Randomised Algorithms
Lecture 1: Introduction to Course & Introduction to Chernoff Bounds

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Outline

Introduction

Topics and Syllabus

A (Very) Brief Reminder of Probability Theory

Basic Examples

Introduction to Chernoff Bounds
**Randomised Algorithms**

**What?** Randomised Algorithms utilise random bits to compute their output.

**Why?** Randomised Algorithms often provide an efficient (and elegant!) solution or approximation to a problem that is costly (or impossible) to solve deterministically. But often: simple algorithm at the cost of a sophisticated analysis!

"... If somebody would ask me, what in the last 10 years, what was the most important change in the study of algorithms I would have to say that people getting really familiar with randomised algorithms had to be the winner."

- Donald E. Knuth (in *Randomization and Religion*)

**How?** This course aims to strengthen your knowledge of probability theory and apply this to analyse examples of randomised algorithms.

What if I (initially) don’t care about randomised algorithms? Many of the techniques in this course (Markov Chains, Concentration of Measure, Spectral Theory) are very relevant to other popular areas of research and employment such as Data Science and Machine Learning.
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Some stuff you should know...

In this course we will assume some basic knowledge of probability:

- random variable
- computing expectations and variances
- notions of independence
- “general” idea of how to compute probabilities (manipulating, counting and estimating)

![Dice Roll](image-url)
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You should also be familiar with basic computer science, mathematics knowledge such as:

- graphs
- basic algorithms (sorting, graph algorithms etc.)
- matrices, norms and vectors
Textbooks

(We will adopt some of the labels (e.g., Theorem 35.6) from this book in Lectures 6-10)
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1 Introduction (Lecture)

- Intro to Randomised Algorithms; Logistics; Recap of Probability; Examples.

Lectures 2-5 focus on probabilistic tools and techniques.

2–3 Concentration (Lectures)

- Concept of Concentration; Recap of Markov and Chebyshev; Chernoff Bounds and Applications; Extensions: Hoeffding's Inequality and Method of Bounded Differences; Applications.

4 Markov Chains and Mixing Times (Lecture)

- Recap; Stopping and Hitting Times; Properties of Markov Chains; Convergence to Stationary Distribution; Variation Distance and Mixing Time

5 Hitting Times and Application to 2-SAT (Lecture)

- Reversible Markov Chains and Random Walks on Graphs; Cover Times and Hitting Times on Graphs (Example: Paths and Grids); A Randomised Algorithm for 2-SAT

Lectures 6-8 introduce linear programming, a (mostly) deterministic but very powerful technique to solve various optimisation problems.

6–7 Linear Programming (Lectures)

- Introduction to Linear Programming, Applications, Standard and Slack Forms, Simplex Algorithm, Finding an Initial Solution, Fundamental Theorem of Linear Programming

8 Travelling Salesman Problem (Interactive Demo)

- Hardness of the general TSP problem, Formulating TSP as an integer program; Classical TSP instance from 1954; Branch & Bound Technique to solve integer programs using linear programs
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We then see how we can efficiently combine linear programming with randomised techniques, in particular, rounding:

9–10 **Randomised Approximation Algorithms** (Lectures)
- MAX-3-CNF and Guessing, Vertex-Cover and Deterministic Rounding of Linear Program, Set-Cover and Randomised Rounding, Concluding Example: MAX-CNF and Hybrid Algorithm
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Lectures 11-12 cover a more advanced topic with ML flavour:

11–12 **Spectral Graph Theory and Spectral Clustering**  (Lectures)

- Eigenvalues, Eigenvectors and Spectrum; Visualising Graphs; Expansion; Cheeger’s Inequality; Clustering and Examples; Analysing Mixing Times
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Recap: Probability Space

In probability theory we wish to evaluate the likelihood of certain results from an experiment. The setting of this is the probability space \((\Omega, \Sigma, P)\).
Recap: Probability Space

In probability theory we wish to evaluate the likelihood of certain results from an experiment. The setting of this is the **probability space** \((\Omega, \Sigma, P)\).

### Components of the Probability Space \((\Omega, \Sigma, P)\)

- **The Sample Space** \(\Omega\) contains all the possible outcomes \(\omega_1, \omega_2, \ldots\) of the experiment.
- **The Event Space** \(\Sigma\) is the power-set of \(\Omega\) containing events, which are combinations of outcomes (subsets of \(\Omega\) including \(\emptyset\) and \(\Omega\)).
- **The Probability Measure** \(P\) is a function from \(\Sigma\) to \(\mathbb{R}\) satisfying
  - (i) \(0 \leq P[\mathcal{E}] \leq 1\), for all \(\mathcal{E} \in \Sigma\)
  - (ii) \(P[\Omega] = 1\)
  - (iii) If \(\mathcal{E}_1, \mathcal{E}_2, \ldots \in \Sigma\) are pairwise disjoint (\(\mathcal{E}_i \cap \mathcal{E}_j = \emptyset\) for all \(i \neq j\)) then

\[
P\left[\bigcup_{i=1}^{\infty} \mathcal{E}_i\right] = \sum_{i=1}^{\infty} P[\mathcal{E}_i].
\]
Recap: Random Variables

A random variable $X$ on $(\Omega, \Sigma, P)$ is a function $X : \Omega \rightarrow \mathbb{R}$ mapping each sample “outcome” to a real number.

Intuitively, random variables are the “observables” in our experiment.
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Examples of random variables

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  $$X_1 + X_2 + X_3$$
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- The indicator random variable $1_{\mathcal{E}}$ of an event $\mathcal{E} \in \Sigma$ given by

  $$1_{\mathcal{E}}(\omega) = \begin{cases} 
  1 & \text{if } \omega \in \mathcal{E} \\
  0 & \text{otherwise.}
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### Examples of random variables

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- The indicator random variable $1_E$ of an event $E \in \Sigma$ given by
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  1_E(\omega) = \begin{cases} 
  1 & \text{if } \omega \in E \\
  0 & \text{otherwise}
  \end{cases}
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  For the indicator random variable $1_E$ we have $E[1_E] = P[E]$.

