ is real; [2 marks]

Answer: $X(-f) = [X(f)]^*$ (where * denotes the complex conjugate)

(ii) $x(t) = -x(-t)$. [2 marks]

Answer: $X(f) = -X(-f)$

(b) Give the result of the Fourier transform $X(f) = \int_{-\infty}^{\infty} x(t)\, \mathrm{e}^{-2\pi \mathrm{j} f t}\, \mathrm{d}t$, using Dirac's delta where appropriate, of

(i) $x(t) = 1$; [1 mark]

Answer: $X(f) = \delta(f)$

(ii) $x(t) = \cos(2\pi t)$; [2 marks]

Answer: $X(f) = \frac{1}{2}[\delta(f-1) + \delta(f+1)]$

(iii) $x(t) = \mathrm{rect}(t)$; [2 marks]

Answer: $X(f) = \mathrm{sinc}(f) = \frac{\sin \pi f}{\pi f}$

(iv) $x(t) = [\frac{1}{2} + \frac{1}{2} \cdot \cos(2\pi t)] \cdot \mathrm{rect}(t)$. [3 marks]

Answer: $X(f) = \frac{1}{2}\mathrm{sinc}(f) + \frac{1}{4}\mathrm{sinc}(f-1) + \frac{1}{4}\mathrm{sinc}(f+1)$
(follows from $(b)(i)$–$(b)(iii)$ via convolution theorem and linearity of Fourier transform)

stochastic signals

(c) When is a random sequence $\{x_n\}$ called a "white noise" signal? [2 marks]

Answer: if its autocorrelation sequence $\phi_{xx}(k) = \mathcal{E}(x_{n+k} \cdot x_n^*) = 0$ for all $k \neq 0$.

Karhunen–Loève transform

(d) Consider an n-dimensional random vector variable $\mathbf{X}$.

(i) How is its covariance matrix defined? [2 marks]

Answer: $(\mathrm{Cov}(\mathbf{X}))_{i,j} = \mathcal{E}((X_i - \mathcal{E}(X_i)) \cdot (X_j - \mathcal{E}(X_j)))$

(*ii*) How can you change its representation without loss of information into a random vector of equal dimensionality in which all elements are mutually uncorrelated? [4 marks]

Answer: Since $\mathrm{Cov}(\mathbf{X})$ is symmetric, it can be diagonalized into $\mathrm{Cov}(\mathbf{X}) = BDB^T$, where B is an orthonormal matrix containing the eigenvectors of $\mathrm{Cov}(\mathbf{X})$ and D is a diagonal matrix containing the corresponding eigenvalues. Then $B^T\mathbf{X}$ is the decorrelated representation with $\mathrm{Cov}(B^T\mathbf{X}) = D$.

COMPUTER SCIENCE TRIPOS Part II – 2011 – Paper 9

4 Digital Signal Processing (MGK)

(*a*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

discrete sequences

(*i*) A square-summable sequence is also called power signal. [1 mark]

Answer: A square-summable sequence is also called *energy* signal.

sine function

(*ii*) Adding together sine waves of the same frequency always results in another sine wave of the same phase. [1 mark]

Answer: Adding together sine waves of the same frequency always results in another sine wave of the same *frequency*.

Dirac combs

(*iii*) The Fourier transform of a Dirac comb is a Dirac impulse. [1 mark]

Answer: The Fourier transform of a Dirac comb is a Dirac *comb*.

decibel

(*iv*) 60 dBm = 1 W [1 mark]

Answer: *30* dBm = 1 W or 60 dBm = *1000* W or 60 dBm = 1 *kW*

filter design

(*b*) Before resampling a digital image at a quarter of its original resolution, you want to apply an anti-aliasing low-pass filter.

(*i*) If you apply a 1-dimensional filter with impulse response $\{h_n\}$ both horizontally and vertically to image pixels $I_{x,y}$, what are the resulting filtered pixel values $\tilde{I}_{x,y}$? [4 marks]

Answer: $\tilde{I}_{x,y} = \sum_{\Delta x} \sum_{\Delta y} I_{x-\Delta x, y-\Delta y} \cdot h_{\Delta x} \cdot h_{\Delta y}$

(*ii*) What would be the discrete impulse response $\{h_n\}$ of an ideal low-pass filter for this application, if its length were of no concern? [4 marks]

Answer: $h_n = \mathrm{sinc}(n/4) = \dfrac{\sin \pi n/4}{\pi n/4}$

(The easiest way to arrive at this solution is to recall that an ideal anti-aliasing filter is sinc scaled such that the minimum distance between its zero crossings matches the new sampling period.)

(*iii*) You decide to truncate the impulse response $\{h_n\}$ at its second zero-crossing on each side, resulting in a new impulse response $\{\bar{h}_n\}$. In the frequency domain, this results in $\bar{H} = H * T$ for what function T? [4 marks]

Answer: First zeros: $h_4 = h_{-4} = 0$, second zeros: $h_8 = h_{-8} = 0$.
Therefore new filter: $\bar{h}_n = \mathrm{sinc}(n/4) \cdot \mathrm{rect}(n/16)$.
In frequency domain (neglecting here the periodicity in the spectrum of a sampled signal): Let $f = 1$ be the original sampling frequency, then the ideal filter has $H(f) = \mathrm{rect}(4f)$ and the truncated version has the spectrum $\mathrm{rect}(4f) * \mathrm{sinc}(16f)$, i.e. $T = \mathrm{sinc}(16f)$

(iv) In order to make the frequency-domain response of your filter $\{\bar{h}_n\}$ smoother, you convolve it in the frequency domain with a rectangular pulse, the width of which is twice the distance between the zero crossings of T. What does the resulting time-domain impulse response $\{\breve{h}_n\}$ look like?
[4 marks]

Answer: The $\mathrm{sinc}(ax)$ function has zeros at distance $1/a$, and the width of the rectangular pulse $\mathrm{rect}(ax/2)$ is $2/a$.

Therefore (with $a = 16$) our smoothed filter will in the frequency domain be

$$\mathrm{rect}(4f) * \mathrm{sinc}(16f) * \mathrm{rect}(8f)$$

which is in the time domain

$$\breve{h}_n = \mathrm{sinc}(n/4) \cdot \mathrm{rect}(n/16) \cdot \mathrm{sinc}(n/8)$$

In other words, we have applied the main lobe of the sinc function as a time-domain window (Lanczos window). (The amplitude of all these filters can be normalized such that $\sum h_n = 1$.)

COMPUTER SCIENCE TRIPOS Part II – 2010 – Paper 8

6 Digital Signal Processing (MGK)

sampling, properties of the Fourier transform, convolution theorem

The *Purpletoe* standard for trouser-area networking uses a radio signal with a bandwidth of less than 1 MHz. The carrier frequency is $f_c(k) = (2400 + 2k)$ MHz, where $k \in \{1, 2, 3, \ldots, 40\}$ is the channel number. Consider a receiver design in which the antenna signal is first multiplied with a sine wave of *fixed* frequency f_m, is then band-pass filtered to eliminate frequencies outside the range 1 MHz to 100 MHz, and is finally sampled by an analogue-to-digital converter with sampling frequency f_s for further digital processing.

