# The Principle of Duality

Whenever one defines a concept / proves a theorem in terms of commutative diagrams in a category $\mathbf{C}$, one obtains another concept / theorem, called its dual, by reversing the direction or morphisms throughout, that is, by replacing $\mathbf{C}$ by its opposite category $\mathbf{C}^{\mathrm{op}}$.

For example, "isomorphism" is a self-dual concept.

# Initial object

(the dual notion to "terminal object")

An object $0$ of a category $\mathbf{C}$ is initial if for all $X \in \mathbf{C}$, there is a unique $\mathbf{C}$-morphism $0 \to X$, which we write as $[]_X : 0 \to X$.

So we have $\begin{cases} \forall X \in \mathbf{C},\ \ []_X \in \mathbf{C}(0, X) \\ \forall X \in \mathbf{C}, \forall f \in \mathbf{C}(0, X),\ \ f = []_X \end{cases}$

(In particular, $\mathtt{id}_0 = []_0$.)

NB: By duality, we have that initial objects are unique up to unique isomorphism and that any object isomorphic to an initial object is itself initial.

# Examples of initial objects

- The empty set is initial in **Set**.
- Any singleton set has a uniquely determined monoid structure and is initial in **Mon**. (why?)

  So initial and terminal objects coincide in **Mon**

  An object that is both initial and terminal in a category is called a zero object.
- A preorder $\underline{P} = (P, \sqsubseteq)$, regarded as a category $\mathbf{C}_{\underline{P}}$, has an initial object iff it has a least element $\bot$, that is: $\forall x \in P, \bot \sqsubseteq x$.

# Free monoids as initial objects

The free monoid on a set $X$ is
$\underline{\texttt{List}}\, X = (\texttt{List}\, X, @, \texttt{nil})$ where

$$\begin{aligned} \texttt{List}\, X &= \text{set of finite lists of elements of } X \\ @ &= \text{list concatenation} \\ \texttt{nil} &= \text{empty list} \end{aligned}$$

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The singleton-list function

$$\begin{aligned} \texttt{s}_X : X &\to \texttt{List}\,X \\ x &\mapsto [x] = x :: \texttt{nil} \end{aligned}$$

has the following (initial) universal property ...

# Free monoids as initial objects

**Theorem.** For any monoid $\underline{M} = (M, \bullet, \iota)$ and function $f : X \to M$, there is a unique monoid morphism $f^{\#} \in \mathbf{Mon}(\underline{\mathtt{List}\,X}, \underline{M})$ making

$$\begin{array}{ccc} X & \xrightarrow{\mathtt{s}_X} & \mathtt{List}\,X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f^{\#}} \\ & & M \end{array}$$

commute in $\mathbf{Set}$.

# Free monoids as initial objects

**Theorem.** $\forall \underline{M} \in \mathbf{Mon}, \forall f \in \mathbf{Set}(X, M), \exists! f^{\#} \in \mathbf{Mon}(\underline{\mathtt{List}}\, X, \underline{M}),\ f^{\#} \circ \mathtt{s}_X = f$

The theorem just says that $\mathtt{s}_X : X \to \mathtt{List}\, X$ is an initial object in the category $X/\mathbf{Mon}$:

- objects: $(\underline{M}, f)$ where $\underline{M} \in \mathtt{obj}\ \mathbf{Mon}$ and $f \in \mathbf{Set}(X, M)$
- morphisms in $X/\mathbf{Mon}((\underline{M}_1, f_1), (\underline{M}_2, f_2))$ are $h \in \mathbf{Mon}(\underline{M}_1, \underline{M}_2)$ such that $h \circ f_1 = f_2$
- identities and composition as in $\mathbf{Mon}$

# Free monoids as initial objects

**Theorem.** $\forall \underline{M} \in \mathbf{Mon}, \forall f \in \mathbf{Set}(X, M), \exists! f^{\#} \in \mathbf{Mon}(\underline{\mathtt{List}\, X}, \underline{M}),\ f^{\#} \circ \mathtt{s}_X = f$

The theorem just says that $\mathtt{s}_X : X \to \mathtt{List}\, X$ is an initial object in the category $X/\mathbf{Mon}$:

So this "universal property" determines the monoid $\mathtt{List}\, X$ uniquely up to isomorphism in $\mathbf{Mon}$.

