Functors

morphisms of categories

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $\boxed{F : \mathbf{C} \to \mathbf{D}}$ is specified by:

- a function $\mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{D}$ whose value at X is written $\boxed{F\,X}$
- for all $X, Y \in \mathbf{C}$, a function $\mathbf{C}(X, Y) \to \mathbf{D}(F\,X, F\,Y)$ whose value at $f : X \to Y$ is written $\boxed{F\,f : F\,X \to F\,Y}$ and which is required to preserve composition and identity morphisms:

$$\begin{aligned} F(g \circ f) &= F\,g \circ F\,f \\ F(\mathtt{id}_X) &= \mathtt{id}_{F\,X} \end{aligned}$$

Examples of functors

"Forgetful" functors from categories of set-with-structure back to **Set**.

E.g. $U : \mathbf{Mon} \to \mathbf{Set}$

$$\begin{cases} U(M, \bullet, \iota) & = M \\ U((M_1, \bullet_1, \iota_1) \xrightarrow{f} (M_2, \bullet_2, \iota_2)) & = M_1 \xrightarrow{f} M_2 \end{cases}$$

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Similarly $U : \mathbf{Preord} \to \mathbf{Set}$.

Examples of functors

Free monoid functor $\boxed{F : \mathbf{Set} \longrightarrow \mathbf{Mon}}$

Given $A \in \mathbf{Set}$,

$$F\,A = (\mathtt{List}\,A, @, \mathtt{nil})\text{, the free monoid on } A$$

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Given $A \in \mathbf{Set}$,

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Given a function $f : A \to B$, we get a function $F\,f : \mathtt{List}\,A \to \mathtt{List}\,B$ by mapping f over finite lists:

$$F\,f\,[a_1, \ldots, a_n] = [f\,a_1, \ldots, f\,a_n]$$

This gives a monoid morphism $F\,A \to F\,B$; and mapping over lists preserves composition ($F(g \circ f) = F\,g \circ F\,f$) and identities ($F\,\mathtt{id}_A = \mathtt{id}_{F\,A}$). So we do get a functor from **Set** to **Mon**.

Examples of functors

If $\mathbf{C}$ is a category with binary products and $X \in \mathbf{C}$, then the function $(\_) \times X : \texttt{obj}\,\mathbf{C} \to \texttt{obj}\,\mathbf{C}$ extends to a functor $(\_) \times X : \mathbf{C} \to \mathbf{C}$ mapping morphisms $f : Y \to Y'$ to

$$f \times \texttt{id}_X : Y \times X \to Y' \times X$$

$$\left(\text{recall that } f \times g \text{ is the unique morphism with } \begin{cases} \pi_1 \circ (f \times g) & = f \circ \pi_1 \\ \pi_2 \circ (f \times g) & = g \circ \pi_2 \end{cases}\right)$$

since it is the case that

$$\begin{cases} \texttt{id}_X \times \texttt{id}_Y & = \texttt{id}_{X \times Y} \\ (f' \circ f) \times \texttt{id}_X & = (f' \times \texttt{id}_X) \circ (f \times \texttt{id}_X) \end{cases}$$

Examples of functors

If $\mathbf{C}$ is a cartesian closed category and $X \in \mathbf{C}$, then the function $(\_)^X : \mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$ extends to a functor $(\_)^X : \mathbf{C} \to \mathbf{C}$ mapping morphisms $f : Y \to Y'$ to

$$f^X \triangleq \mathtt{cur}(f \circ \mathtt{app}) : Y^X \to Y'^X$$

since it is the case that

$$\begin{cases} (\mathtt{id}_Y)^X & = \mathtt{id}_{Y^X} \\ (g \circ f)^X & = g^X \circ f^X \end{cases}$$

Contravariance

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ is called a contravariant functor from $\mathbf{C}$ to $\mathbf{D}$.

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{C}$, then $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathbf{C}^{\mathrm{op}}$

so $F\,X \xleftarrow{F\,f} F\,Y \xleftarrow{F\,g} F\,Z$ in $\mathbf{D}$ and hence

$$\boxed{F(g \circ_{\mathbf{C}} f) = F\,f \circ_{\mathbf{D}} F\,g}$$

(contravariant functors reverse the order of composition)

A functor $\mathbf{C} \to \mathbf{D}$ is sometimes called a covariant functor from $\mathbf{C}$ to $\mathbf{D}$.