(*a*) What is the largest set of frequencies from which f_m can be chosen such that no information is lost from any of the 40 channels? [4 marks]

Answer: Multiplication with $\cos(2\pi t f_m)$ will result in two copies of the input spectrum, shifted both upwards and downwards by f_m. The frequencies occupied by all 40 channels range from ±2401.5 MHz to ±2480.5 MHz (centre frequency 2402 MHz to 2480 MHz, 0.5 MHz sideband on each side). Therefore we can choose either 2380.5 MHz $\leq f_m \leq$ 2400.5 MHz (if we aim for the positive input frequencies, $2480.5 - 100 = 2380.5$, $2401.5 - 1 = 2400.5$) or 2481.5 MHz $\leq f_m \leq$ 2501.5 MHz (if we aim for the negative input frequencies, whose spectrum is a mirror image of the former, $2480.5 + 1 = 2481.5$, $2401.5 + 100 = 2501.5$). The respective other copy will end up above 4 GHz, and will be eliminated by the band-pass filter.

(*b*) Which of the combinations of f_m and f_s that preserve all information from all 40 channels in the sampled output has the lowest sampling frequency f_s, assuming there is no signal outside these channels? [4 marks]

Answer: With either $f_m = 2400.5$ MHz or $f_m = 2481.5$ MHz, the output of the band-pass filter will be limited to the range 1 MHz to 80 MHz, therefore the Nyquist limit requires a sampling frequency of at least twice 80 MHz, that is $f_s \geq 160$ MHz.

(*c*) To make eavesdropping more difficult, *Purpletoe* transmitters hop several times each second from one channel to another, in a secret pseudo-random order that is cryptographically pre-agreed and shared only with intended receivers. Consider for your receiver a special eavesdropping mode that exploits aliasing such that transmissions of a data packet using different channel numbers k all look the same after sampling (assuming that there is only a single transmitter in range). Which combination of f_s and f_m achieves that, and how? [8 marks]

Answer: Channels are 2 MHz apart, so all channels will appear, after sampling, as aliases of each other with $f_s = 2$ MHz. In addition, we need to avoid that aliases of the positive and negative input frequencies overlap. This is avoided if each channel k fulfils, after the frequency shift, the band-pass sampling requirement $n \cdot f_s/2 < |f_c(k) \pm f_m \pm 0.5\ \text{MHz}| < (n + 1) \cdot f_s/2$ (for some non-negative integer n). This in turn is the case with either $f_m = (2401.5 - n)$ MHz or $f_m = (2480.5 + n)$ MHz, with integer $1 \leq n \leq 21$ to stay within the pass-band.

(For full marks, only one valid value needs to be given, e.g. $f_m = 2400.5$ MHz, and the non-overlap criterion can be demonstrated graphically.)

(*d*) Cost pressures force you to use a cheaper circuit that multiplies the radio signal with a square wave of frequency f_{m}, instead of a sine wave. How does this affect the design of your receiver? [4 marks]

Answer: The Fourier spectrum of a square wave (or indeed any other periodic wave form) of frequency f_{m} differs from that of a sine wave of the same frequency in that it can have non-zero values not only at $\pm f_{\mathrm{m}}$, but also at integer multiples $\pm n f_{\mathrm{m}}$. Multiplying with such a waveform will result in additional copies of the input spectrum. However, in our receiver design, all of these additional copies end up above 4 GHz, and are eliminated by the band-pass filter, and therefore <u>no changes to the design are needed</u>.

(Engineering remark: practical, non-ideal, real-world, analogue band-pass filters may lose attenuation above some frequency, due to inadequate shielding, and might therefore require additional shielding or filtering here.)

COMPUTER SCIENCE TRIPOS Part II – 2010 – Paper 9

7 Digital Signal Processing (MGK)

types of sequences and systems

(*a*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

z-transform

(*i*) An absolutely summable discrete sequence will have in the corresponding z-transform plane at $z = 1$ a positive value. [1 mark]

Answer: An absolutely summable discrete sequence will have in the corresponding z-transform plane at $z = 1$ a *finite* value.

(*ii*) A memory-less system depends only on the next input value. [1 mark]

Answer: A memory-less system depends only on the *current* input value.

convolution

(*b*) Define the convolution operator on discrete sequences. [2 marks]

Answer: $\{p_n\} * \{q_n\} = \{r_n\} \quad \Longleftrightarrow \quad \forall n \in \mathbb{Z} : r_n = \sum_{k=-\infty}^{\infty} p_k \cdot q_{n-k}$

(*c*) Prove that convolution of discrete sequences is an associative operation. [6 marks]

Answer: $\{r_n\} = (\{u_n\} * \{v_n\}) * \{w_n\} \Longleftrightarrow r_n = \sum_{l=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} u_k \cdot v_{l-k} \right) \cdot w_{n-l} =$

$$\sum_{k=-\infty}^{\infty} u_k \cdot \left(\sum_{l=-\infty}^{\infty} v_{l-k} \cdot w_{n-l} \right) \overset{m:=l-k}{=} \sum_{k=-\infty}^{\infty} u_k \cdot \left(\sum_{m=-\infty-k}^{\infty-k} v_m \cdot w_{n-(m+k)} \right) =$$

$$\sum_{k=-\infty}^{\infty} u_k \cdot \left(\sum_{m=-\infty}^{\infty} v_m \cdot w_{(n-k)-m} \right) \Longleftrightarrow \{r_n\} = \{u_n\} * (\{v_n\} * \{w_n\})$$

band-pass signal sampling

(*d*) Given samples $x_n = x(t_s \cdot n)$ for all integers n, where $x(t)$ is a continuous signal whose Fourier transform has non-zero values only at frequencies f with $f_l < |f| < f_h$,

(*i*) under which condition can the original waveform $x(t)$ be reconstructed; [4 marks]

Answer: There will be no aliasing if there is a non-negative integer m such that $\frac{m}{2t_s} \le f_l$ and $\frac{m+1}{2t_s} \ge f_h$.

(*ii*) and how can this be done? [6 marks]

Answer: Bandpass filter the Dirac-pulse representation

$$\hat{x}(t) = t_s \cdot \sum_{n=-\infty}^{\infty} x_n \cdot \delta(t - t_s \cdot n)$$

of the sequence $\{x_n\}$ to the frequency interval from f_l to f_h. Such a band-pass filter has the impulse response

$$h(t) = 2f_\mathrm{h}\mathrm{sinc}(2f_\mathrm{h}t) - 2f_\mathrm{l}\mathrm{sinc}(2f_\mathrm{l}t)$$

and applying it by convolution $x = \hat{x} * h$ leads to the interpolation function

$$x(t) = t_\mathrm{s} \cdot \sum_{n=-\infty}^{\infty} x_n \cdot h(t - t_\mathrm{s} \cdot n).$$

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6 Digital Signal Processing (MGK)

phasors, convolution, properties of the Fourier transform, FIR lowpass filters, sampling

While reverse-engineering a radio receiver, you find in its firmware the following two discrete systems implemented:

$$y_n := x_n \mathrm{e}^{\mathrm{j}\pi n/2}$$
$$z_n := \sum_{k=-4000}^{4000} y_{n-k-4000} \times 10^{-3}\mathrm{sinc}(k/10^3) \times \left(0.54 - 0.46\cos\left(2\pi\frac{k+4000}{8000}\right)\right)$$

The discrete sequence $\{x_n\}$ emerges from an analog-to-digital converter operating at sampling frequency $f_\mathrm{s} = 240$ kHz, whose input is connected via a 100 kHz low-pass filter and linear amplifier directly to a radio antenna.