We will see later that $X \mapsto \mathtt{List}\, X$ is part of a functor (= morphism of categories) which is left adjoint to the "forgetful functor" $\mathbf{Mon} \to \mathbf{Set} : \underline{M} \mapsto M$.

# Products

Problem: In a category, find a universal construction specifying a product object $X \times Y$ that internalises pairs of generalised elements of objects $X$ and $Y$.

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That is,

$$\frac{C \longrightarrow X \times Y}{C \longrightarrow X \quad C \longrightarrow Y}$$

where the passage from top to bottom is given by projecting on the first and second components.

More precisely,

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

such that

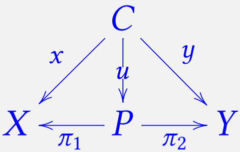
$$\hom(C, X \times Y) \xrightarrow{\langle \pi_1 \circ \_, \pi_2 \circ \_ \rangle} \hom(C, X) \times \hom(C, Y)$$

is an isomorphism.

# Binary products

In a category $\mathbf{C}$, a product for objects $X, Y \in \mathbf{C}$ is a diagram $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ with the universal property:

For all $X \xleftarrow{x} C \xrightarrow{y} Y$ in $\mathbf{C}$, there is a unique $\mathbf{C}$-morphism $u : C \to P$ such that the following diagram commutes in $\mathbf{C}$:



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For all $X \xleftarrow{x} C \xrightarrow{y} Y$ in $\mathbf{C}$, there is a unique $\mathbf{C}$-morphism $u : C \to P$ such that
$x = \pi_1 \circ u$ and $y = \pi_2 \circ u$

So $(P, \pi_1, \pi_2)$ is a terminal object in the category with

- objects: $(C, x, y)$ where $X \xleftarrow{x} C \xrightarrow{y} Y$ in $\mathbf{C}$
- morphisms $f : (C_1, x_1, y_1) \to (C_2, x_2, y_2)$ are $f \in \mathbf{C}(C_1, C_2)$ such that $x_1 = x_2 \circ f$ and $y_1 = y_2 \circ f$
- composition and identities as in $\mathbf{C}$

So if it exists, the binary product of two objects in a category is unique up to (unique) isomophism.

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**N.B.** products of objects in a category do not always exist. For example in the category


two objects, no non-identity morphisms

the objects $0$ and $1$ do not have a product, because there is no diagram of the form $0 \leftarrow ? \rightarrow 1$ in this category.

# Notation for binary products

Assuming $\mathbf{C}$ has binary products of objects, the product of $X, Y \in \mathbf{C}$ is written

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

and given $X \xleftarrow{x} C \xrightarrow{y} Y$, the unique $u : C \to X \times Y$ with $\pi_1 \circ u = x$ and $\pi_2 \circ u = y$ is written

$$\langle x\,,y\rangle : C \to X \times Y$$

## Examples:

- In **Set**, category-theoretic products are given by the usual cartesian product of sets (set of all ordered pairs) and their projections:

$$
\begin{aligned}
X \times Y &= \{(x, y) \mid x \in X \wedge y \in Y\} \\
\pi_1(x, y) &= x \\
\pi_2(x, y) &= y
\end{aligned}
$$

- In **Mon**, can take product of $(M_1, \cdot_1, e_1)$ and $(M_2, \cdot_2, e_2)$ to be

$$(M_1 \times M_2, \cdot, (e_1, e_2))$$

product in **Set**

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot_1 y_1, x_2 \cdot_2 y_2)$$

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The projection functions $M_1 \xleftarrow{\pi_1} M_1 \times M_2 \xrightarrow{\pi_2} M_2$ are monoid morphisms for this monoid structure on $M_1 \times M_2$ and have the universal property needed for a product in **Mon** (check).