- The number of sixes of two dice throws $X_1, X_2 \in \{1, 2, \ldots, 6\}$ is
  \[
  1_{X_1=6} + 1_{X_2=6}
  \]
Recap: Boole’s Inequality (Union Bound)

Let $E_1, \ldots, E_n$ be a collection of events in $\Sigma$. Then

$$P\left[ \bigcup_{i=1}^{n} E_i \right] \leq \sum_{i=1}^{n} P[E_i].$$

Union Bound

---

A Proof using Indicator Random Variables:

1. Let $1_{E_i}$ be the random variable that takes value 1 if $E_i$ holds, 0 otherwise.
2. $E[1_{E_i}] = P[E_i]$ (Check this).
3. It is clear that $1_{\bigcup_{i=1}^{n} E_i} \leq \sum_{i=1}^{n} 1_{E_i}$ (Check this).
4. Taking expectation completes the proof.
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A Randomised Algorithm for MAX-CUT (1/2)

$E(A, B)$: set of edges with one endpoint in $A \subseteq V$ and the other in $B \subseteq V$. 

Given: Undirected graph $G = (V, E)$
Goal: Find $S \subseteq V$ such that $e(S, Sc) := |E(S, Sc)|$ is maximised.

**MAX-CUT Problem**

Applications:
- network design
- VLSI design
- clustering
- statistical physics

Comments:
- This problem will appear again in the course.
- MAX-CUT is NP-hard.
- It is different from the clustering problem, where we want to find a sparse cut.
- Note that the MIN-CUT problem is solvable in polynomial time!
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S &= \{a, b, e\} \\
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$S = \{a, b, e\}$
$e(S, S^c) = 6$
A Randomised Algorithm for MAX-CUT (2/2)

\[
\text{RANDMAXCut}(G)
\]

1: Start with \( S \leftarrow \emptyset \)
2: For each \( v \in V \), add \( v \) to \( S \) with probability \( 1/2 \)
3: Return \( S \)

RAND MAX CUT (G) gives a 2-approximation using time \( O(n) \).

Proposition
More details on approximation algorithms from Lecture 9 onwards!

This kind of "random guessing" will appear often in this course!

Later: learn stronger tools that imply concentration around the expectation!

Proof:
We need to analyse the expectation of \( e(S, S_c) \):

\[
E[e(S, S_c)] = \sum_{\{u, v\} \in E} E[\{u \in S, v \in S_c\} \cup \{u \in S_c, v \in S\}]
\]

\[
= \sum_{\{u, v\} \in E} P[\{u \in S, v \in S_c\} \cup \{u \in S_c, v \in S\}]
\]

\[
= \sum_{\{u, v\} \in E} P[\{u \in S\}] \cdot P[\{v \in S_c\}]
\]

\[
= |E|/2.
\]

Since for any \( S \subseteq V \), we have \( e(S, S_c) \leq |E| \), the proof is complete.
A Randomised Algorithm for MAX-CUT (2/2)

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**Proposition**

RANDMaxCut($G$) gives a 2-approximation using time $O(n)$.

**Question:**

1. What is the sample space $\Omega$ here?
2. Which quantity do we need to analyse?
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\textbf{RANDMAXCUT}(G) gives a \textit{2-approximation} using time \( O(n) \).

**Proof:**

- We need to analyse the \textit{expectation} of \( e(S, S^c) \):

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E \left[ e(S, S^c) \right] = E \left[ \sum_{\{u,v\} \in E} 1_{\{u \in S, v \in S^c\} \cup \{u \in S^c, v \in S\}} \right]
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Since for any \( S \subseteq V \), we have \( e(S, S^c) \leq |E| \), the proof is complete.
A Randomised Algorithm for MAX-CUT (2/2)

**RANDMAXCUT(G)**

1: Start with $S \leftarrow \emptyset$

2: For each $v \in V$, add $v$ to $S$ with probability $1/2$

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A Randomised Algorithm for MAX-CUT (2/2)

\[ \text{RN\textsc{dMaxCut}(G)} \]

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Basic Examples 15
A Randomised Algorithm for MAX-CUT (2/2)

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Suppose that there are $n$ coupons to be collected from the cereal box. Every morning you open a new cereal box and get one coupon. We assume that each coupon appears with the same probability in the box.
Example: Coupon Collector

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Source: https://www.express.co.uk/life-style/life/567954/Discount-codes-money-saving-vouchers-coupons-mum

This is a very important example in the design and analysis of randomised algorithms.

**Coupon Collector Problem**

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Example Sequence for \( n = 8 \): 7, 6, 3, 3, 2, 5, 4, 2, 4, 1, 4, 2, 1, 4, 3, 1, 4, 8 ✓
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Exercise ( [Ex. 1.11] )

1. Prove it takes \( n \sum_{k=1}^{n} \frac{1}{k} \approx n \log n \) expected boxes to collect all coupons

2. Use Union Bound to prove that the probability it takes more than \( n \log n + cn \) boxes to collect all \( n \) coupons is \( \leq e^{-c} \).

Hint: It is useful to remember that \( 1 - x \leq e^{-x} \) for all \( x \).

