Applications of Continuations
We have seen that:

- Classical logic has a beautiful inference system
- Embeds into constructive logic via double-negation translations
- This yields an operational interpretation

**What can we program with continuations?**
The Typed Lambda Calculus with Continuations

Types  \[ X ::= 1 \mid X \times Y \mid 0 \mid X + Y \mid X \rightarrow Y \mid \neg X \]

Terms  \[ e ::= x \mid \langle \rangle \mid \langle e, e \rangle \mid \text{fst } e \mid \text{snd } e \]
\[ \text{abort} \mid L e \mid R e \mid \text{case}(e, Lx \rightarrow e', Ry \rightarrow e'') \]
\[ \lambda x : X. e \mid e e' \]
\[ \text{throw}(e, e') \mid \text{letcont } x. e \]

Contexts  \[ \Gamma ::= \cdot \mid \Gamma, x : X \]
Continuation Typing

\[ \frac{\Gamma, u : \neg X \vdash e : X}{\Gamma \vdash \text{letcont } u : \neg X. e : X} \quad \text{CONT} \]

\[ \frac{\Gamma \vdash e : \neg X \quad \Gamma \vdash e' : X}{\Gamma \vdash \text{throw}_Y(e, e') : Y} \quad \text{THROW} \]
signature CONT = sig
  type 'a cont
  val callcc : ('a cont -> 'a) -> 'a
  val throw : 'a cont -> 'a -> 'b
end

<table>
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<th>SML</th>
<th>Type Theory</th>
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<td>'a cont</td>
<td>¬A</td>
</tr>
<tr>
<td>throw k v</td>
<td>throw(k, v)</td>
</tr>
<tr>
<td>callcc (fn x =&gt; e)</td>
<td>letcont x : ¬X. e</td>
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An Inefficient Program

val mul : int list -> int

fun mul [] = 1
  | mut (n :: ns) = n * mul ns

• This function multiplies a list of integers
• If 0 occurs in the list, the whole result is 0
A Less Inefficient Program

val mul' : int list -> int

fun mul' [] = 1
  | mul' (0 :: ns) = 0
  | mul' (n :: ns) = n * mul' ns

• This function multiplies a list of integers
• If 0 occurs in the list, it immediately returns 0
  • mul' [0,1,2,3,4,5,6,7,8,9] will immediately return
  • mul' [1,2,3,4,5,6,7,8,9,0] will multiply by 0, 9 times
Even Less Inefficiency, via Escape Continuations

```ocaml
val loop = fn : int cont -> int list -> int
fun loop return [] = 1
  | loop return (0 :: ns) = throw return 0
  | loop return (n :: ns) = n * loop return ns

val mul_fast : int list -> int
fun mul_fast ns = callcc (fn ret => loop ret ns)
```

- `loop` multiplies its arguments, unless it hits 0
- In that case, it throws 0 to its continuation
- `mul_fast` captures its continuation, and passes it to `loop`
- So if `loop` finds 0, it does no multiplications!
In 1961, John McCarthy (inventor of Lisp) proposed a language construct \texttt{amb}.

This was an operator for \textit{angelic nondeterminism}.

```plaintext
let val x = amb [1,2,3]
val y = amb [4,5,6]

in

assert (x * y = 10);
(x, y)

end

(* Returns (2,5) *)
```

- Does search to find a successful assignment of values.
- Can be implemented via backtracking – \textit{using continuations}.
signature AMB = sig
  (* Internal implementation *)
  val stack : int option cont list ref
  val fail : unit -> 'a

  (* External API *)
  exception AmbFail
  val assert : bool -> unit
  val amb : int list -> int
end
exception AmbFail
val stack : int option cont list ref = ref []

fun fail () =
  case !stack of
  [] => raise AmbFail
  | (k :: ks) => (stack := ks;
    throw k NONE)

fun assert b =
  if b then () else fail()
fun amb [] = fail ()
| amb (x :: xs) =
  let
    fun next y k =
      (stack := k :: !stack;
       SOME y)
  in
    case callcc (next x) of
      SOME v => v
    | NONE => amb xs
  end

• amb [] backtracks immediately!
• next y k pushes k onto the backtrack stack, and returns SOME y
• Save the backtrack point, then see if we immediately return, or
  if we are resuming from a backtrack point and must try the other values
fun test2() =
  let val x = amb [1,2,3,4,5,6]
  val y = amb [1,2,3,4,5,6]
  val z = amb [1,2,3,4,5,6]
  in
  assert(x + y + z >= 13);
  assert(x > 1);
  assert(y > 1);
  assert(z > 1);
  (x, y, z)
  end