(*a*) Explain the function of *both* discrete systems in the frequency domain and their main parameters (e.g., type of filter, cutoff frequency, type of window). [12 marks]

Answer: $y_n := x_n \cdot \mathrm{e}^{\mathrm{j}\pi n/2}$: Multiplying a signal in the time domain with a phasor corresponds to shifting a signal in the frequency domain, that is if $X(f)$ is the Fourier transform of $x(t)$, then $X(f - \Delta f)$ will be the Fourier transform of $x(t) \cdot \mathrm{e}^{2\pi\mathrm{j}\Delta f t}$. We can rewrite the given phasor as $\mathrm{e}^{\mathrm{j}\pi n/2} = \mathrm{e}^{2\pi\mathrm{j}n/4} = \mathrm{e}^{2\pi\mathrm{j}\Delta f n/f_\mathrm{s}} = \mathrm{e}^{2\pi\mathrm{j}\Delta f t}$ with $t = n/f_\mathrm{s}$ and $\Delta f = f_\mathrm{s}/4 = 240$ kHz$/4 = 60$ kHz. Therefore the first system shifts the spectrum of the antenna signal by 60 kHz.

$z_n := \sum_{k=-4000}^{4000} y_{n-k-4000} \times 10^{-3}\mathrm{sinc}(k/10^3) \times \left(0.54 - 0.46\cos\left(2\pi\frac{k+4000}{8000}\right)\right)$:
Convolving a signal in the time domain corresponds to multiplying the signal with a function in the Fourier domain, so this is a filter. The impulse response of an ideal low-pass filter with cut-off frequency f_c is proportional to $\mathrm{sinc}(2f_\mathrm{c} \cdot t)$. We can rewrite the term $\mathrm{sinc}(k/10^3) = \mathrm{sinc}(2f_\mathrm{c}k/f_\mathrm{s}) = \mathrm{sinc}(2f_\mathrm{c} \cdot t)$ with $t = k/f_\mathrm{s}$ and $f_\mathrm{c} = f_\mathrm{s}/2000 = 240$ kHz$/2000 = 120$ Hz. Therefore, this system is a low-pass filter with a cutoff frequency of 120 Hz. It also adds a delay of 4000 samples, for causality. It was constructed using a Hamming window and has order 8000.

(*b*) In approximately which frequency range will antenna signals substantially influence the resulting sequence $\{z_n\}$? [4 marks]

Answer: After being shifted by 60 kHz, the spectrum from the antenna is filtered down to the range -120 Hz to $+120$ Hz, therefore only frequencies in the range 60 kHz $\pm$ 120 Hz will be passed through.

(*c*) Will the subsequent application of the discrete system

$$b_n := z_{n\times 500}$$

cause aliasing, and why? [4 marks]

Answer: If the double-sided bandwidth of $\{z_n\}$ was reduced by an ideal low-pass filter to 240 Hz (i.e., from -120 Hz to $+120$ Hz), any sampling frequency of 240 Hz or more would not cause aliasing. The sequence $\{b_n\}$ subsamples this signal with $f_\mathrm{s}/500 = 480$ Hz, which is twice the Nyquist limit, leaving room for filter imperfections.

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12 Digital Signal Processing (MGK)

(*a*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

filters

(*i*) In the stopband, a filter design approximates a gain of −1. [1 mark]

Answer: In the stopband, a filter design approximates a gain of *0*.

z-transform

(*ii*) For infinite sequences the z-transform always converges across the entire complex plane. [1 mark]

Answer: For *impulse* sequences the z-transform always converges across the entire complex plane.

window functions

(*iii*) The Barlett window is the product of a rectangular window and a raised cosine function. [1 mark]

Answer: The *Hann* window is the product of a rectangular window and a raised cosine function.

complex numbers

(*iv*) Multiplying two complex variables can be implemented with two real-valued multiplications and five real-valued additions. [1 mark]

Answer: Multiplying two complex variables can be implemented with *three* real-valued multiplications and five real-valued additions.

sampling

(*v*) As a continuous signal is sampled, its Fourier spectrum becomes non-linear. [1 mark]

Answer: As a continuous signal is sampled, its Fourier spectrum becomes *periodic*.

(*b*) Briefly explain

JPEG

(*i*) the zigzag ordering of DCT coefficients in JPEG; [3 marks]

Answer: The 8×8 matrix of DCT coefficients usually has the highest values in the low-frequency corner, and after quantization, the highest probability of zero values away from the low-frequency corner. The zigzag order in which the 8×8 matrix is encoded achieves that the coefficients which are zero with a high probability are mostly adjacent, and can therefore be encoded efficiently by the following runlength encoder.

MPEG

(*ii*) the difference between I-, P- and B-frames in MPEG; [3 marks]

Answer: An I-frame (“independent”) is encoded independently of any other image frame (similar to a JPEG image), and therefore can serve as a starting point for a decoder. A P-frame ("predicted") encodes the difference to a previously encoded I- or P-frame, whereas a B-frame ("bidirectional") can encode the difference to both past and future frames.

JPEG

(*iii*) the relationship between RGB and YCrCb colour coordinates. [4 marks]

Answer: The brightness $Y = 0.3R + 0.6G + 0.1B$ of a sample point in an image is perceived by the human eye with much higher spatial resolution than the remaining linearly independent colour components $V = R - Y$ and $U = B - Y$. Therefore, many image coding schemes apply a linear transform into a color space with a Y axis. The $Cr = V/1.6+0.5$ and $Cb = U/2+0.5$ form of the V and U components just renormalizes their range back into the interval $[0, 1]$.

convolution theorem, sampling

(*c*) A 300 Hz sine wave is sampled at 1000 Hz. This discrete sequence is then multiplied, sample by sample, with the discrete sequence

$$\ldots, 0, +1, 0, -1, 0, +1, 0, -1, 0, +1, 0, -1, \ldots$$

Which frequencies appear in the Fourier transform of the result? [5 marks]

Answer: The given sequence is a sine wave with a frequency of 1/4 of the sampling frequency, therefore its spectrum has peaks at ±250 Hz. Multiplication in the time domain is convolution in the frequency domain. The convolution of peaks at ±250 Hz and ±300 Hz results in peaks at ±50 Hz and ±550 Hz, and the latter alias at ±450 Hz. Therefore, the spectrum of the result has peaks at ±50 Hz and ±450 Hz.

COMPUTER SCIENCE TRIPOS Part II – 2008 – Paper 8

11 Digital Signal Processing (MGK)

rect function

(*a*) What is the Fourier transform of a rectangular pulse of amplitude A and duration $d > 0$, centred around $t = 0$? [4 marks]

Answer:

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}\{\mathrm{rect}(t)\}(f) = \mathrm{sinc}(f)$$

$$\Rightarrow \quad \mathcal{F}\{A \cdot \mathrm{rect}(t/d)\}(f) = Ad \cdot \mathrm{sinc}(fd) = \underline{\underline{Ad \cdot \frac{\sin \pi f d}{\pi f d}}}$$

convolution

(*b*) Calculate the Fourier transform of the triangular pulse

$$\Lambda(t) = \begin{cases} 1 - |t|, & \text{for } |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

[*Hint:* Think of $\Lambda(t)$ as the result of a convolution.] [4 marks]

Answer: $\Lambda = \mathrm{rect} * \mathrm{rect}$, i.e. $\Lambda(t) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\,\mathrm{rect}(t-\tau)\,\mathrm{d}\tau$
$\Rightarrow \quad \mathcal{F}\{\Lambda(t)\}(f) = \mathcal{F}^2\{\mathrm{rect}(t)\}(f) = \mathrm{sinc}^2(f) = (\sin \pi f d)^2/(\pi f d)^2$

sampling

(*c*) A 2 kHz sine wave is sampled at 12 kHz. The resulting values are later converted back into a continuous signal using *linear interpolation*.