In this course: \( \log n = \ln n \)
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Outline

Introduction

Topics and Syllabus

A (Very) Brief Reminder of Probability Theory

Basic Examples

Introduction to Chernoff Bounds
Concentration Inequalities

- **Concentration** refers to the phenomena where random variables are very close to their mean.
- This is very useful in randomised algorithms as it ensures an **almost** deterministic behaviour.
Concentration Inequalities

- **Concentration** refers to the phenomena where random variables are very close to their mean.
- This is very useful in randomised algorithms as it ensures an almost deterministic behaviour.
- It gives us the best of two worlds:
  1. **Randomised Algorithms**: Easy to Design and Implement
  2. **Deterministic Algorithms**: They do what they claim
Chernoff Bounds: A Tool for Concentration (1952)

- Chernoff’s bounds are “strong” bounds on the tail probabilities of sums of independent random variables
- Random variables can be discrete (or continuous)
- Usually these bounds decrease exponentially as opposed to a polynomial decrease in Markov’s or Chebyshev’s inequality (see example)

Hermann Chernoff (1923-)

\[(1 + \delta)\mu(1 - \delta)\mu\]

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Introduction to Chernoff Bounds

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Hermann Chernoff (1923-)

![Image of Chernoff](image-url)

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Introduction to Chernoff Bounds 19
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- have found various applications in:
  - Randomised Algorithms
  - Statistics
  - Random Projections and Dimensionality Reduction
  - Learning Theory (e.g., PAC-learning)

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Introduction to Chernoff Bounds
Recap: Markov and Chebyshev

**Markov’s Inequality**

If $X$ is a non-negative random variable, then for any $a > 0$,

$$P[X \geq a] \leq \frac{E[X]}{a}.$$ 

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If $X$ is a random variable, then for any $a > 0$,

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- Let $f : \mathbb{R} \rightarrow [0, \infty)$ and **increasing**, then $f(X) \geq 0$, and thus

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P[X \geq a] \leq P[f(X) \geq f(a)] \leq \frac{E[f(X)]}{f(a)}.
  \]
- Similarly, if $g : \mathbb{R} \to [0, \infty)$ and decreasing, then $g(X) \geq 0$, and thus
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P[X \leq a] \leq P[g(X) \geq g(a)] \leq \frac{E[g(X)]}{g(a)}.
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Chebyshev’s inequality (or Markov) can be obtained by choosing $f(X) := (X - \mu)^2$ (or $f(X) := X$, respectively).
Markov and Chebyshev use the first and second moment of the random variable. Can we keep going?
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- Yes!
Markov and Chebyshev use the **first and second moment** of the random variable. Can we keep going?

- Yes!

We can consider the first, second, **third and more** moments! That is the basic idea behind the **Chernoff Bounds**.
Our First Chernoff Bound

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then, for any $\delta > 0$ it holds that

$$
P[X \geq (1 + \delta)\mu] \leq \left[ \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right]^\mu. \quad (\star)
$$

Chernoff Bounds (General Form, Upper Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then, for any $\delta > 0$ it holds that

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While (★) is one of the easiest (and most generic) Chernoff bounds to derive, the bound is complicated and hard to apply...
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$$\mathbb{P}[X \geq (1 + \delta)\mu] \leq \left[\frac{e^{\delta}}{(1 + \delta)^{(1+\delta)}}\right]^{\mu}. \quad (\star)$$

This implies that for any $t > \mu$,

$$\mathbb{P}[X \geq t] \leq e^{-\mu} \left(\frac{e\mu}{t}\right)^{t}.$$

While $(\star)$ is one of the easiest (and most generic) Chernoff bounds to derive, the bound is complicated and hard to apply...
Consider throwing a fair coin $n$ times and count the total number of heads.
Example: Coin Flips (1/3)

- Consider throwing a fair coin $n$ times and count the total number of heads $X_i \in \{0, 1\}$, $X = \sum_{i=1}^{n} X_i$ and $E[X] = n \cdot 1/2 = n/2$
Consider throwing a fair coin \( n \) times and count the total number of heads

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The above expression equals 1 only for $\delta = 0$, and then it gives a value strictly less than 1 (check this!)

⇒ The inequality is **exponential in** $n$, (for fixed $\delta$) which is much better than Chebyshev’s inequality.
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⇒ The inequality is **exponential in** \( n \), (for fixed \( \delta \)) which is much better than Chebyshev’s inequality.

What about a concrete value of \( n \), say \( n = 100 \)?
Example: Coin Flips (2/3)

\[ P[ \text{Bin}(100, 1/2) = x ] \]
Example: Coin Flips (3/3)

Consider \( n = 100 \) independent coin flips. We wish to find an upper bound on the probability that the number of heads is greater or equal than 75.
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- **Markov’s inequality**: \( \mathbb{E}[X] = 100/2 = 50. \)

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P[X \geq 3/2 \cdot \mathbb{E}[X]] \leq 2/3 = 0.666.
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\[
P[|X - \mu| \geq t] \leq \frac{\mathbb{V}[X]}{t^2},
\]

and plugging in \( t = 25 \) gives an upper bound of \( 25/25^2 = 1/25 = 0.04 \), much better than what we obtained by Markov’s inequality.
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  and plugging in \( t = 25 \) gives an upper bound of \( 25/25^2 = 1/25 = 0.04 \), much better than what we obtained by Markov’s inequality.

- **Chernoff bound:** setting \( \delta = 1/2 \) gives
  
  \[
  P[X \geq 3/2 \cdot \mathbb{E}[X]] \leq \left(\frac{e^{1/2}}{(3/2)^{3/2}}\right)^{50} = 0.004472.
  \]
Consider $n = 100$ independent coin flips. We wish to find an upper bound on the probability that the number of heads is greater or equal than 75.