(* Returns (2, 5, 6) *)
Conclusions

- **amb** required the *combination* of state and continuations
- Theorem of Andrzej Filinski that this is *universal*
- Any “definable monadic effect” can be expressed as a combination of state and first-class control:
  - Exceptions
  - Green threads
  - Coroutines/generators
  - Random number generation
  - Nondeterminism
Dependent Types
The Curry Howard Correspondence

<table>
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<th>Logic</th>
<th>Language</th>
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<td>Intuitionistic Propositional Logic</td>
<td>STLC</td>
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<tr>
<td>Classical Propositional Logic</td>
<td>STLC + 1&lt;sup&gt;st&lt;/sup&gt; class continuations</td>
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<tr>
<td>Pure Second-Order Logic</td>
<td>System F</td>
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- Each logical system has a corresponding computational system
- One thing is missing, however
- Mathematics uses quantification over *individual elements*
- Eg, $\forall x, y, z, n \in \mathbb{N}$. if $n > 2$ then $x^n + y^n \neq z^n$
A Logical Curiosity

\[ \Gamma \vdash z : \mathbb{N} \quad \Gamma \vdash s(e) : \mathbb{N} \]

\[ \Gamma \vdash e_0 : \mathbb{N} \quad \Gamma \vdash e_1 : X \quad \Gamma, x : X \vdash e_2 : X \]

\[ \Gamma \vdash \text{iter}(e_0, z \rightarrow e_1, s(x) \rightarrow e_2) : X \]

- \( \mathbb{N} \) is the type of natural numbers
- Logically, it is equivalent to the unit type:
  - \((\lambda x : 1. z) : 1 \rightarrow \mathbb{N}\)
  - \((\lambda x : \mathbb{N}. ()') : \mathbb{N} \rightarrow 1\)
- Language of types has no way of distinguishing \( z \) from \( s(z) \).
Dependent Types

- Language of types has no way of distinguishing $z$ from $s(z)$.
- So let’s fix that: let types refer to values
- Type grammar and term grammar mutually recursive
- Huge gain in expressive power
• Much of earlier course leaned on prior knowledge of ML for motivation
• Before we get to the theory of dependent types, let’s look at an implementation
• Agda: a dependently-typed functional programming language
• http://wiki.portal.chalmers.se/agda/pmwiki.php
• Datatype declarations give constructors and their types
• Functions given type signature, and clausal definition

```agda
data Bool : Set where
  true : Bool
  false : Bool

not : Bool → Bool
not true = false
not false = true
```
data Nat : Set where
  z : Nat
  s : Nat → Nat

_+_ : Nat → Nat → Nat
z + m = m
s n + m = s (n + m)

_×_ : Nat → Nat → Nat
z × m = z
s n × m = m + (n × m)

• Datatype constructors can be recursive
• Functions can be recursive, but checked for termination
Datatypes can be polymorphic

- `app` has F-style explicit polymorphism
- `app'` has implicit, inferred polymorphism
data Vec (A : Set) : Nat → Set where

[] : Vec A z

_,_ : {n : Nat} → A → Vec A n → Vec A (s n)

• This is a length-indexed list
• Cons takes a head and a list of length \(n\), and produces a list of length \(n + 1\)
• The empty list has a length of 0
Agda: Indexed Datatypes

```agda
data Vec (A : Set) : Nat → Set where
  [] : Vec A z
  _,_ : {n : Nat} → A → Vec A n → Vec A (s n)

head : {A : Set} → {n : Nat} → Vec A (s n) → A
head (x , xs) = x

• head takes a list of length > 0, and returns an element
• No [] pattern present
• Not needed for coverage checking!
• Note that {n:Nat} is also an implicit (inferred) argument
```
\textbf{data Vec (A : Set) : Nat → Set where}

\noindent \textbullet\ \texttt{[]} : Vec A z

\noindent \textbullet\ \texttt{_,_} : \{n : Nat\} → A → Vec A n → Vec A (s n)

\noindent \textbullet\ \texttt{app} : \{A : Set\} → \{n m : Nat\} →

\noindent \hspace{1cm} Vec A n → Vec A m → Vec A (n + m)