(*i*) At what other frequencies besides 2 kHz is there signal energy in the resulting continuous waveform? [4 marks]

Answer: The Fourier transform of the original signal has components at ± 2 kHz. Sampling at 12 kHz will replicate this spectrum every 12 kHz, leading to components at $n \cdot (12 \text{ kHz}) \pm 2$ kHz, that is aliases at 10 kHz, 14 kHz, 22 kHz, 26 kHz, …

(*ii*) Consider among those other components the one with the lowest frequency. By what factor is its voltage lower compared to the 2 kHz component? [4 marks]

Answer: Linear interpolation of a 12 kHz sampled signal is equivalent to convolving the 12 kHz Dirac-puls sequence with a triangular function of width 2/12 kHz, i.e. $\Lambda(t \cdot 12 \text{ kHz})$. This is in the frequency domain equivalent to multiplying with $\mathrm{const} \cdot \mathcal{F}\{\Lambda(t)\}(f/12 \text{ kHz})$, therefore the voltage ratio between the original signal at 2 kHz and the first alias at 10 kHz will be

$$\frac{\mathrm{sinc}^2(2 \text{ kHz}/12 \text{ kHz})}{\mathrm{sinc}^2(10 \text{ kHz}/12 \text{ kHz})} = \left(\frac{\sin(\pi\frac{1}{6})/(\pi\frac{1}{6})}{\sin(\pi\frac{5}{6})/(\pi\frac{5}{6})}\right)^2 = \left(\frac{\frac{5}{6}}{\frac{1}{6}}\right)^2 = \frac{25}{1}$$

spectral analysis

(iii) Your colleague records with a PC soundcard at 44.1 kHz sampling frequency 1024 samples of the continuous waveform, loads these into MATLAB as vector `x` and then attempts to plot an amplitude spectrum with the command

```
plot(real(fft(x)));
```

Name two problems that need to be fixed in this command before the resulting plot is likely to agree with the result of (*c*)(*ii*). [4 marks]

Answer:

- use `abs` instead of `real` on the complex-valued output of `fft`, to obtain an amplitude spectrum;
- multiply `x` with a windowing function before applying `fft`, to avoid "scalloping" and ensure a more predictable height of spectral peaks.

COMPUTER SCIENCE TRIPOS Part II – 2008 – Paper 9

11 Digital Signal Processing (MGK)

bandpass sampling

(*a*) A radio system outputs signals with frequency components only in the range 2.5 MHz to 3.5 MHz. The analog-to-digital converter that you want to use to digitize such signals can be operated at sampling frequencies that are an integer multiple of 1 MHz. What is the *lowest* sampling frequency that you can use without destroying information through aliasing? [5 marks]

Answer: Find lowest $f_s = m \times 1$ MHz, such that there exists an integer n with $n \cdot f_s/2 <$ 2.5 MHz and $(n+1) \cdot f_s/2 >$ 3.5 MHz. The first one that works is $f_s =$ 4 MHz with $n = 1$.

IIR filters, z-transform

(*b*) Consider a digital filter with an impulse response for which the z-transform is

$$H(z) = \frac{(z+1)^2}{(z - 0.7 - 0.7\mathrm{j})(z - 0.7 + 0.7\mathrm{j})}$$

(*i*) Draw the location of zeros and poles of this function in relation to the complex unit circle. [2 marks]

Answer: H has a pole at $0.7 + 0.7\mathrm{j}$ and $0.7 - 0.7\mathrm{j}$ and two zeros at -1.

(*ii*) If this filter is operated at a sampling frequency of 48 kHz, which (approximate) input frequency will experience the lowest attenuation? [2 marks]

Answer: The poles are closest to the points of the complex unit circle at radian angles $\pi/4$ and $-\pi/4$. With radian angle 0 corresponding to 0 Hz and radian angle π corresponding to half the sampling frequency (24 kHz), the points on the unit circle closest to the poles correspond to ± 6 kHz. As the zeros are far away, the highest values of H on the unit circle will therefore occur very close to the location that corresponds to 6 kHz.

(*iii*) Draw a direct form I block-diagram representation of this digital filter. [5 marks]

Answer:

$$H(z) = \frac{z^2 + 2z + 1}{z^2 - 1.4z + 0.98} = \frac{1 + 2z^{-1} + z^{-2}}{1 - 1.4z^{-1} + 0.98z^{-2}}$$

corresponds to an IIR filter with the constant-coefficient difference equation $y_n = 1.4y_{n-1} - 0.98y_{n-2} + x_n + 2x_{n-1} + x_{n-2}$. (Equivalent block diagram omitted here.)

audio-visual coding techniques

(*c*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

(*i*) Statistical independence implies negative covariance. [1 mark]

Answer: Statistical independence implies *zero* covariance.

(*ii*) Group 3 MH fax code uses a form of arithmetic coding. [1 mark]

Answer: Group 3 MH fax code uses a form of *Huffman* coding.

(*iii*) Steven's law states that rational scales follow a logarithmic law. [1 mark]

Answer: Steven's law states that rational scales follow a *power* law.

(*iv*) The Karhunen-Loève transform is commonly approximated by the z-transform. [1 mark]

Answer: The Karhunen-Loève transform is commonly approximated by the *DCT*.

(*v*) 40 dB corresponds to an 80$\times$ increase in voltage. [1 mark]

Answer: 40 dB corresponds to an *100*$\times$ increase in voltage.

(*vi*) The human ear has about 480 critical bands. [1 mark]

Answer: The human ear has about *24* critical bands.

COMPUTER SCIENCE TRIPOS Part II – 2007 – Paper 8

10 Digital Signal Processing (MGK)

transform coding (Daubechies wavelets)

(*a*) The DAUB4 wavelet transform involves a pair of 4-point FIR filters.

(*i*) Explain the properties that these filters are designed to have and provide a system of equations that defines the two impulse responses accordingly. [8 marks]

Answer: The filter pair is designed to be orthogonal, meaning that the two filters split up the signal into two subband signals, from which every second sample can be discarded without loss of information. If c_0, c_1, c_2, c_3 is the impulse response of the low-pass filter, then the impulse response of the corresponding high-pass filter is chosen as $c_3, -c_2, c_1, -c_0$. Negating every second coefficient inverts the spectral characteristic of the filter and reversing the coefficients makes an orthogonal solution possible. Achieving the orthogonality property is equivalent to the matrix

$$F = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & & & & & & \\ c_3 & -c_2 & c_1 & -c_0 & & & & & & \\ & & c_0 & c_1 & c_2 & c_3 & & & & \\ & & c_3 & -c_2 & c_1 & -c_0 & & & & \\ \vdots & \vdots & & & & & \ddots & & & \\ & & & & & & c_0 & c_1 & c_2 & c_3 \\ & & & & & & c_3 & -c_2 & c_1 & -c_0 \\ c_2 & c_3 & & & & & & & c_0 & c_1 \\ c_1 & -c_0 & & & & & & & c_3 & -c_2 \end{pmatrix}$$

being orthogonal, that is $FF^\top = I$, which is the case iff

$$\begin{aligned} c_0^2 + c_1^2 + c_2^2 + c_3^2 &= 1 \\ c_2c_0 + c_3c_1 &= 0 \end{aligned}$$

In addition, the high-pass filter should eliminate any constant and linear components of a signal (all polynomials of degree 1), which is the case iff

$$\begin{aligned} c_3 - c_2 + c_1 - c_0 &= 0 \\ 0c_3 - 1c_2 + 2c_1 - 3c_0 &= 0 \end{aligned}$$

These four equations determine the four filter coefficients c_0, c_1, c_2, c_3.