- **Markov’s inequality:** $\mathbb{E}[X] = \frac{100}{2} = 50$.

  $$\Pr[X \geq \frac{3}{2} \cdot \mathbb{E}[X]] \leq \frac{2}{3} = 0.666.$$

- **Chebyshev’s inequality:** $\mathbb{V}[X] = \sum_{i=1}^{100} \mathbb{V}[X_i] = 100 \cdot (1/2)^2 = 25$.

  $$\Pr[|X - \mu| \geq t] \leq \frac{\mathbb{V}[X]}{t^2},$$

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- **Remark:** The exact probability is $0.00000028 \ldots$
Consider \( n = 100 \) independent coin flips. We wish to find an upper bound on the probability that the number of heads is greater or equal than 75.

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  \]

- **Remark**: The exact probability is \( 0.00000028 \ldots \)

**Chernoff bound yields a much better result (but needs independence!)**
Randomised Algorithms
Lecture 2: Concentration Inequalities, Application to Balls-into-Bins

Thomas Sauerwald (tms41@cam.ac.uk)
How to Derive Chernoff Bounds

Application 1: Balls into Bins
The three main steps in deriving Chernoff bounds for sums of independent random variables $X = X_1 + \cdots + X_n$ are:

1. Instead of working with $X$, we switch to the moment generating function $e^{\lambda X}$, $\lambda > 0$ and apply Markov’s inequality.
2. Compute an upper bound for $E[e^{\lambda X}]$ (using independence).
3. Optimise value of $\lambda$ to obtain best tail bound.
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Recipe
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The three main steps in deriving Chernoff bounds for sums of independent random variables $X = X_1 + \cdots + X_n$ are:

1. Instead of working with $X$, we switch to the moment generating function $e^{\lambda X}$, $\lambda > 0$ and apply Markov's inequality $\sim \mathbb{E}[e^{\lambda X}]$

2. Compute an upper bound for $\mathbb{E}[e^{\lambda X}]$ (using independence)

3. Optimise value of $\lambda$ to obtain best tail bound
Chernoff Bound: Proof

Suppose \( X_1, \ldots, X_n \) are independent Bernoulli random variables with parameter \( p_i \). Let \( X = X_1 + \ldots + X_n \) and \( \mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i \). Then, for any \( \delta > 0 \) it holds that

\[
\mathbb{P}[X \geq (1 + \delta)\mu] \leq \left[ \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right]^{\mu}.
\]

Proof:
Suppose \( X_1, \ldots, X_n \) are independent Bernoulli random variables with parameter \( p_i \). Let \( X = X_1 + \ldots + X_n \) and \( \mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i \). Then, for any \( \delta > 0 \) it holds that

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P \left[ X \geq (1 + \delta)\mu \right] \leq \left( \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^\mu.
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Proof:

1. For \( \lambda > 0 \),

\[
P \left[ X \geq (1 + \delta)\mu \right] \leq \mathbb{P} \left[ e^{\lambda X} \geq e^{\lambda (1+\delta)\mu} \right] \leq e^{-\lambda(1+\delta)\mu} \mathbb{E} \left[ e^{\lambda X} \right]
\]

where \( e^{\lambda X} \) is increasing.
Chernoff Bound: Proof

Suppose \( X_1, \ldots, X_n \) are independent Bernoulli random variables with parameter \( p_i \). Let \( X = X_1 + \ldots + X_n \) and \( \mu = \mathbb{E}[X] = \sum_{i=1}^n p_i \). Then, for any \( \delta > 0 \) it holds that

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P \left[ X \geq (1 + \delta)\mu \right] \leq \left[ \frac{e^{\delta}}{(1 + \delta)^{(1+\delta)}} \right]^{\mu}.
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\]

2. \( \mathbb{E} \left[ e^{\lambda X} \right] = \mathbb{E} \left[ e^{\lambda \sum_{i=1}^n X_i} \right] = \prod_{i=1}^n \mathbb{E} \left[ e^{\lambda X_i} \right] \)
Chernoff Bound: Proof

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P[X \geq (1 + \delta)\mu] \leq \left[\frac{e^\delta}{(1 + \delta)^{(1+\delta)}}\right]^{\mu}.
$$

Proof:

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P[X \geq (1 + \delta)\mu] \leq P[e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}] \leq e^{-\lambda(1+\delta)\mu} \mathbb{E}[e^{\lambda X}].$$

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3. $\mathbb{E}[e^{\lambda X_i}] = e^{\lambda p_i} + (1 - p_i) = 1 + p_i(e^{\lambda} - 1) \leq e^{p_i(e^{\lambda} - 1)}$
Chernoff Bound: Proof

1. For \( \lambda > 0 \),
\[
P[X \geq (1 + \delta)\mu] = e^{\lambda x} \text{ is incr}
\]
\[
P[e^{\lambda X} \geq e^{\lambda (1 + \delta)\mu}] \leq Markov e^{-\lambda (1 + \delta)\mu} E[e^{\lambda X}]
\]

2. \( E[e^{\lambda X}] = E[\prod_{i=1}^{n} e^{\lambda X_i}] = \prod_{i=1}^{n} E[e^{\lambda X_i}] = \prod_{i=1}^{n} e^{\lambda p_i} \)
\[
\leq 1 + x \leq e^x \leq e^{p_i(e^{\lambda} - 1)}
\]

3. Putting all together
\[
P[X \geq (1 + \delta)\mu] \leq e^{-\lambda (1 + \delta)\mu} \prod_{i=1}^{n} e^{p_i(e^{\lambda} - 1)} = e^{-\lambda (1 + \delta)\mu} e^{\mu(e^{\lambda} - 1)}
\]

4. Choose \( \lambda = \log(1 + \delta) > 0 \) to get the result.
Chernoff Bound: Proof

1. For $\lambda > 0$,

$$
P [ X \geq (1 + \delta)\mu ] = e^{\lambda x} \text{ is incr } P [ e^{\lambda X} \geq e^{\lambda (1 + \delta)\mu} ] \leq e^{-\lambda (1 + \delta)\mu} E [ e^{\lambda X} ] \text{ Markov}$$

