Complexity Theory

Lecture 6: Reductions beyond graphs

Tom Gur

Recap

- A problem is $\mathcal{NP}$-hard if any language in $\mathcal{NP}$ is reducible to it.
- A problem is $\mathcal{NP}$-complete if it is: (1) $\mathcal{NP}$-hard, (2) in $\mathcal{NP}$.
- Cook-Levin Theorem: 3SAT is $\mathcal{NP}$-complete.
- Using 3SAT, we can establish NP-completeness of many problems (e.g., IS, Clique, Hamiltonicity, TSP).

Beyond graph problems

Sets, Numbers and Scheduling

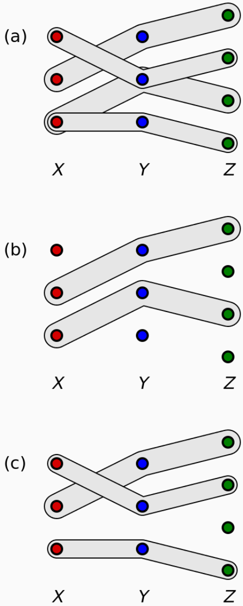
It is not just problems about formulas and graphs that turn out to be NP-complete.

Literally thousands of naturally arising problems have been proved NP-complete, in areas involving network design, scheduling, optimisation, data storage and retrieval, artificial intelligence and many others.

Such problems arise naturally whenever we have to construct a solution within constraints, and the most effective way appears to be an exhaustive search of an exponential solution space.

We now examine three more NP-complete problems, whose significance lies in that they have been used to prove a large number of other problems NP-complete, through reductions.

3D Matching



3D Matching

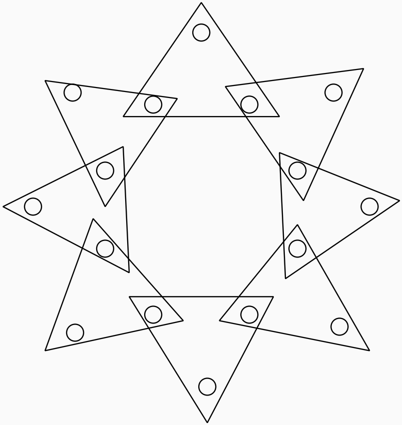
The decision problem of *3D Matching* is defined as:

> *Given three disjoint sets X, Y and Z, and a set of triples $M \subseteq X \times Y \times Z$, does M contain a matching?*
> *I.e. is there a subset $M' \subseteq M$, such that each element of X, Y and Z appears in exactly one triple of M'?*

We can show that 3DM is NP-complete by a reduction from 3SAT.

Reduction

If a Boolean expression ϕ in 3CNF has n variables, and m clauses, we construct for each variable v the following gadget.



In addition, for every clause c, we have two elements x_c and y_c.

If the literal v occurs in c, we include the triple

$$(x_c, y_c, z_{vc})$$

in M.

Similarly, if $\neg v$ occurs in c, we include the triple

$$(x_c, y_c, \bar{z}_{vc})$$

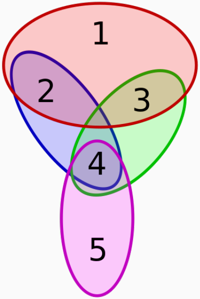
in M.

Finally, we include extra dummy elements in X and Y to make the numbers match up.

Turing fact of the day

Alan Turing chained his mug to a radiator in his office to keep it from getting pinched...

Set Cover



Exact Set Covering

Two other well known problems are proved NP-complete by immediate reduction from 3DM.

Exact Cover by 3-Sets is defined by:

> *Given a set U with $3n$ elements, and a collection $S = \{S_1, \ldots, S_m\}$ of three-element subsets of U, is there a sub-collection containing exactly n of these sets whose union is all of U?*

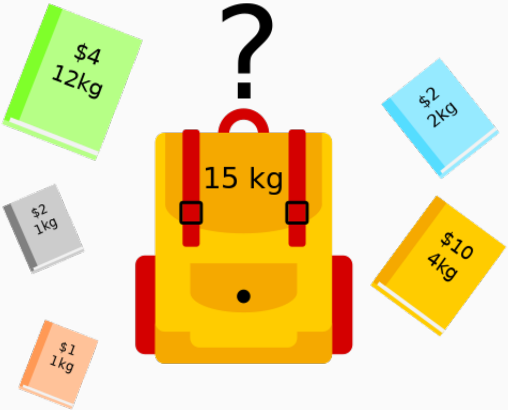
The reduction from 3DM simply takes $U = X \cup Y \cup Z$, and S to be the collection of three-element subsets resulting from M.

Set Covering

More generally, we have the *Set Covering* problem:

> *Given a set U, a collection $S = \{S_1, \ldots, S_m\}$ of subsets of U and an integer budget B, is there a collection of B sets in S whose union is U?*

Knapsack



Knapsack

KNAPSACK is a problem which generalises many natural scheduling and optimisation problems, and through reductions has been used to show many such problems NP-complete.

In the problem, we are given n items, each with a positive integer value v_i and weight w_i.

We are also given a maximum total weight W, and a minimum total value V.

> *Can we select a subset of the items whose total weight does not exceed W, and whose total value is at least V?*

Reduction

The proof that KNAPSACK is NP-complete is by a reduction from the problem of Exact Cover by 3-Sets.

Given a set $U = \{1, \ldots, 3n\}$ and a collection of 3-element subsets of U, $S = \{S_1, \ldots, S_m\}$.

We map this to an instance of KNAPSACK with m elements each corresponding to one of the S_i, and having weight and value

$$\sum_{j \in S_i} (m+1)^{j-1}$$

and set the target weight and value both to

$$\sum_{j=0}^{3n-1} (m+1)^j$$

Scheduling

Some examples of the kinds of scheduling tasks that have been proved NP-complete include:

Timetable Design

> *Given a set H of work periods, a set W of workers each with an associated subset of H (available periods), a set T of tasks and an assignment $r : W \times T \to \mathbb{N}$ of required work, is there a mapping $f : W \times T \times H \to \{0, 1\}$ which completes all tasks?*

Scheduling

Sequencing with Deadlines

Given a set T of tasks and for each task a length $l \in \mathbb{N}$, a release time $r \in \mathbb{N}$ and a deadline $d \in \mathbb{N}$, is there a work schedule which completes each task between its release time and its deadline?

Job Scheduling

Given a set T of tasks, a number $m \in \mathbb{N}$ of processors a length $l \in \mathbb{N}$ for each task, and an overall deadline $D \in \mathbb{N}$, is there a multi-processor schedule which completes all tasks by the deadline?

What's next

1) coNP
2) Cryptography
3) Space Complexity
4) Space and Time Hierarchy
5) Quantum Complexity