(*ii*) Explain briefly how this filter pair is used in the wavelet transform. [4 marks]

Answer: The input signal is passed through both filters, and every second sample in the output of both filters is discarded, resulting in the same number of samples as there were originally. The process is then reapplied, but only to the remaining samples of the low-pass filter output. This is repeated recursively, until only one single sample emerges from the low-pass filter.

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sampling and reconstruction of low-pass and pass-band signals

(*b*) Consider a digital radio designed to receive all signals in the frequency range 90–105 MHz. Its antenna amplifier includes a bandpass filter that eliminates any signals outside this frequency range. The filtered antenna signal is directly fed into a digital-to-analogue converter, such that all subsequent demodulation steps can be performed in software.

(*i*) What is the lowest sampling frequency that can be used without risking loss of information due to aliasing? Explain briefly why. [5 marks]

Answer: Sampled signals can be reconstructed without aliasing if their spectral components remain entirely within the interval $n \cdot f_\mathrm{s}/2 < |f| < (n+1) \cdot f_\mathrm{s}/2$ for some $n \in \mathbb{N}$. The frequency band 90–105 MHz is 15 MHz wide and the band boundaries are exactly aligned between 90 MHz $= 6 \cdot 15$ MHz and 105 MHz $= 7 \cdot 15$ MHz, therefore with $f_\mathrm{s}/2 = 15$ MHz, $f_\mathrm{s} = 30$ MHz is the lowest suitable sampling frequency. With this sampling frequency, the 90–105 MHz band will alias every 30 MHz at 0–15, 30–45, 60–75, 120–135, etc. MHz, while the inverted (-105)–(-90) MHz band will alias – without overlap – at 15–30, 45–60, 75–90, 105–120, etc. MHz.

(*ii*) If the resulting discrete sequence were turned into a continuous baseband signal through sinc interpolation, what relationship would there be between the spectra of the input and output signal? In particular, what would a 94 MHz sine-wave antenna signal be converted into? [3 marks]

Answer: The sinc interpolator is a low-pass filter for frequencies in the range (-15)–15 MHz. Signals in the 90–105 MHz range will be aliased $3 \cdot 30$ MHz lower at 0–15 MHz. In particular, a 94 MHz signal would be aliased at 94 MHz $+ n \cdot f_\mathrm{s}$ for all integers n, which is in the 0–15 MHz baseband of the sinc-interpolation filter for $n = -3$ at 94 MHz $- 3 \cdot 30$ MHz $= 4$ MHz.

COMPUTER SCIENCE TRIPOS Part II – 2007 – Paper 9

10 Digital Signal Processing (MGK)

complex phasors or z-transform

(*a*) Write an efficient microcontroller program (pseudo code) that outputs a continuous sine wave of frequency $f = 440$ Hz with values y_n in the range -1 to 1 at a sampling frequency $f_s = 32$ kHz. The programming language you have available lacks complex-number arithmetic, the runtime environment offers only basic floating-point arithmetic (i.e., no trigonometric functions), addition is much faster than multiplication, and there is insufficient memory to store a precomputed waveform. [10 marks]

Answer:

Approach 1: A sine wave of frequency $f = 440$ Hz at a sample rate of $f_s = 32$ kHz can be generated as the imaginary part of a complex phasor that advances by $\phi = 2\pi f/f_s$ radians per sample. In other words, the phasor has initially the value $z_0 = 1$ and is then multiplied for each sample with the complex number $r = e^{2\pi j f/f_s}$, that is $z_{n+1} = z_n \cdot r$, to obtain the output $y_n = \Im\{z_n\} = \Im\{r^n\} = \sin(2\pi n f/f_s)$. Since our programming language does not support complex arithmetic, we precompute the constants $r_1 = \cos\phi$ and $r_2 = \sin\phi$ such that $r = r_1 + j r_2$. We then use the variables x_n and y_n such that $z_n = x_n + j y_n$, where $x_0 = 1$, $y_0 = 0$, $x_{n+1} = x_n r_1 - y_n r_2$, $y_{n+1} = x_n r_2 + y_n r_1$.

There exists a variant implementation of the complex multiplication that requires one multiplication less, at the expense of three more additions. However, the following alternative is even more efficient.

Approach 2: The second-order FIR filter $H(z) = \frac{z^2}{(z-r)(z-r^*)} = \frac{1}{1-2r_1z^{-1}+z^{-2}}$ with poles at $r = e^{2\pi j f/f_s}$ and $r^* = e^{-2\pi j f/f_s}$ will have the desired waveform as its impulse response. It can be implemented as $y_n = 2r_1 y_{n-1} - y_{n-2}$ and at the required amplitude the initial values of the waveform are $y_0 = 0$ and $y_1 = \Im\{r\} = r_2$.

sampling

(*b*) The discrete sequence $y_n = \cos(2\pi n f_1/f_s) + A \cdot \cos(2\pi n f_2/f_s)$ is fed into a (hypothetical) digital-to-analogue converter that outputs a constant voltage $y(t) = y_n$ during the time interval $n/f_s \le t < (n+1)/f_s$ for all integers n.

(*i*) Explain how this behaviour of the digital-to-analogue converter affects the amplitude spectrum of the resulting signal. [5 marks]

Answer: If we represent the discrete sequence as a continuous sequence $\sum y_n \cdot \delta(t - n/f_s)$ of Dirac pulses, then its amplitude spectrum for frequencies below $f_s/2$ would show two peaks of equal amplitude at $\pm f_1$ and $\pm f_2$ for $A = 1$. The behaviour of the digital-to-analogue converter is equivalent to convolving in the time domain the continuous sequence with a rectangular pulse of width $1/f_s$ and amplitude 1. This is in the frequency domain equivalent to multiplying the amplitude spectrum with the function $f_s^{-1}\sin(\pi f/f_s)/(\pi f/f_s)$. (We can neglect here the $1/(2f_s)$ phase shift caused by the rectangular pulse not being centred around 0, because we are only interested in the amplitude spectrum.)

(*ii*) What amplitude A has to be chosen for the second term such that the resulting amplitude spectrum shows equally high peaks at both $f_1 = 1$ kHz and $f_2 = 2$ kHz if the sampling frequency is $f_s = 6$ kHz. [5 marks]

Answer:

$$\frac{1}{f_\mathrm{s}} \cdot \frac{\sin(\pi f_1/f_\mathrm{s})}{\pi f_1/f_\mathrm{s}} = A \cdot \frac{1}{f_\mathrm{s}} \cdot \frac{\sin(\pi f_2/f_\mathrm{s})}{\pi f_2/f_\mathrm{s}} \quad \Longrightarrow \quad A = \frac{\sin(\pi f_1/f_\mathrm{s}) f_2}{\sin(\pi f_2/f_\mathrm{s}) f_1}$$

$$A = \frac{\sin(\pi \times 1\ \mathrm{kHz}/6\ \mathrm{kHz}) \times 2\ \mathrm{kHz}}{\sin(\pi \times 2\ \mathrm{kHz}/6\ \mathrm{kHz}) \times 1\ \mathrm{kHz}} = \frac{2\sin(\pi/6)}{\sin(\pi/3)} = \frac{2\sin(30^\circ)}{\sin(60^\circ)} = \frac{2}{\sqrt{3}}$$

COMPUTER SCIENCE TRIPOS Part II – 2006 – Paper 8

10 Digital Signal Processing (MGK)

quantisation

(*a*) Consider a software routine that converts and records the audio samples received in a digital telephone network call (8 kHz sampling frequency, 8 bit/sample) into a WAV file (8 kHz sampling frequency, 16 bit/sample, uniform quantisation). Your colleague attempted to write a very simple conversion routine for this task, but the resulting audio is very distorted.

