Complexity Theory

Lecture 10: Time vs Space Complexity

Tom Gur

http://www.cl.cam.ac.uk/teaching/2324/Complexity

Sublinear Complexity!

Space Complexity

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As we are only counting work space, it makes sense to consider bounding functions f that are less than linear.

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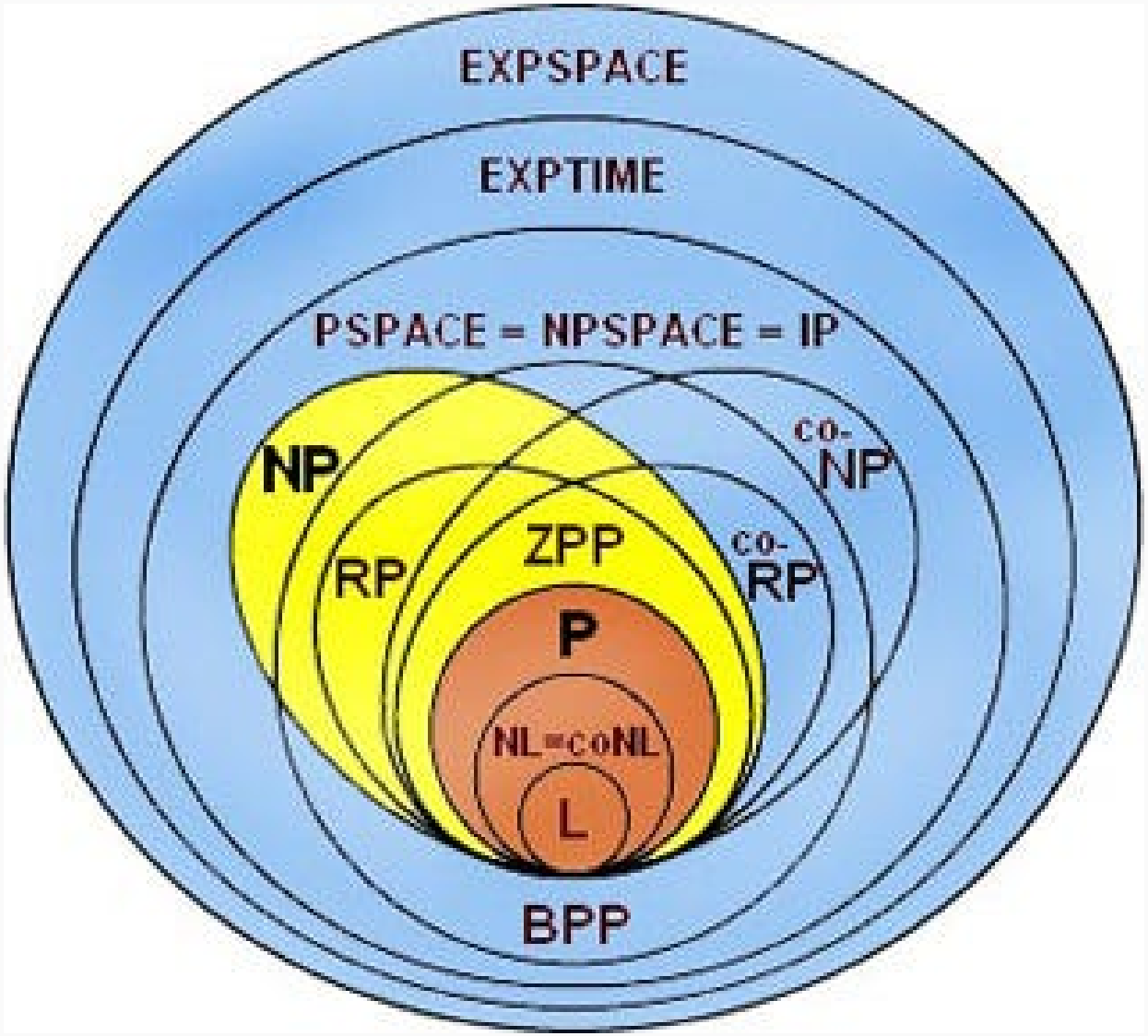

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We have the following inclusions:

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It would be easier to prove a more general statement!

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Definition

A function $f : \mathbb{N} \to \mathbb{N}$ is *constructible* if:

- f is non-decreasing, i.e. $f(n+1) \geq f(n)$ for all n; and
- there is a deterministic machine M which, on any input of length n, replaces the input with the string $0^{f(n)}$, and M runs in time $O(n + f(n))$ and uses $O(f(n))$ *work space*.

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If f and g are constructible functions, then so are $f + g$, $f \cdot g$, 2^f and $f(g)$ (this last, provided that $f(n) > n$).

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Also, given a Turing machine M and a constructible function f, we can define a machine that simulates M for $f(n)$ steps.

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The first two are straightforward from definitions.

The third is an easy simulation.

The last requires some more work.

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Bonus: Can you do it in NL?

Nondeterministic space vs deterministic time

We can use the $O(n^2)$ time algorithm for st-Connectivity to show that:

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for some constant k.

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> *Here, $c^{f(n)}$ represents the number of different possible contents of the work space, and n different head positions on the input.*

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Then, M accepts x if, and only if, some accepting configuration is reachable from the starting configuration $(s, \triangleright, x, \triangleright, \varepsilon)$ in the configuration graph of M, x.

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In particular, this establishes that NL $\subseteq$ P and NPSPACE $\subseteq$ EXP.

Scaling up complexity results

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A similar argument shows that if $\mathsf{P} = \mathsf{NP}$, then $\mathsf{EXP} = \mathsf{NEXP}$.

Bonus: Interactive Proofs (IP) and PCPs