- In **Preord**, we can take the product of $(P_1, \sqsubseteq_1)$ and $(P_2, \sqsubseteq_2)$ to be

$$(P_1 \times P_2, \sqsubseteq)$$

product in **Set**

$$(x_1, x_2) \sqsubseteq (y_1, y_2) \overset{\triangle}{\Leftrightarrow} x_1 \sqsubseteq_1 y_1 \land x_2 \sqsubseteq_2 y_2$$

- In **Preord**, we can take the product of $(P_1, \sqsubseteq_1)$ and $(P_2, \sqsubseteq_2)$ to be

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product in **Set**

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The projection functions $P_1 \xleftarrow{\pi_1} P_1 \times P_2 \xrightarrow{\pi_2} P_2$ are monotone for this preorder on $P_1 \times P_2$ and have the universal property needed for a product in **Preord** (check).

- Recall that each preorder $\underline{P} = (P, \sqsubseteq)$ determines a category $\mathbf{C}_{\underline{P}}$.
  Given $p, q \in P = \mathtt{obj}\, \mathbf{C}_{\underline{P}}$, the product $p \times q$ (if it exists) is a greatest lower bound (or glb, or meet) for $p$ and $q$ in $\underline{P}$:
  **lower bound**:
  $$p \times q \sqsubseteq p \;\wedge\; p \times q \sqsubseteq q$$
  **greatest** among all lower bounds:
  $$\forall \ell \in P,\ \ell \sqsubseteq p \;\wedge\; \ell \sqsubseteq q \;\Rightarrow\; \ell \sqsubseteq p \times q$$
  **Notation:** glbs are often written $\boxed{p \wedge q}$ or $\boxed{p \sqcap q}$

# Binary product of morphisms

Suppose a category $\mathbf{C}$ has binary products; that is, for every pair of $\mathbf{C}$-objects $X$ and $Y$ there is a product diagram $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$.

Given $f \in \mathbf{C}(A, X)$ and $g \in \mathbf{C}(B, Y)$, then

$$\boxed{f \times g : A \times B \longrightarrow X \times Y}$$

stands for $\langle f \circ \pi_1 \,, g \circ \pi_2 \rangle$; that is, the unique morphism $u \in \mathbf{C}(A \times B, X \times Y)$ satisfying $\pi_1 \circ u = f \circ \pi_1$ and $\pi_2 \circ u = g \circ \pi_2$.

# Binary coproducts

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Thus the coproduct of $X, Y \in \mathbf{C}$ if it exists, is a diagram $X \xrightarrow{\iota_1} X + Y \xleftarrow{\iota_2} Y$ with the universal property:

$$\forall\ (X \xrightarrow{f} Z \xleftarrow{g} Y),$$
$$\exists!\ (X + Y \xrightarrow{[f,g]} Z),$$
$$f = [f,g] \circ \iota_1 \ \wedge\ g = [f,g] \circ \iota_2$$

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$$\langle \_\circ\iota_1\, ,\, \_\circ\iota_2 \rangle : \mathbf{C}(X+Y, Z) \xrightarrow{\cong} \mathbf{C}(X,Z)\times\mathbf{C}(Y,Z)$$

Examples:

- In **Set**, the coproduct of $X$ and $Y$

$$X \xrightarrow{\iota_1} X + Y \xleftarrow{\iota_2} Y$$

is given by their disjoint union (tagged sum)

$$\begin{aligned} X + Y &= \{(1, x) \mid x \in X\} \cup \{(2, y) \mid y \in Y\} \\ \iota_1(x) &= (1, x) \\ \iota_2(y) &= (2, y) \end{aligned}$$

(prove this)

- Recall that each preorder $\underline{P} = (P, \sqsubseteq)$ determines a category $\mathbf{C}_{\underline{P}}$.
  Given $p, q \in P = \texttt{obj}\,\mathbf{C}_{\underline{P}}$, the coproduct $p + q$ (if it exists) is a least upper bound (or lub, or join) for $p$ and $q$ in $\underline{P}$:
  **upper bound**:
  $$p \sqsubseteq p + q \;\wedge\; q \sqsubseteq p + q$$
  **least** among all upper bounds:
  $$\forall u \in P,\; p \sqsubseteq u \;\wedge\; q \sqsubseteq u \;\Rightarrow\; p + q \sqsubseteq u$$
  **Notation:** lubs are often written $p \vee q$ or $p \sqcup q$

# Binary coproduct of morphisms

Suppose a category $\mathbf{C}$ has binary coproducts; that is, for every pair of $\mathbf{C}$-objects $X$ and $Y$ there is a coproduct diagram $X \xrightarrow{\iota_1} X + Y \xleftarrow{\iota_2} Y$.