2. $E [ e^{\lambda X} ] = E [ e^{\lambda \sum_{i=1}^{n} X_i} ] = \prod_{i=1}^{n} E [ e^{\lambda X_i} ]$

3. $E [ e^{\lambda X_i} ] = e^{\lambda p_i} + (1 - p_i) = 1 + p_i(e^{\lambda} - 1) \leq e^{p_i(e^{\lambda} - 1)}$ with $1 + x \leq e^x$
Chernoff Bound: Proof

1. For \( \lambda > 0, \)

\[ P \left[ X \geq (1 + \delta)\mu \right] = e^{\lambda x} \text{ incr} \]

\[ \leq e^{-\lambda(1+\delta)\mu} E \left[ e^{\lambda X} \right] \quad \text{Markov} \]

2. \[ E \left[ e^{\lambda X} \right] = E \left[ e^{\lambda \sum_{i=1}^{n} X_i} \right] = \prod_{i=1}^{n} E \left[ e^{\lambda X_i} \right] \text{ indep} \]

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\[ P \left[ X \geq (1 + \delta)\mu \right] \leq e^{-\lambda(1+\delta)\mu} \prod_{i=1}^{n} e^{p_i(e^{\lambda} - 1)} = e^{-\lambda(1+\delta)\mu} e^{\mu(e^\lambda - 1)} \]
1. For $\lambda > 0$,

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P [ X \geq (1 + \delta)\mu ] = P \left[ e^{\lambda X} \geq e^{\lambda(1+\delta)\mu} \right] \leq e^{-\lambda(1+\delta)\mu} E \left[ e^{\lambda X} \right] \tag{Markov}
\]

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3. \( E \left[ e^{\lambda X_i} \right] = e^{\lambda p_i} + (1 - p_i) = 1 + p_i(e^\lambda - 1) \leq e^{p_i(e^\lambda - 1)} \tag{1+x \leq e^x} \)

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P [ X \geq (1 + \delta)\mu ] \leq e^{-\lambda(1+\delta)\mu} \prod_{i=1}^{n} e^{p_i(e^\lambda - 1)} = e^{-\lambda(1+\delta)\mu} e^{\mu(e^\lambda - 1)}
\]

5. Choose $\lambda = \log(1 + \delta) > 0$ to get the result.
Chernoff Bounds: Lower Tails

We can also use Chernoff Bounds to show a random variable is not too small compared to its mean:

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then, for any $0 < \delta < 1$ it holds that

$$P[X \leq (1 - \delta)\mu] \leq \left[ \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right]^\mu,$$

and thus, by substitution, for any $t < \mu$,

$$P[X \leq t] \leq e^{-\mu} \left( \frac{e^\mu}{t} \right)^t.$$

Exercise on Supervision Sheet
Hint: multiply both sides by $-1$ and repeat the proof of the Chernoff Bound
Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then, for all $t > 0$,

$$
P[X \geq \mathbb{E}[X] + t] \leq e^{-2t^2/n}
$$

$$
P[X \leq \mathbb{E}[X] - t] \leq e^{-2t^2/n}
$$

For $0 < \delta < 1$,

$$
P[X \geq (1 + \delta)\mathbb{E}[X]] \leq \exp(-\delta^2\mathbb{E}[X]^3)
$$

$$
P[X \leq (1 - \delta)\mathbb{E}[X]] \leq \exp(-\delta^2\mathbb{E}[X]^2)
$$
Nicer Chernoff Bounds

“Nicer” Chernoff Bounds

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then,

For all $t > 0$,

$$P[X \geq \mathbb{E}[X] + t] \leq e^{-\frac{2t^2}{n}}$$

$$P[X \leq \mathbb{E}[X] - t] \leq e^{-\frac{2t^2}{n}}$$

For $0 < \delta < 1$,

$$P[X \geq (1 + \delta)\mathbb{E}[X]] \leq \exp\left(-\frac{\delta^2 \mathbb{E}[X]^3}{3}\right)$$

$$P[X \leq (1 - \delta)\mathbb{E}[X]] \leq \exp\left(-\frac{\delta^2 \mathbb{E}[X]^2}{2}\right)$$
Nicer Chernoff Bounds

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Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then,

- For all $t > 0$,
  \[
  \mathbb{P}[X \geq \mathbb{E}[X] + t] \leq e^{-\frac{2t^2}{n}}
  \]
  \[
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  \]

- For \( 0 < \delta < 1 \),
  \[
  \mathbb{P}[X \geq (1 + \delta)\mathbb{E}[X]] \leq \exp\left(-\frac{\delta^2\mathbb{E}[X]}{3}\right)
  \]
  \[
  \mathbb{P}[X \leq (1 - \delta)\mathbb{E}[X]] \leq \exp\left(-\frac{\delta^2\mathbb{E}[X]}{2}\right)
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  \]

All upper tail bounds hold even under a relaxed independence assumption: For all $1 \leq i \leq n$ and $x_1, x_2, \ldots, x_{i-1} \in \{0, 1\}$,

\[
\mathbb{P}[X_i = 1 \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq p_i.
\]
How to Derive Chernoff Bounds

Application 1: Balls into Bins
**Balls into Bins**

You have \( m \) balls and \( n \) bins. Each ball is allocated in a bin picked independently and uniformly at random.

---

**Balls into Bins Model**

You have \( m \) balls and \( n \) bins. Each ball is allocated in a bin picked independently and uniformly at random.
Balls into Bins

You have \( m \) balls and \( n \) bins. Each ball is allocated in a bin picked independently and uniformly at random.