(*i*) Name two variants of the method used for quantising the amplitude of audio samples in digital telephone networks and explain one of them. [4 marks]

Answer: The two logarithmic quantisation conventions commonly used in digital telephone networks are known as A-law (used in Europe and most other countries) and μ-law (used e.g. in the United States and Japan).
Answer option 1 – The A-law encoding applies the following logarithmic transform to the input voltage $x \in [-V, V]$ before uniformly quantising the result $y \in [-V, V]$ (with parameter $A = 87.6$):

$$y = \begin{cases} \frac{A|x|}{1+\log A}\,\mathrm{sgn}(x) & \text{for } 0 \le |x| \le \frac{V}{A} \\ \frac{V\left(1+\log\frac{A|x|}{V}\right)}{1+\log A}\,\mathrm{sgn}(x) & \text{for } \frac{V}{A} \le |x| \le V \end{cases}$$

This transform extends the logarithm into an odd function and bridges across the pole at $\log 0$ with linear interpolation over $[-V/A, V/A]$.
Answer option 2 – The μ-law encoding applies the following logarithmic transform instead (with parameter $\mu = 255$):

$$y = \frac{V \log(1 + \mu|x|/V)}{\log(1 + \mu)}\,\mathrm{sgn}(x) \quad \text{for } -V \le x \le V$$

(*ii*) Your colleague's routine right-pads each 8-bit data word from the telephone network with eight additional least-significant zero bits to obtain 16 bit values. Explain how this distorts the signal by discussing which frequencies could appear at the output when the incoming telephone signal consists of a pure 1 kHz sine tone. [4 marks]

Answer: If the A/μ-law logarithmic transform is not undone, the output will be a non-linear distortion of a sine wave. This waveform is no longer a pure sine tone, but it is still periodic with the same 1 ms period length. The Fourier spectrum of such periodic functions contains only frequencies that are multiples of 1 kHz. Therefore, this routine could output, in addition to the original 1 kHz tone, also signals at 0 kHz, 2 kHz, 3 kHz and 4 kHz. (With 8 kHz sampling frequency, any higher multiples of 1 kHz will alias back to these.)

---

filtered random sequences

(*b*) A real-valued discrete random sequence $\{x_i\}$ is fed into a linear time-invariant filter with impulse response $h_0 = 1$, $h_3 = 1$, and $h_i = 0$ for all other i. We observe for the resulting output sequence $\{y_i\}$ the expected value

$$\mathcal{E}(y_{i+k} \cdot x_i) = \begin{cases} 1 & \text{for } k = -1 \\ 2 & \text{for } k = 0 \\ 1 & \text{for } k = 1 \\ 1 & \text{for } k = 2 \\ 2 & \text{for } k = 3 \\ 1 & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of the autocorrelation sequence $\{\phi_{xx}(k)\}$? [4 marks]

Answer: We know:

$$\begin{aligned} \{\ldots, h_{-2}, h_{-1}, h_0, h_1, h_2, h_3, h_4, h_5, \ldots\} &= \{\ldots, 0, 0, 1, 0, 0, 1, 0, 0, \ldots\} \\ \phi_{yx}(k) = \mathcal{E}(y_{i+k} \cdot x_i), \quad \{\phi_{yx}(k)\} &= \{\ldots, 0, 1, 2, 1, 1, 2, 1, 0, \ldots\} \\ \{\phi_{yx}(k)\} &= \{h_n\} * \{\phi_{xx}(k)\} \end{aligned}$$

Therefore

$$\{\ldots, 0, 1, 2, 1, 1, 2, 1, 0, \ldots\} = \{\ldots, 0, 0, 1, 0, 0, 1, 0, 0, \ldots\} * \{\phi_{xx}(k)\}$$

gives us

$$\Rightarrow \{\phi_{xx}(k)\} = \{\ldots, 0, 1, 2, 1, 0, 0, 0, 0, \ldots\}$$

that is $\phi_{xx}(-1) = \phi_{xx}(1) = 1$, $\phi_{xx}(0) = 2$, and $\phi_{xx}(k) = 0$ for $|k| > 1$.

---

colour coordinates

(*c*) The $YCrCb$ colour encoding is used in many image compression methods.

(*i*) How is it defined and why is it used? [4 marks]

Answer: The YUV coordinate system defined through $Y = 0.3R + 0.6G + 0.1B$, $V = R - Y$, $U = B - Y$ rotates and shears the RGB colour cube such that the luminosity component Y, in which the human eye has the highest spatial resolution, is separated from the remaining colour information. The $YCrCb$ coordinate system rescales the U and V coordinates into $Cb = U/2.0 + 0.5$ and $Cr = V/1.6 + 0.5$, such that all $YCrCb$ coordinates fit within the same range of values [0,1] as the original coordinates.

(*ii*) Is the conversion from 3×8-bit RGB to 3×8-bit $YCrCb$ coordinates fully reversible? Why? [4 marks]

Answer: A 3×8-bit RGB-to-$YCrCb$ transform takes values from a RGB cube with 256^3 possible positions and maps it into a rotated parallelepiped that fits inside the RGB cube. The $YCrCb$ values are quantised the same way as the RGB cube, but the parallelepiped volume that they occupy is smaller than the volume of the RGB cube. Therefore, there are less than 256^3 possible $YCrCb$ values. Consequently, many pairs of RGB exist that map to the same $YCrCb$ value and these cannot be unambiguously mapped back to a single original RGB value. The transform is not fully reversible and discards some information.

COMPUTER SCIENCE TRIPOS Part II – 2006 – Paper 9

10 Digital Signal Processing (MGK)

sampling and aliasing

Consider a software routine that converts the sampling rate of digital audio data from 8 kHz to 48 kHz, without changing the represented sound. It reads an input sequence $\{x_i\}$ and produces an output sequence $\{y_i\}$. The routine first inserts five samples of value 0 between each consecutive pair of input samples. This results in a new intermediate sequence $\{x'_i\}$ with $x'_{6i} = x_i$ and $x'_{6i+k} = 0$ for all $k \in \{1, \ldots, 5\}$. The sequence $\{x'_i\}$ is then low-pass filtered, resulting in $\{y_i\}$.

(*a*) How can the process of taking discrete-time samples $\{x_i\}$ from a continuous waveform $x(t)$ be modelled through a function $\hat{x}(t)$ that represents the sampling result but can still be analysed using the continuous Fourier transform? [2 marks]

Answer: Sampling a continuous signal $x(t)$ at frequency $f_s = 8$ kHz is equivalent to multiplying it with the impulse comb $\sum_i \delta(t - i/f_s)$, resulting in $\hat{x}(t) = \sum_i x_i \delta(t - i/f_s)$.

(*b*) What effect does sampling with 8 kHz have on the Fourier spectrum of the signal? [2 marks]

Answer: The function $\hat{x}(t)$ has a periodic spectrum with a period of f_s. Sampling is equivalent to convolving the signal spectrum with the impulse comb $\sum_i \delta(f - i \cdot f_s)$, therefore $\hat{X}(f) = \sum_i X(f - i \cdot f_s)$.

(*c*) How and under what condition can this sampling process be reversed? [2 marks]

Answer: If the original signal $x(t)$ was limited to frequencies in the range $[-f_s/2, f_s/2]$, we can reconstruct it from $\hat{x}(t)$ with a low-pass filter that eliminates any frequencies outside that band.