Given $f \in \mathbf{C}(A, X)$ and $g \in \mathbf{C}(B, Y)$, then

$$\boxed{f + g : A + B \longrightarrow X + Y}$$

stands for $[\iota_1 \circ f, \iota_2 \circ g]$; that is, the unique morphism $u \in \mathbf{C}(A + B, X + Y)$ satisfying $u \circ \iota_1 = \iota_1 \circ f$ and $u \circ \iota_2 = \iota_2 \circ g$.

# Exponentials

Problem: In a category with binary products, find a universal construction specifying an exponential object (or internal hom) $X \Rightarrow Y$ with generalised elements corresponding to parameterised morphisms from $X$ to $Y$.

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That is,

$$\frac{C \longrightarrow X \Rightarrow Y}{C \times X \longrightarrow Y}$$

where the passage from top to bottom is given by application.

More precisely,

$$\mathtt{app} : (X \Rightarrow Y) \times X \to Y$$

such that

$$\hom(C, X \Rightarrow Y) \xrightarrow{\mathtt{app} \circ (\_ \times \mathtt{id}_X)} \hom(C \times X, Y)$$

is an isomorphism.

# Exponential objects

Suppose a category $\mathbf{C}$ has binary products

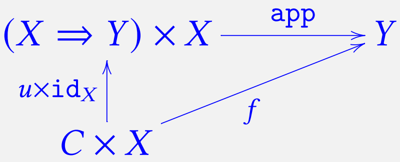
An exponential for $\mathbf{C}$-objects $X$ and $Y$ is specified by

a $\mathbf{C}$-object $X \Rightarrow Y$

a $\mathbf{C}$-morphism $\mathtt{app} : (X \Rightarrow Y) \times X \to Y$

satisfying the universal property

for all $C \in \mathbf{C}$ and $f \in \mathbf{C}(C \times X, Y)$, there is a unique $u \in \mathbf{C}(C, X \Rightarrow Y)$ such that




commutes in $\mathbf{C}$.

**Notation:** we write $\boxed{\mathtt{cur}\, f}$ for the unique $u$ such that $\mathtt{app} \circ (u \times \mathtt{id}_X) = f$.

# Exponential objects

The universal property of $\mathtt{app} : (X \Rightarrow Y) \times X \to Y$ says that there is a bijection

$$
\begin{aligned}
\hom(C, X \Rightarrow Y) &\cong \hom(C \times X, Y) \\
g &\mapsto \mathtt{app} \circ (g \times \mathtt{id}_X) \\
\mathtt{cur}\, f &\leftarrow f \\
\mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_X) &= f \\
g &= \mathtt{cur}(\mathtt{app} \circ (g \times \mathtt{id}_X))
\end{aligned}
$$

# Exponential objects

## The universal property of $\mathtt{app} : (X \Rightarrow Y) \times X \to Y$ says that there is a bijection…

It also says that $(X \Rightarrow Y, \mathtt{app})$ is a terminal object in the following category:

- objects: $(C, f)$ where $f \in \mathbf{C}(C \times X, Y)$
- morphisms $g : (C, f) \to (C', f')$ are $g \in \mathbf{C}(Z, Z')$ such that $f' \circ (g \times \mathtt{id}_X) = f$
- composition and identities as in $\mathbf{C}$.

So when they exist, exponential objects are unique up to (unique) isomorphism.

Example: Exponential objects in **Set**.

Given $X, Y \in \mathbf{Set}$, let $(X \Rightarrow Y) \in \mathbf{Set}$ denote the set of all functions from $X$ to $Y$.