- A very natural but also rich mathematical model
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- In computer science, there are several interpretations:
**Balls into Bins Model**

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- In computer science, there are several interpretations:
  1. Bins are a hash table, balls are items
  2. Bins are processors and balls are jobs
  3. Bins are data servers and balls are queries
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Exercise: Think about the relation between the Balls into Bins Model and the Coupon Collector Problem.
You have $m$ balls and $n$ bins. Each ball is allocated in a bin picked independently and uniformly at random.

Balls into Bins Model

You have $m$ balls and $n$ bins. Each ball is allocated in a bin picked independently and uniformly at random.
Balls into Bins: Bounding the Maximum Load (1/4)

You have $m$ balls and $n$ bins. Each ball is allocated in a bin picked independently and uniformly at random.

**Question 1:** How large is the maximum load if $m = 2n \log n$?
You have \( m \) balls and \( n \) bins. Each ball is allocated in a bin picked independently and uniformly at random.

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- Focus on an arbitrary single bin. Let \( X_i \) the indicator variable which is 1 iff ball \( i \) is assigned to this bin. Note that \( p_i = P[X_i = 1] = 1/n \).
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- Focus on an arbitrary single bin. Let \( X_i \) the indicator variable which is 1 iff ball \( i \) is assigned to this bin. Note that \( p_i = \Pr [ X_i = 1 ] = 1 / n \).
- The total balls in the bin is given by \( X := \sum_{i=1}^{n} X_i \).
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- Since $m = 2n \log n$, then $\mu = \mathbf{E}[X] = 2 \log n$
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- Since $m = 2n \log n$, then $\mu = E[X] = 2 \log n$.
- By the Chernoff Bound,
  \[
P[X \geq t] \leq e^{-\mu}(e^{\mu/t})^t
  \]
  \[
P[X \geq 6 \log n] \leq e^{-2 \log n} \left(\frac{2e \log n}{6 \log n}\right)^{6 \log n} \leq e^{-2 \log n} = n^{-2}
  \]
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- By the Chernoff Bound,
  \[ \mathbb{P}[X \geq t] \leq e^{-\mu} (e^{\mu/t})^t \]
  \[ \mathbb{P}[X \geq 6 \log n] \leq e^{-2 \log n} \left(\frac{2e \log n}{6 \log n}\right)^{6 \log n} \leq e^{-2 \log n} = n^{-2} \]

"here we could have used the “nicer” bounds as well!"

By the Chernoff Bound,
Let $E_j := \{X(j) \geq 6 \log n\}$, that is, bin $j$ receives at least $6 \log n$ balls.

- By the Union Bound, 
  $$
  \Pr\left[ \bigcup_{j=1}^{n} E_j \right] \leq \sum_{j=1}^{n} \Pr[E_j] \leq n - 1.
  $$

Therefore whp, no bin receives at least $6 \log n$ balls.

By pigeonhole principle, the max loaded bin receives at least $2 \log n$ balls. Hence our bound is pretty sharp.

whp stands for with high probability:
An event $E$ (that implicitly depends on an input parameter $n$) occurs whp if
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\Pr[E] \to 1 \text{ as } n \to \infty.
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This is a very standard notation in randomised algorithms but it may vary from author to author. Be careful!
Let $E_j := \{X(j) \geq 6 \log n\}$, that is, bin $j$ receives at least $6 \log n$ balls.

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**whp** stands for *with high probability*:

An event $\mathcal{E}$ (that implicitly depends on an input parameter $n$) occurs whp if $\Pr [\mathcal{E}] \rightarrow 1$ as $n \rightarrow \infty$.

This is a very standard notation in randomised algorithms but it may vary from author to author. Be careful!
Question 2: How large is the maximum load if \( m = n \)?

Using the Chernoff Bound:

\[
P[X \geq t] \leq e^{-1}(et)^t \leq (e^{\mu}(e^{\mu/t}))^t
\]

By setting \( t = 4\log n / \log \log n \), we claim to obtain

\[
P[X \geq t] \leq n^{-2}.
\]

Indeed:

\[
4\log n \log \log n \cdot (\log(e/4) + \log \log \log n - \log \log n) \leq 4\log n \log \log n (\log \log \log n),
\]

obtaining that

\[
P[X \geq t] \leq n^{-2}.
\]

This inequality only works for large enough \( n \).
Question 2: How large is the maximum load if $m = n$?

- Using the Chernoff Bound:
  \[ P[X \geq t] \leq e^{-\mu}(e\mu/t)^t \]

\[ P[X \geq t] \leq e^{-1} \left( \frac{e}{t} \right)^t \leq \left( \frac{e}{t} \right)^t \]
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Application 1: Balls into Bins 12
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- By setting \( t = 4 \log n/ \log \log n \), we claim to obtain \( P[X \geq t] \leq n^{-2} \).

- Indeed:
  \[ \left( \frac{e \log \log n}{4 \log n} \right)^{4 \log n/ \log \log n} = \exp \left( \frac{4 \log n}{\log \log n} \cdot \log \left( \frac{e \log \log n}{4 \log n} \right) \right) \]
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  \]
  
  \[
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This inequality only works for large enough \( n \).
We just proved that

\[ P \left[ X \geq 4 \log n / \log \log n \right] \leq n^{-2}, \]

thus by the Union Bound, no bin receives more than \( \Omega \left( \log n / \log \log n \right) \) balls with probability at least \( 1 - 1/n \).
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thus by the Union Bound, no bin receives more than \( \Omega \left( \log n/ \log \log n \right) \) balls with probability at least \( 1 - 1/n \).

- As mentioned on the to prove that \( \mathsf{whp} \) at least one bin receives at least \( c \log n/ \log \log n \) balls, for some constant \( c > 0 \).
Conclusions

- If the number of balls is $2 \log n$ times $n$ (the number of bins), then to distribute balls at random is a good algorithm.