(*d*) Can $\hat{x}(t)$ also model another sampling process that results in the discrete sequence $\{x'_i\}$, and if so, what is its sampling frequency? [2 marks]

Answer: Yes, we can interpret $\hat{x}(t)$ as a function that was sampled at $6f_s = 48$ kHz, leading to the discrete sequence $\{x'_i\}$.

(*e*) How does the continuous spectrum associated with $\{x'_i\}$ relate to that of $\{x_i\}$? [2 marks]

Answer: The function $\hat{x}(t)$ models both the sampling sequence $\{x_i\}$ at f_s and the sampling sequence $\{x'_i\}$ at $6f_s$, therefore the continuous Fourier spectrum associated with both these sequences is identical (and the spectrum of $\{x'_i\}$ also has a period length f_s).

(*f*) What purpose serves the low-pass filter that the routine applies? In particular, what would happen to a 1 kHz sine tone input if this filter were not applied and

$\{y_i\} = \{x'_i\}$ were output instead? What cut-off frequency must the filter have? [5 marks]

Answer: A discrete sequence $\{x'_i\}$ with sampling frequency $6f_\mathrm{s}$ will normally be converted back into a continuous signal by removing any frequencies outside $[-6f_\mathrm{s}/2, 6f_\mathrm{s}/2]$, i.e. using a low-pass filter with cut-off frequency $6f_\mathrm{s}/2$. In our case, this band still contains six copies of the original $[-f_\mathrm{s}/2, f_\mathrm{s}/2]$ signal spectrum of $x(t)$. For example, with $f_\mathrm{s} = 8$ kHz, a 1 kHz sine tone in $\{x_i\}$ would have aliases at 7, 9, 15, 17, and 23 kHz in $\{x'_i\}$ [i.e., at $(8k \pm 1)$ kHz (modulo 48/2 kHz) for all $k \in \mathbb{Z}$]. To ensure that $\{y_i\}$ is an undistorted sampling sequence of $x(t)$, we have to remove these copies of the spectrum from $\{x'_i\}$, by applying to $\{x'_i\}$ a discrete low-pass filter with cut-off frequency $f_\mathrm{s}/2 = 4$ kHz, which eliminates frequencies f with $f_\mathrm{s}/2 < |f| \leq 6f_\mathrm{s}/2$.

FIR low-pass filter design

(*g*) Provide a formula for calculating a 25-sample long causal finite impulse response $\{h_i\}$ of a low-pass filter suitable for this routine, based on the Hamming windowing function. [5 marks]

Answer: A Hamming-window-based FIR low-pass filter of order $n = 24$ with cut-off frequency $f_\mathrm{s}/2 = 4$ kHz for sampling frequency $6f_\mathrm{s} = 48$ kHz:

$$h_i = \frac{\sin[2\pi(i-12)/12]}{2\pi(i-12)/12} \cdot [0.54 - 0.46 \times \cos(2\pi i/24)], \qquad 0 \leq i \leq 24$$

(It consists of a 4 kHz sinc-function, which is the ideal but infinitely long 4 kHz low-pass filter, sampled at 48 kHz, is multiplied with a modified raised-cosine (Hamming) window function to limit its length, and is shifted by half the window size to make it causal.)

COMPUTER SCIENCE TRIPOS Part II – 2005 – Paper 7

10 Digital Signal Processing (MGK)

types of discrete systems

(*a*) Characterise the systems below as linear/nonlinear, causal/noncausal, and time invariant/time varying:

(*i*) $y_n = ax_{3n-2}$ [2 marks]

Answer: linear, noncausal, time varying

(*ii*) $y_n = y_{n-1} + 6x_{n-2}$ [2 marks]

Answer: linear, causal, time invariant

(*iii*) $y_n = y_{n-1} - x_{n+5} + x_{n-5}$ [2 marks]

Answer: linear, noncausal, time invariant

(*iv*) $y_n = \frac{x_n}{x_{n-3}y_{n-2}}$ [2 marks]

Answer: nonlinear, causal, time invariant

(*v*) $y_n = x_n - \cos\left(\frac{\pi}{2}n\right)$ [2 marks]

Answer: nonlinear (but affine linear), causal, time varying

linear time-invariant systems

(*b*) Consider the system $h : \{x_n\} \to \{y_n\}$ with $y_n - y_{n-1} = x_n - x_{n-3}$.

(*i*) Give the impulse response of this system. [2 marks]

Answer: $y_n - y_{n-1} = x_n - x_{n-3} \quad \Longrightarrow \quad y_n = y_{n-1} + x_n - x_{n-3}$

Input: $\ldots, x_{-1}, x_0, x_1, \ldots = \ldots, 0, 0, 0, 1, 0, 0, 0, 0, 0, \ldots$
Output: $\ldots, y_{-1}, y_0, y_1, \ldots = \ldots, 0, 0, 0, 1, 1, 1, 0, 0, 0, \ldots$

$\Longrightarrow$ impulse response = 1, 1, 1

(*ii*) Give one sine-wave input sequence of the form

$$x_n = a \cdot \sin(b \cdot n + c)$$

(with $a \neq 0, b \neq 0$) for which $y_n = 0$ for all n. [2 marks]

Answer: From the impulse response of h, we can see that its defining equation can be rewritten as $y_n = x_n + x_{n-1} + x_{n-2}$. Any sequence with a period of three samples will be turned by h into a constant sequence and all y_n will equal the sum of the three samples from a single period of the input sequence. An example of a sine-wave sequence with a period of three samples is $x_n = \sin\left(\frac{1}{3}2\pi n\right)$, that is $\ldots, 0, \sqrt{3}/2, -\sqrt{3}/2, 0, \sqrt{3}/2, -\sqrt{3}/2, \ldots$

(Also a valid solution is to make b a multiple of π, e.g. $x_n = \sin(\pi n) = 0$.)

---

polynomial representation of filters

(iii) Express the system h as a rational function $H(z)$. [3 marks]

Answer:

$$H(z) = \frac{1 - z^{-3}}{1 - z^{-1}} = \frac{z^3(1 - z^{-3})}{z^3(1 - z^{-1})} = \frac{z^3 - 1}{z^2(z-1)} = \frac{z^2 + z + 1}{z^2} = 1 + z^{-1} + z^{-2}$$

---

zeros and poles

(iv) Determine the values $z \in \mathbb{C}$ for which $H(z) = 0$. [3 marks]

Answer: Any z for which

$$H(z) = \frac{z^3 - 1}{z^2(z-1)} = 0$$

will have to fulfill $z^3 - 1 = 0$ and will therefore have to be one of the three cubic roots of 1: $e^{j\frac{0}{3}2\pi} = 1$, $e^{j\frac{1}{3}2\pi} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$, and $e^{j\frac{2}{3}2\pi} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$. The first of these is eliminated by the factor $(z-1)$ in the denominator, leaving $H\left(e^{j\frac{1}{3}2\pi}\right) = 0$ and $H\left(e^{j\frac{2}{3}2\pi}\right) = 0$.

(Alternative approach: simply apply the quadratic formula to $z^2 + z + 1 = 0$.)

COMPUTER SCIENCE TRIPOS Part II – 2005 – Paper 8

10 Information Theory and Coding (MGK)

[...]

(*d*) Briefly explain

decibels

(*i*) how 10 V is expressed in dBμV; [1 mark]

Answer: 10 V = 10^7 μV = (20 × 7) dBμV = 140 dBμV

---

colour coordinates

(*ii*) the YCrCb coordinate system. [4 marks]

Answer: Human colour vision splits the red/green/blue input signal into separate luminosity and colour channels. Compression algorithms can approximate this by taking a linear combination of about 30% red, 60% green, and 10% blue as the luminance signal $Y = 0.3R+0.6G+0.1B$ (the exact coefficients differ between standards and do not matter here). The remaining colour information can be preserved, without adding redundancy, in the form of the difference signals $R-Y$ and $B-Y$. These are usually encoded scaled as $Cb = (B-Y)/2+0.5$ and $Cr = (R-Y)/1.6+0.5$, such that the colour cube remains, after this "rotation", entirely within the encoded unit cube, assuming that the original RGB values were all in the interval $[0,1]$.