Function application gives a morphism
$\mathtt{app} : (X \Rightarrow Y) \times X \to Y$ in **Set**

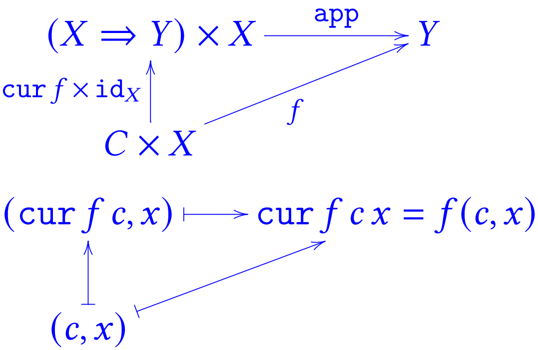
$$\mathtt{app}(f, x) = f\,x \qquad (f \in (X \Rightarrow Y), x \in X)$$

The Currying operation transforms morphisms
$f : C \times X \to Y$ in **Set** to morphisms

$$\mathtt{cur}\,f : C \to X \Rightarrow Y \text{ in } \mathbf{Set}$$

$$\mathtt{cur}\,f\,c\,x = f(c, x) \qquad (f \in (X \Rightarrow Y), c \in C, x \in X)$$

For each function $f : C \times X \to Y$ we get a commutative diagram in **Set**:



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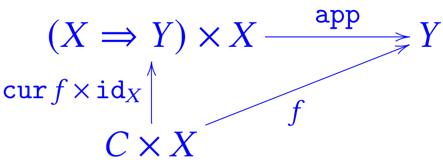
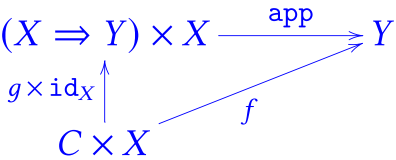



Furthermore, if any function $g : C \to X \Rightarrow Y$ also satisfies




then $g = \mathtt{cur}\, f$, because of function extensionality.

Indeed,

$$\begin{aligned}
&\mathtt{app} \circ (g \times \mathtt{id}_X) = f \\
&\quad\Longrightarrow \forall (c, x) \in C \times X, \mathtt{app}(g\,c, x) = f(c, x) \\
&\quad\Longrightarrow \forall x \in X, \forall c \in C, g\,c\,x = \mathtt{cur}\,f\,c\,x \\
&\quad\Longrightarrow \forall c \in C, g\,c = \mathtt{cur}\,f\,c \\
&\quad\Longrightarrow g = \mathtt{cur}\,f
\end{aligned}$$

# Cartesian closed category

**Definition.** $\mathbf{C}$ is a cartesian closed category (ccc) if it is a category with a terminal object, binary products, and exponentials of any pair of objects.

This is a key concept for the semantics of lambda calculus and for the foundations of functional programming languages.

Examples:

- **Set** is a ccc — as we have seen.
- **Preord** is a ccc: we already saw that it has a terminal object and binary products; the exponential of $(P_1, \sqsubseteq_1)$ and $(P_2, \sqsubseteq_2)$ is $(P_1 \Rightarrow P_2, \sqsubseteq)$ where

  $$P_1 \Rightarrow P_2 \triangleq \mathbf{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$$

  $$f \sqsubseteq g \overset{\triangle}{\Leftrightarrow} \forall x \in P_1,\ f\,x \sqsubseteq_2 g\,x$$

  (check that this is a pre-order and does give an exponential in **Preord**)
- $\mathbf{DiGph}(\mathbf{Set})$ is a ccc.

# Bicartesian closed category

**Definition.** $\mathcal{C}$ is a bicartesian category if it is a category with a terminal and initial object, and binary products and coproducts of any pair of objects.

**Definition.** $\mathcal{C}$ is a bicartesian closed category (biccc) if it is a bicartesian category with exponentials of any pair of objects.

This is a key concept for the semantics of lambda calculus and for the foundations of functional programming languages.

Examples: **Set**, **Preord**, **DiGph(Set)** are bicccs.