A Better Load Balancing Approach

This is called the power of two choices: It is a common technique to improve the performance of randomised algorithms (covered in Chapter 17 of the textbook by Mitzenmacher and Upfal).
Conclusions

- If the number of balls is $2 \log n$ times $n$ (the number of bins), then to distribute balls at random is a good algorithm.
  - This is because the worst case maximum load is whp. $6 \log n$, while the average load is $2 \log n$.
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For any $m \geq n$, we can improve this by sampling two bins in each step and then assign the ball into the bin with lesser load.
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A Better Load Balancing Approach

For any $m \geq n$, we can improve this by sampling **two bins** in each step and then assign the ball into the bin with lesser load.

$\Rightarrow$ for $m = n$ this gives a maximum load of $\log_2 \log n + \Theta(1)$ w.p. $1 - 1/n$. 
Conclusions

- If the number of balls is \(2 \log n\) times \(n\) (the number of bins), then to distribute balls at random is a good algorithm
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A Better Load Balancing Approach

For any \(m \geq n\), we can improve this by sampling two bins in each step and then assign the ball into the bin with lesser load.
\[\Rightarrow \text{for } m = n \text{ this gives a maximum load of } \log_2 \log n + \Theta(1) \text{ w.p. } 1 - 1/n.\]

This is called the **power of two choices**: It is a common technique to improve the performance of randomised algorithms (covered in Chapter 17 of the textbook by Mitzenmacher and Upfal)
For “the discovery and analysis of balanced allocations, known as the power of two choices, and their extensive applications to practice.”

“These include i-Google’s web index, Akamai’s overlay routing network, and highly reliable distributed data storage systems used by Microsoft and Dropbox, which are all based on variants of the power of two choices paradigm. There are many other software systems that use balanced allocations as an important ingredient.”
Simulation

Sampled two bins u.a.r.

https://www.dimitrioslos.com/balls_and_bins/visualiser.html
Outline

Application 2: Randomised QuickSort

Extensions of Chernoff Bounds

Applications of Method of Bounded Differences

Appendix: More on Moment Generating Functions (non-examinable)
QuickSort

QUICKSORT (Input A[1], A[2], . . . , A[n])
1: Pick an element from the array, the so-called pivot
2: If |A| = 0 or |A| = 1 then
3: return A
4: else
5: Create two subarrays A_1 and A_2 (without the pivot) such that:
6: A_1 contains the elements that are smaller than the pivot
7: A_2 contains the elements that are greater (or equal) than the pivot
8: QUICKSORT(A_1)
9: QUICKSORT(A_2)
10: return A
QuickSort

1: Pick an element from the array, the so-called pivot
2: If $|A| = 0$ or $|A| = 1$ then
3: \hspace{1em} return $A$
4: else
5: \hspace{1em} Create two subarrays $A_1$ and $A_2$ (without the pivot) such that:
6: \hspace{2em} $A_1$ contains the elements that are smaller than the pivot
7: \hspace{2em} $A_2$ contains the elements that are greater (or equal) than the pivot
8: \hspace{1em} QUICKSORT($A_1$)
9: \hspace{1em} QUICKSORT($A_2$)
10: return $A$

- Example: Let $A = (2, 8, 9, 1, 7, 5, 6, 3, 4)$ with $A[7] = 6$ as pivot.
QuickSort

1: Pick an element from the array, the so-called pivot
2: If $|A| = 0$ or $|A| = 1$ then
3: \hspace{1cm} return $A$
4: else
5: \hspace{1cm} Create two subarrays $A_1$ and $A_2$ (without the pivot) such that:
6: \hspace{1.5cm} $A_1$ contains the elements that are smaller than the pivot
7: \hspace{1.5cm} $A_2$ contains the elements that are greater (or equal) than the pivot
8: \hspace{1cm} QUICKSORT($A_1$)
9: \hspace{1cm} QUICKSORT($A_2$)
10: \hspace{1cm} return $A$

- Example: Let $A = (2, 8, 9, 1, 7, 5, 6, 3, 4)$ with $A[7] = 6$ as pivot.
  $\Rightarrow A_1 = (2, 1, 5, 3, 4)$ and $A_2 = (8, 9, 7)$
QuickSort

**QuickSort** (Input $A[1], A[2], \ldots, A[n]$)

1. Pick an element from the array, the so-called **pivot**
2. If $|A| = 0$ or $|A| = 1$ then
   3. return $A$
4. else
   5. Create two subarrays $A_1$ and $A_2$ (without the pivot) such that:
      6. $A_1$ contains the elements that are smaller than the pivot
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   8. QuickSort($A_1$)
   9. QuickSort($A_2$)
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- **Example**: Let $A = (2, 8, 9, 1, 7, 5, 6, 3, 4)$ with $A[7] = 6$ as pivot.
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- **Worst-Case Complexity** (number of comparisons) is $\Theta(n^2)$,
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9: \hspace{1em} QUICKSORT($A_2$)
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- Example: Let $A = (2, 8, 9, 1, 7, 5, 6, 3, 4)$ with $A[7] = 6$ as pivot.
  \Rightarrow $A_1 = (2, 1, 5, 3, 4)$ and $A_2 = (8, 9, 7)$
- Worst-Case Complexity (number of comparisons) is $\Theta(n^2)$, while Average-Case Complexity is $O(n \log n)$. 
QuickSort

1: Pick an element from the array, the so-called pivot
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- Worst-Case Complexity (number of comparisons) is $\Theta(n^2)$,
  while Average-Case Complexity is $O(n \log n)$.

We will now give a proof of this “well-known” result!
QuickSort: How to Count Comparisons

What is the number of comparisons?