Example of a contravariant functor

If $\mathbf{C}$ is a cartesian closed category and $X \in \mathbf{C}$, then the function $X^{(\_)} : \mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$ extends to a functor $X^{(\_)} : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ mapping morphisms $f : Y \to Y'$ to

$$X^f \triangleq \mathtt{cur}(\mathtt{app} \circ (\mathtt{id}_{X^{Y'}} \times f)) : X^{Y'} \to X^Y$$

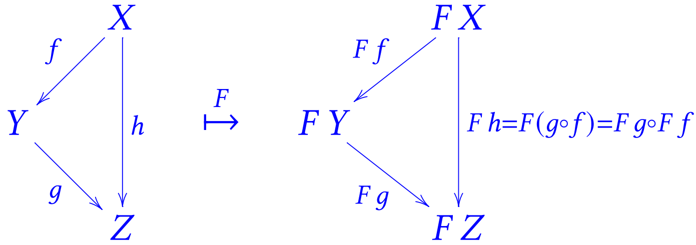
since it is the case that

$$\begin{cases} X^{\mathtt{id}_Y} & = \mathtt{id}_{X^Y} \\ X^{g \circ f} & = X^f \circ X^g \end{cases}$$

Note that since a functor $F : \mathbf{C} \to \mathbf{D}$ preserves domains, codomains, composition, and identity morphisms

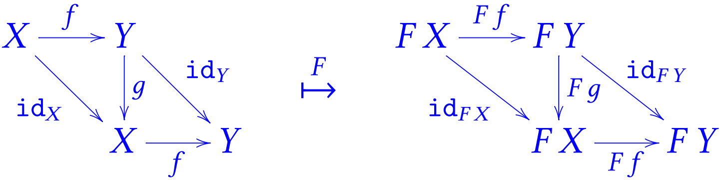
it sends commutative diagrams in $\mathbf{C}$ to commutative diagrams in $\mathbf{D}$

E.g.



Note that since a functor $F : \mathbf{C} \to \mathbf{D}$ preserves domains, codomains, composition, and identity morphisms

it sends isomorphisms in $\mathbf{C}$ to isomorphisms in $\mathbf{D}$, because



so $\boxed{F(f^{-1}) = (F\,f)^{-1}}$

Composing functors

Given functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$, we get a functor $\boxed{G \circ F : \mathbf{C} \to \mathbf{E}}$ with

$$G \circ F \left(\begin{array}{c} X \\ \big\downarrow f \\ Y \end{array} \right) = \begin{array}{c} G(F\,X) \\ \big\downarrow G(F\,f) \\ G(F\,Y) \end{array}$$

(this preserves composition and identity morphisms, because F and G do)

Identity functor

on a category $\mathbf{C}$ is $\boxed{\mathtt{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}}$ where

$$\mathtt{id}_{\mathbf{C}}\left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array}\right) = \begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

Functor composition and identity functors satisfy

associativity	$H \circ (G \circ F) = (H \circ G) \circ F$
unity	$\mathtt{id}_{\mathbf{D}} \circ F = F = F \circ \mathtt{id}_{\mathbf{C}}$

So we can get categories whose objects are categories and whose morphisms are functors

but we have to be a bit careful about size…

Size

One of the axioms of set theory is

set membership is a well-founded relation, that is, there is no infinite sequence of sets $X_0, X_1, X_2, \ldots$ with

$$\cdots \in X_{n+1} \in X_n \in \cdots \in X_2 \in X_1 \in X_0$$

So in particular there is no set X with $X \in X$.

So we cannot form the “set of all sets” or the “category of all categories”.

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So in particular there is no set X with $X \in X$.

So we cannot form the "set of all sets" or the "category of all categories".

But we do assume there are (lots of) big sets

$$\mathscr{U}_0 \in \mathscr{U}_1 \in \mathscr{U}_2 \in \cdots$$

where "big" means each $\mathscr{U}_n$ is a Grothendieck universe...