COMPUTER SCIENCE TRIPOS Part II – 2004 – Paper 7

10 Digital Signal Processing (MGK)

IIR filters and averaging

(*a*) You have designed a digital water-level display installed on the river Cam. A sensor measures the current height of a small floating ball once every minute. In order to reduce the fluctuations that small waves would otherwise cause in the displayed value, you implemented a digital filter $y_i = 0.8y_{i-1} + 0.2x_i$, where the x_i are the measured and the y_i are the displayed water levels.

(*i*) What type of filter is this? [2 marks]

Answer: This is a single-tap infinite impulse response low-pass filter (also known as exponential averaging filter).

(*ii*) The standard deviation caused by small waves in the measurements is 30 mm. There is no measurable correlation between these added noise values. Calculate the standard deviation caused by small waves in the displayed water levels. [8 marks]

Answer: With $y_i = 0.8y_{i-1} + 0.2x_i$, we have

$$y_i = 0.2 \cdot \sum_{j=0}^{i} 0.8^j \cdot x_{i-j}.$$

Recalling that if A and B are independent random variables then $\mathrm{Var}(\alpha A + \beta B) = \alpha^2 \,\mathrm{Var}(A) + \beta^2 \,\mathrm{Var}(B)$, we get

$$\mathrm{Var}(y_i) = 0.2^2 \cdot \sum_{j=0}^{i} 0.8^{2j} \cdot \mathrm{Var}(x_{i-j})$$

and looking only at the noise variance $\mathrm{Var}(x_i) = (30\text{ mm})^2$ and the limit $i \to \infty$, we obtain

$$\begin{aligned}\mathrm{Var}(y_i) &= (30\text{ mm})^2 \cdot 0.2^2 \cdot \sum_{j=0}^{\infty} 0.8^{2j} = (30\text{ mm})^2 \cdot 0.2^2 \cdot \frac{1}{1-0.8^2} \\ &= (30\text{ mm})^2 \cdot \frac{(1-0.8)^2}{1-0.8^2} = (30\text{ mm})^2 \cdot \frac{1-0.8}{1+0.8} = (30\text{ mm})^2 \cdot \frac{1}{9} = (10\text{ mm})^2\end{aligned}$$

The standard deviation caused by small waves in the displayed values will be 10 mm.

FIR filter design, spectral inversion

(*b*) Let H be a digital low-pass filter with finite impulse response $h_0, h_1, \ldots, h_7$. Let f_s be the sampling frequency. Give the impulse response $h'_0, h'_1, \ldots, h'_7$ of a filter H' with frequency response $|H'(f)| = |H(f_\mathrm{s}/2 - f)|$. [4 marks]

Answer: Shifting a spectrum by $f_\mathrm{s}/2$ is equivalent to multiplying a signal with a cosine function of frequency $f_\mathrm{s}/2$, therefore $h'_i = h_i \cdot \cos(2\pi \frac{f_\mathrm{s}}{2} \frac{i}{f_\mathrm{s}}) = h_i \cdot \cos(\pi i) = h_i \cdot (-1)^i$.

spectral estimation

(*c*) A programmer cuts a block out of a digitized sound signal and applies the

Discrete Fourier Transform to estimate its spectral power distribution.

(i) What effect distorts the resulting power spectrum? [3 marks]

Answer: The DFT will represent the power spectrum of a signal accurately only for tones whose period divides the block length of the DFT evenly, that is whose frequency matches exactly one of the discrete frequencies in the DFT output. Sine waves of other frequencies suffer a discontinuity at the DFT block boundary, which leads to spectral energy leakage into neighbour frequencies. Cutting a finite DFT block out of a periodic signal is equivalent to multiplying the signal in the time domain with a rectangular function, or convolving it in the frequency domain with the corresponding $\sin(x)/x$-shaped spectrum.

(ii) Describe briefly one technique to reduce these distortions. [3 marks]

Answer: The signal block can be multiplied with a windowing function (e.g., raised-cosine/Hanning window $0.5 - 0.5\cos(2\pi n/(N-1))$ for $0 \le n < N$) before applying the DFT. This smooths the block boundary to zero, eliminating any discontinuity there. A good windowing function has a compacter spectrum than the equivalent rectangular function and the convolution with it in the time domain will leak less energy to distant neighbour frequencies.

COMPUTER SCIENCE TRIPOS Part II – 2004 – Paper 7

8 Information Theory and Coding (MGK)

[. . .]

psychophysics of perception

(*c*) Explain briefly:

(*i*) sensation limit; [1 mark]

Answer: The sensation limit of a sense is the lowest amplitude of a stimulus that can be perceived.

(*ii*) critical band; [1 mark]

Answer: If two audio tones fall within the same critical band, the ear is unable to recognize two separate tones and perceives a single tone with the average of their frequency instead. (The human ear has approximately 24 non-overlapping critical bands.)

(*iii*) Bark scale. [1 mark]

Answer: The Bark scale is a non-linear transform of an audible frequency into the number range 0 to 24, such that if two frequencies are less than 1 apart on this scale, they are within the same critical band.

(*d*) Which different aspects of perception do Weber's law and Steven's law model? [2 marks]

Answer: Weber's law is concerned with how the difference limit, the smallest amplitude change of a stimulus that can be distinguished, depends on the amplitude of the stimulus. Steven's law on the other hand is concerned with how the amplitude of a stimulus is perceived in relation to other amplitudes, for example how much must the amplitude raise such that the stimulus is perceived as being twice as strong.

COMPUTER SCIENCE TRIPOS Part II – 2004 – Paper 8

10 Information Theory and Coding (MGK)

[. . .]

correlation coding

(*c*) You are asked to compress a collection of files, each of which contains several thousand photographic images. All images in a single file show the same scene. Everything in this scene is static (no motion, same camera position, etc.) except for the intensity of the five light sources that illuminate everything. The intensity of each of the five light sources changes in completely unpredictable and uncorrelated ways from image to image. The intensity of each pixel across all photos in a file can be described as a linear combination of the intensity of these five light sources.

(*i*) Which one of the five techniques *discrete cosine transform*, *μ-law coding*, *2-D Gabor transform*, *Karhunen-Loève transform* and *Golomb coding* would be best suited to remove redundancy from these files, assuming your computer is powerful enough for each? [1 mark]

Answer: The Karhunen-Loève transform.

(*ii*) Explain briefly this transform and why it is of use here. [4 marks]

Answer: The Karhunen-Loève transform decorrelates random vectors. Let the values of the random vector $\mathbf{v}$ represent the individual images in one file. All vector elements being linear combinations of five values means that for each file there exists an orthonormal matrix M such that each image vector $\mathbf{v}$ can be represented as $\mathbf{v} = M\mathbf{t}$, where $\mathbf{t}$ is a new random vector whose covariance matrix is diagonal and in which all but the first five elements are zero. The Karhunen-Loève transform provides this matrix M by calculating the spectral decomposition of the covariance matrix of $\mathbf{v}$. The significant part of the transform result $M^\top\mathbf{v} = \mathbf{t}$ are only five numbers, which can be stored compactly for each image, together with the five relevant rows of M per file.