\noindent \texttt{app \texttt{[]} ys = ys}

\noindent \texttt{app (x , xs) ys = (x , app xs ys)}

\noindent \hspace{1cm} \textbullet\ Note the appearance of \texttt{n + m} in the type

\noindent \hspace{1cm} \textbullet\ This type guarantees that appending two vectors yields a vector whose

\hspace{1cm} \textbullet\ length is the sum of the two
data Vec (A : Set) : Nat → Set where

[] : Vec A z
_,_ : {n : Nat} → A → Vec A n → Vec A (s n)

-- Won't typecheck!

app : {A : Set} → {n m : Nat} →
    Vec A n → Vec A m → Vec A (n + m)

app [] ys = ys
app (x , xs) ys = app xs ys

• We forgot to cons x here
• This program won’t type check!
• Static typechecking ensures a runtime guarantee
The Identity Type

data _≡_ {A : Set} (a : A) : A → Set where
  refl : a ≡ a

• \(a \equiv b\) is the type of proofs that \(a\) and \(b\) are equal
• The constructor \(\text{refl}\) says that a term \(a\) is equal to itself
• Equalities arising from evaluation are automatic
• Other equalities have to be proved
An Automatic Theorem

```
data _≡_ {A : Set} (a : A) : A → Set where
  refl : a ≡ a

_+_ : Nat → Nat → Nat
z + m = m
s n + m = s (n + m)

z+-left-unit : (n : Nat) → (z + n) ≡ n
z+-left-unit n = refl

• z + n evaluates to n
  • So Agda considers these two terms to be identical
```
A Manual Theorem

```
data _≡_ {A : Set} (a : A) : A → Set where
    refl : a ≡ a

cong : {A B : Set} → {a a' : A} →
       (f : A → B) → (a ≡ a') → (f a ≡ f a')
cong f refl = refl

z+-right-unit : (n : Nat) → (n + z) ≡ n
z+-right-unit z = refl
z+-right-unit (s n) = cong s (z+-right-unit n)
```

- We prove the right unit law inductively
- Note that *inductive proofs are recursive functions*
- To do this, we need to show that equality is a congruence
data ≡ {A : Set} (a : A) : A → Set where
  refl : a ≡ a

sym : {A : Set} → {a b : A} →
    a ≡ b → b ≡ a
sym refl = refl

trans : {A : Set} → {a b c : A} →
    a ≡ b → b ≡ c → a ≡ c
trans refl refl = refl

cong : {A B : Set} → {a a' : A} →
    (f : A → B) → (a ≡ a') → (f a ≡ f a')
cong f refl = refl

• An equivalence relation is a reflexive, symmetric transitive relation
• Equality is congruent with everything
Commutativity of Addition

\[ z-+-\text{right} : (n : \text{Nat}) \rightarrow (n + z) \equiv n \]
\[ z-+-\text{right} z = \text{refl} \]
\[ z-+-\text{right} (s n) = \]
\[ \text{cong} \ s (z-+-\text{right} n) \]

\[ s-+-\text{right} : (n m : \text{Nat}) \rightarrow \]
\[ (s (n + m)) \equiv (n + (s m)) \]
\[ s-+-\text{right} z m = \text{refl} \]
\[ s-+-\text{right} (s n) m = \text{cong} \ s (s-+-\text{right} n m) \]

\[ +\text{-comm} : (i j : \text{Nat}) \rightarrow (i + j) \equiv (j + i) \]
\[ +\text{-comm} z j = z-+-\text{right} j \]
\[ +\text{-comm} (s i) j = \text{trans} p2 p3 \]
\[ \text{where} \ p1 : (i + j) \equiv (j + i) \]
\[ p1 = +\text{-comm} i j \]
\[ p2 : (s (i + j)) \equiv (s (j + i)) \]
\[ p2 = \text{cong} \ s \ p1 \]
\[ p3 : (s (j + i)) \equiv (j + (s i)) \]
\[ p3 = s-+-\text{right} j i \]

• First we prove that adding zero on the right does nothing
• Then we prove that successor commutes with addition
• Then we use these two facts to inductively prove commutativity of addition
Dependent types permit referring to program terms in types
This enables writing types which state very precise properties of programs
  · Eg, equality is expressible as a type
Writing a program becomes the same as proving it correct
This is hard, like learning to program again!
But also extremely fun…