Grothendieck universes

A Grothendieck universe $\mathscr{U}$ is a set of sets satisfying

- $X \in Y \in \mathscr{U} \Rightarrow X \in \mathscr{U}$
- $X, Y \in \mathscr{U} \Rightarrow \{X, Y\} \in \mathscr{U}$
- $X \in \mathscr{U} \Rightarrow \mathscr{P}X \triangleq \{Y \mid Y \subseteq X\} \in \mathscr{U}$
- $I \in \mathscr{U} \wedge F \in \mathscr{U}^I \Rightarrow$
  $\{x \mid \exists i \in I,\ x \in F\,i\} \in \mathscr{U}$

The above properties are satisfied by $\mathscr{U} = \emptyset$, but we will always assume

- $\mathbb{N} \in \mathscr{U}$

Size

We assume

there is an infinite sequence $\mathscr{U}_0 \in \mathscr{U}_1 \in \mathscr{U}_2 \in \cdots$ of bigger and bigger Grothendieck universes

and revise the previous definition of "the" category of sets and functions:

$\mathbf{Set}_n$ = category whose objects are all the sets in $\mathscr{U}_n$ and with $\mathbf{Set}_n(X, Y) = Y^X =$ all functions from X to Y.

Notation: $\boxed{\mathbf{Set} \triangleq \mathbf{Set}_0}$ — its objects are called small sets (and other sets we call large).

Size

$\mathbf{Set}$ is the category of small sets.

Definition. A category $\mathbf{C}$ is locally small if for all $X, Y \in \mathbf{C}$, the set of $\mathbf{C}$-morphisms $X \to Y$ is small; that is, $\mathbf{C}(X, Y) \in \mathbf{Set}$.

$\mathbf{C}$ is a small category if it is both locally small and $\mathtt{obj}\,\mathbf{C} \in \mathbf{Set}$.

E.g. $\mathbf{Set}$, $\mathbf{Preord}$, and $\mathbf{Mon}$ are all locally small (but not small).

Given $\underline{P} \in \mathbf{Preord}$, the category $\mathbf{C}_{\underline{P}}$ it determines is small; similarly, the category $\mathbf{C}_{\underline{M}}$ determined by $\underline{M} \in \mathbf{Mon}$ is small.

The category of small categories, **Cat**

- objects are all small categories
- morphisms in $\mathbf{Cat}(\mathbf{C}, \mathbf{D})$ are all functors $\mathbf{C} \to \mathbf{D}$
- composition and identity morphisms as for functors

Cat is a locally small category

Problem: Is **Cat** a bicartesian closed category?

Cat has an initial object

The empty category
(with no objects and no morphisms)
is initial in **Cat**.

$\mathbf{Cat}$ has binary coproducts

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their coproduct $\mathbf{C} \xrightarrow{\iota_1} \mathbf{C} + \mathbf{D} \xleftarrow{\iota_2} \mathbf{D}$ is:

Cat has binary coproducts

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their coproduct $\mathbf{C} \xrightarrow{\iota_1} \mathbf{C} + \mathbf{D} \xleftarrow{\iota_2} \mathbf{D}$ is:

- objects: $\mathtt{obj}(\mathbf{C} + \mathbf{D}) \triangleq \mathtt{obj}(\mathbf{C}) + \mathtt{obj}(\mathbf{D})$
- morphisms:

$$
\begin{aligned}
(\mathbf{C} + \mathbf{D})\big(\iota_1(C), \iota_2(C')\big) &\triangleq \mathbf{C}(C, C') \\
(\mathbf{C} + \mathbf{D})\big(\iota_2(D), \iota_2(D')\big) &\triangleq \mathbf{D}(D, D') \\
(\mathbf{C} + \mathbf{D})\big(\iota_1(C), \iota_2(D)\big) &\triangleq \emptyset \\
(\mathbf{C} + \mathbf{D})\big(\iota_2(D), \iota_1(C)\big) &\triangleq \emptyset
\end{aligned}
$$

- composition and identity morphisms are given by those of $\mathbf{C}$ (between objects tagged by ι_1) or $\mathbf{D}$ (between objects tagged by ι_2)

- $$\begin{cases} \iota_1(C \xrightarrow{f} C') & \triangleq & \iota_1(C) \xrightarrow{\iota_1(f)} \iota_1(C') \\ \iota_2(D \xrightarrow{g} D') & \triangleq & \iota_2(D) \xrightarrow{\iota_2(g)} \iota_1(D') \end{cases}$$

Cat has a terminal object

The category

$* \circlearrowright \mathtt{id}_*$

one object, one morphism

is terminal in **Cat**

$\mathbf{Cat}$ has binary products

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their product $\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$ is:

$\mathbf{Cat}$ has binary products

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their product
$\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$ is:

- objects: $\mathtt{obj}(\mathbf{C} \times \mathbf{D}) \triangleq \mathtt{obj}(\mathbf{C}) \times \mathtt{obj}(\mathbf{D})$
- morphisms:

$$(\mathbf{C} \times \mathbf{D})\big((C, D), (C', D')\big) \triangleq \mathbf{C}(C, C') \times \mathbf{D}(D, D')$$

- composition and identity morphisms are given by those of $\mathbf{C}$ (in the first component) and $\mathbf{D}$ (in the second component)

- $$\begin{cases} \pi_1\left((C,D) \xrightarrow{(f,g)} (C',D')\right) = C \xrightarrow{f} C' \\ \pi_2\left((C,D) \xrightarrow{(f,g)} (C',D')\right) = D \xrightarrow{g} D' \end{cases}$$

Cat has exponentials

Exponentials in Cat are called functor categories.

To define them we need to consider natural transformations, which are the appropriate notion of morphism between functors.

Natural transformations

Definition. Given categories and functors $F, G : \mathbf{C} \to \mathbf{D}$, a natural transformation $\boxed{\theta : F \to G}$ is a family of $\mathbf{D}$-morphisms $\theta_X \in \mathbf{D}(F\,X, G\,X)$, one for each $X \in \mathbf{C}$, such that for all $\mathbf{C}$-morphisms $f : X \to Y$, the diagram

$$\begin{array}{ccc} F\,X & \xrightarrow{\;\theta_X\;} & G\,X \\ {\scriptstyle F\,f}\downarrow & & \downarrow{\scriptstyle G\,f} \\ F\,Y & \xrightarrow[\;\theta_Y\;]{} & G\,Y \end{array}$$

commutes in $\mathbf{D}$, that is, $\theta_Y \circ F\,f = G\,f \circ \theta_X$.

Composing natural transformations

Given functors $F, G, H : \mathbf{C} \to \mathbf{D}$ and natural transformations $\theta : F \to G$ and $\varphi : G \to H$,

we get $\boxed{\varphi \circ \theta} : F \to H$ with

$$(\varphi \circ \theta)_X = \left(F\,X \xrightarrow{\theta_X} G\,X \xrightarrow{\varphi_X} H\,X \right)$$

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$$(\varphi \circ \theta)_X = \left(F\,X \xrightarrow{\theta_X} G\,X \xrightarrow{\varphi_X} H\,X \right)$$

Check naturality:

$$\begin{aligned}
H\,f \circ (\varphi \circ \theta)_X &\triangleq H\,f \circ \varphi_X \circ \theta_X \\
&= \varphi_Y \circ G\,f \circ \theta_X && \text{naturality of } \varphi \\
&= \varphi_Y \circ \theta_Y \circ F\,f && \text{naturality of } \theta \\
&\triangleq (\varphi \circ \theta)_Y \circ F\,f
\end{aligned}$$

Identity natural transformation

Given a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$, we get a natural transformation $\boxed{\mathtt{id}_F : F \longrightarrow F}$ with

$$(\mathtt{id}_F)_X = F\,X \xrightarrow{\mathtt{id}_{F\,X}} F\,X$$

Identity natural transformation

Given a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$, we get a natural transformation $\boxed{\mathtt{id}_F : F \longrightarrow F}$ with

$$(\mathtt{id}_F)_X = F\,X \xrightarrow{\mathtt{id}_{F\,X}} F\,X$$

Check naturality:

$$F\,f \circ (\mathtt{id}_F)_X \triangleq F\,f \circ \mathtt{id}_{F\,X} = F\,f = \mathtt{id}_{F\,Y} \circ F\,f \triangleq (\mathtt{id}_F)_Y \circ F\,f$$

Functor categories

It is easy to see that composition and identities for natural transformations satisfy

$$(\psi \circ \varphi) \circ \theta = \psi \circ (\varphi \circ \theta)$$
$$\mathtt{id}_G \circ \theta = \theta \circ \mathtt{id}_F$$

so that we get a category:

Definition. Given categories $\mathbf{C}$ and $\mathbf{D}$, the functor category $\boxed{\mathbf{D}^{\mathbf{C}}}$ has

- objects are all functors $\mathbf{C} \longrightarrow \mathbf{D}$
- given $F, G : \mathbf{C} \longrightarrow \mathbf{D}$, morphism from F to G in $\mathbf{D}^{\mathbf{C}}$ are the natural transformations $F \longrightarrow G$
- composition and identity morphisms as above

