University of Cambridge
2024/25 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 3
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CCCs

1. Show that the category **Preord** of preorders and monotone functions is a cartesian closed category.

2. Show that for any objects X and Y in a cartesian closed category $\mathbf{C}$, there are functions

$$\begin{aligned} f \in \mathbf{C}(X, Y) &\mapsto \ulcorner f \urcorner \in \mathbf{C}(1, Y^X) \\ g \in \mathbf{C}(1, Y^X) &\mapsto \overline{g} \in \mathbf{C}(X, Y) \end{aligned}$$

   that give a bijection between the set $\mathbf{C}(X, Y)$ of $\mathbf{C}$-morphisms from X to Y and the set $\mathbf{C}(1, Y^X)$ of global elements of the exponential Y^X.

3. Show that for any objects X and Y in a cartesian closed category $\mathbf{C}$, the morphism $\mathtt{app} : Y^X \times X \to Y$ satisfies $\mathtt{cur}(\mathtt{app}) = \mathtt{id}_{Y^X}$.

4. Suppose $f : Y \times X \to Z$ and $g : W \to Y$ are morphisms in a cartesian closed category $\mathbf{C}$. Prove that
$$\mathtt{cur}(f \circ (g \times \mathtt{id}_X)) = (\mathtt{cur}\, f) \circ g \in \mathbf{C}(W, Z^X)$$

5. Let $\mathbf{C}$ be a cartesian closed category. For each $\mathbf{C}$-object X and $\mathbf{C}$-morphism $f : Y \to Z$, define
$$f^X \triangleq \mathtt{cur}(Y^X \times X \xrightarrow{\mathtt{app}} Y \xrightarrow{f} Z) \in \mathbf{C}(Y^X, Z^X)$$
   (a) Prove that $(\mathtt{id}_Y)^X = \mathtt{id}_{Y^X}$.
   (b) Given $f \in \mathbf{C}(Y \times X, Z)$ and $g \in \mathbf{C}(Z, W)$, prove that
$$\mathtt{cur}(g \circ f) = g^X \circ \mathtt{cur}\, f \in \mathbf{C}(Y, W^X)$$
   (c) Deduce that if $u \in \mathbf{C}(Y, Z)$ and $v \in \mathbf{C}(Z, W)$, then $(v \circ u)^X = v^X \circ u^X \in \mathbf{C}(Y^X, W^X)$.

6. Let $\mathbf{C}$ be a cartesian closed category. For each $\mathbf{C}$-object X and $\mathbf{C}$-morphism $f : Y \to Z$, define
$$X^f \triangleq \mathtt{cur}(X^Z \times Y \xrightarrow{\mathtt{id} \times f} X^Z \times Z \xrightarrow{\mathtt{app}} X) \in \mathbf{C}(X^Z, X^Y)$$
   (a) Prove that $X^{\mathtt{id}_Y} = \mathtt{id}_{X^Y}$.
   (b) Given $g \in \mathbf{C}(W, X)$ and $f \in \mathbf{C}(Y \times X, Z)$, prove that
$$\mathtt{cur}(f \circ (\mathtt{id}_Y \times g)) = Z^g \circ \mathtt{cur}\, f \in \mathbf{C}(Y, Z^W)$$
   (c) Deduce that if $u \in \mathbf{C}(Y, Z)$ and $v \in \mathbf{C}(Z, W)$, then $X^{(v \circ u)} = X^u \circ X^v \in \mathbf{C}(X^W, X^Y)$.

7. For $f : B \longrightarrow A$ and $g : X \longrightarrow Y$ in a cartesian closed category show that

$$X^A \xrightarrow{X^f} X^B \xrightarrow{g^B} Y^B \; = \; X^A \xrightarrow{g^A} Y^A \xrightarrow{Y^f} Y^B$$

Show further that these composites equal

$$g^f \triangleq \mathtt{cur}(\, g \circ \mathtt{app} \circ (\mathtt{id}_{X^A} \times f)\,) : X^A \longrightarrow Y^B$$

8. Let $\mathbf{C}$ be a cartesian closed category in which every pair of objects X and Y possesses a binary coproduct $X \xrightarrow{\mathtt{inl}_{X,Y}} X + Y \xleftarrow{\mathtt{inr}_{X,Y}} Y$. For all objects $X, Y, Z \in \mathbf{C}$ construct an isomorphism $(Y \times X) + (Z \times X) \xrightarrow{\cong} (Y + Z) \times X$.

9. (a) State, without proof, what the product in $\mathbf{Set}^{\mathrm{op}}$ of two objects is.
   (b) Show by example that there are objects X and Y in $\mathbf{Set}^{\mathrm{op}}$ for which there is no exponential and hence that $\mathbf{Set}^{\mathrm{op}}$ is not a cartesian closed category.

IPL and STLC

1. Using the natural deduction rules for Intuitionistic Propositional Logic, give proofs of the following judgements. In each case write down a corresponding typing judgement of the Simply Typed Lambda Calculus.
   (a) $\diamond, \psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$
   (b) $\diamond, \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$
   (c) $\diamond, ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi \vdash \varphi \Rightarrow \psi$

2. (a) Given simple types A, B, C, give terms s and t of the Simply Typed Lambda Calculus that satisfy the following typing and $\beta\eta$-equality judgements:

$$\diamond, x : (A \times B) \to C \vdash s : A \to (B \to C)$$
$$\diamond, y : A \to (B \to C) \vdash t : (A \times B) \to C$$
$$\diamond, x : (A \times B) \to C \vdash t[s/y] =_{\beta\eta} x : (A \times B) \to C$$
$$\diamond, y : A \to (B \to C) \vdash s[t/x] =_{\beta\eta} y : A \to (B \to C)$$

   (b) Explain why the above implies that for any three objects X, Y, Z in a cartesian closed category $\mathbf{C}$, there are morphisms

$$f : Z^{(X \times Y)} \longrightarrow (Z^Y)^X$$
$$g : (Z^Y)^X \longrightarrow Z^{(X \times Y)}$$

   that give an isomorphism $Z^{(X \times Y)} \cong (Z^Y)^X$ in $\mathbf{C}$.

3. Make up and solve a question like the above ending with an isomorphism $X^1 \cong X$ for any object X in a cartesian closed category with terminal object 1.

4. Given types A', A, B, B' in Simply Typed Lambda Calculus, give a term t satisfying

$$\diamond \vdash t : (A' \to A) \to (B \to B') \to (A \to B) \to (A' \to B')$$

If the semantics in a cartesian closed category of A', A, B and B' are the objects X', X, Y, Y' respectively, what is the semantics of t?