If $\mathscr{U}$ is a Grothendieck universe, then for each $I \in \mathscr{U}$ and $F \in \mathscr{U}^I$ we have that their dependent product and dependent function sets

$$\textstyle\sum_{i \in I} F\,i \triangleq \{(i, x) \mid i \in I \wedge x \in F\,i\}$$

$$\textstyle\prod_{i \in I} F\,i \triangleq \{f \subseteq \sum_{i \in I} F\,i \mid f \text{ is single-valued and total}\}$$

are also in $\mathscr{U}$; and, as a special case (of $\prod$, when F is a constant function with value X) we also have that $I, X \in \mathscr{U}$ implies $X^I \in \mathscr{U}$.

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If $\mathbf{C}$ and $\mathbf{D}$ are small categories, then so is $\mathbf{D}^{\mathbf{C}}$.

because

$$\texttt{obj}(\mathbf{D}^{\mathbf{C}}) \subseteq \textstyle\sum_{F \in (\texttt{obj}\,D)^{\texttt{obj}\,\mathbf{C}}} \prod_{X, Y \in \texttt{obj}\,\mathbf{C}} \mathbf{D}(F\,X, F\,Y)^{\mathbf{C}(X,Y)}$$

$$\mathbf{D}^{\mathbf{C}}(F, G) \subseteq \textstyle\prod_{X \in \texttt{obj}\,\mathbf{C}} \mathbf{D}(F\,X, G\,X)$$

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Aim to show that functor category $\mathbf{D}^{\mathbf{C}}$ is the exponential of $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{Cat}$ …

Theorem. There is an application functor

$$\mathtt{app} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \longrightarrow \mathbf{D}$$

that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{Cat}$.

Given $(F, X) \in \mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\mathtt{app}(F, X) \triangleq F\,X$$

and given $(\theta, f) : (F, X) \to (G, Y)$ in $\mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\begin{aligned}\mathtt{app}\left((F, X) \xrightarrow{(\theta, f)} (G, Y)\right) &\triangleq F\,X \xrightarrow{F\,f} F\,Y \xrightarrow{\theta_Y} G\,Y \\ &= F\,X \xrightarrow{\theta_X} G\,X \xrightarrow{G\,f} G\,Y\end{aligned}$$

Check: $\begin{cases}\mathtt{app}(\mathtt{id}_F, \mathtt{id}_X) & = \mathtt{id}_{F\,X} \\ \mathtt{app}(\varphi \circ \theta, g \circ f) & = \mathtt{app}(\varphi, g) \circ \mathtt{app}(\theta, f)\end{cases}$

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that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{Cat}$.

Definition of currying: given functor $F : \mathbf{E} \times \mathbf{C} \longrightarrow \mathbf{D}$, we get a functor $\mathtt{cur}\, F : \mathbf{E} \longrightarrow \mathbf{D}^{\mathbf{C}}$ as follows. For each $Z \in \mathbf{E}$, $\mathtt{cur}\, F\, Z \in \mathbf{D}^{\mathbf{C}}$ is the functor

$$\mathtt{cur}\, F\, Z \left(\begin{array}{c} X \\ \downarrow f \\ X' \end{array} \right) \triangleq \begin{array}{c} F(Z, X) \\ \downarrow F(\mathtt{id}_Z, f) \\ F(Z, X') \end{array}$$

For each $g : Z \longrightarrow Z'$ in $\mathbf{E}$, $\mathtt{cur}\, F\, g : \mathtt{cur}\, F\, Z \longrightarrow \mathtt{cur}\, F\, Z'$ is the natural transformation whose component at each $X \in \mathbf{C}$ is

$$(\mathtt{cur}\, F\, g)_X \triangleq F(g, \mathtt{id}_X) : F(Z, X) \longrightarrow F(Z', X)$$

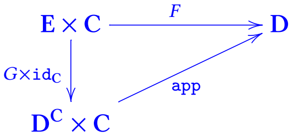
(Check that this is natural in X; and that $\mathtt{cur}\, F$ preserves composition and identities in $\mathbf{E}$.)

Theorem. There is an application functor

$$\mathtt{app} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \longrightarrow \mathbf{D}$$

that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{Cat}$.

Have to check that $\mathtt{cur}\, F$ is the unique functor $G : \mathbf{E} \longrightarrow \mathbf{D}^{\mathbf{C}}$ that makes



commute in $\mathbf{Cat}$ (exercise).