Note that the number of comparison by QUICKSORT is equivalent to the sum of the depths of all nodes in the tree (why?). In this case:

$$0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 + 4 = 19.$$
QuickSort: How to Count Comparisons

2, 8, 9, 1, 7, 5, 6, 3, 4
2, 1, 5, 3, 4
2, 5, 3, 4
2, 3
2
5
8, 9, 7
8, 9
8
What is the number of comparisons?

Note that the number of comparison by QuickSort is equivalent to the sum of the depths of all nodes in the tree (why?). In this case:

0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 + 4 = 19.

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How to pick a good pivot? We don’t, *just pick one at random.*
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This should be your standard answer in this course 😊
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Let us analyse **QUICKSORT** with *random* pivots.
How to pick a good pivot? We don’t, *just pick one at random.*

This should be your standard answer in this course 😊

Let us analyse Quicksort with random pivots.

1. Assume $A$ consists of $n$ different numbers, w.l.o.g., $\{1, 2, \ldots, n\}$

2. Let $H_i$ be the deepest level where element $i$ appears in the tree.

3. We will prove that there exists $C > 0$ such that $P[H_i \leq C \log n] \geq 1 - \frac{1}{n}$.

4. Actually, we will prove something slightly stronger:

$$P[n \cap \bigcap_{i=1}^{n} \{H_i \leq C \log n\}] \geq 1 - \frac{1}{n}.$$
How to pick a good pivot? We don’t, just pick one at random.

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Let us analyse QuartzSort with random pivots.
1. Assume A consists of $n$ different numbers, w.l.o.g., \{1, 2, \ldots, n\}
2. Let $H_i$ be the deepest level where element $i$ appears in the tree.
   Then the number of comparison is $H = \sum_{i=1}^{n} H_i$
How to pick a good pivot? We don’t, just pick one at random.

This should be your standard answer in this course 😊

Let us analyse QUICKSORT with random pivots.

1. Assume $A$ consists of $n$ different numbers, w.l.o.g., $\{1, 2, \ldots, n\}$
2. Let $H_i$ be the deepest level where element $i$ appears in the tree.
   Then the number of comparison is $H = \sum_{i=1}^{n} H_i$
3. We will prove that there exists $C > 0$ such that
   $$\Pr[H \leq Cn \log n] \geq 1 - n^{-1}.$$
Randomised QuickSort: Analysis (1/4)

How to pick a good pivot? We don’t, just pick one at random.

This should be your standard answer in this course 😊

Let us analyse QUICKSORT with random pivots.

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4. Actually, we will prove sth slightly stronger:
   $$P\left[ \bigcap_{i=1}^{n} \{H_i \leq C \log n\} \right] \geq 1 - n^{-1}.$$
Let $P$ be a path from the root to the deepest level of some element. A node in $P$ is called good if the corresponding pivot partitions the array into two subarrays each of size at most $2/3$ of the previous one. Otherwise, the node is bad. Further let $s_t$ be the size of the array at level $t$ in $P$. The elements are sorted as follows:

Element 2: $(2, 8, 9, 1, 7, 5, 6, 3, 4)$ → $(2, 1, 5, 3, 4)$ → $(2, 5, 3, 4)$ → $(2, 3)$ → $(2)$
Let $P$ be a path from the root to the deepest level of some element. A node in $P$ is called **good** if the corresponding pivot partitions the array into two subarrays each of size at most $2/3$ of the previous one. Otherwise, the node is **bad**.

Further let $s_t$ be the size of the array at level $t$ in $P$.

<table>
<thead>
<tr>
<th>Level</th>
<th>Size $s_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Element 2: $(2, 8, 9, 1, 7, 5, 6, 3, 4) \rightarrow (2, 1, 5, 3, 4) \rightarrow (2, 5, 3, 4) \rightarrow (2, 3) \rightarrow (2)$
Randomised QuickSort: Analysis (2/4)

- Let $P$ be a path from the root to the deepest level of some element

\[
\begin{align*}
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- A node in $P$ is called **good** if the corresponding pivot partitions the array into two subarrays each of size at most $\frac{2}{3}$ of the previous one
- otherwise, the node is **bad**

**Element 2:** $(2, 8, 9, 1, 7, 5, 6, 3, 4) \rightarrow (2, 1, 5, 3, 4) \rightarrow (2, 5, 3, 4) \rightarrow (2, 3) \rightarrow (2)$
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- Further let $s_t$ be the size of the array at level $t$ in $P$.

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Let $P$ be a path from the root to the deepest level of some element.
- A node in $P$ is called **good** if the corresponding pivot partitions the array into two subarrays each of size at most $2/3$ of the previous one.
- Otherwise, the node is **bad**.

Further let $s_t$ be the size of the array at level $t$ in $P$.

- Element 2: $(2, 8, 9, 1, 7, 5, 6, 3, 4) \rightarrow (2, 1, 5, 3, 4) \rightarrow (2, 5, 3, 4) \rightarrow (2, 3) \rightarrow (2)$
Consider now any element $i \in \{1, 2, \ldots, n\}$ and construct the path $P = P(i)$ one level by one.
Consider now any element \( i \in \{1, 2, \ldots, n\} \) and construct the path \( P = P(i) \) one level by one.

For \( P \) to proceed from level \( k \) to \( k + 1 \), the condition \( s_k > 1 \) is necessary.
Consider now any element \( i \in \{1, 2, \ldots, n\} \) and construct the path \( P = P(i) \) one level by one. For \( P \) to proceed from level \( k \) to \( k + 1 \), the condition \( s_k > 1 \) is necessary. How far could such a path \( P \) possibly run until we have \( s_k = 1 \)?
Consider now any element $i \in \{1, 2, \ldots, n\}$ and construct the path $P = P(i)$ one level by one.

- For $P$ to proceed from level $k$ to $k + 1$, the condition $s_k > 1$ is necessary.

How far could such a path $P$ possibly run until we have $s_k = 1$?

- We start with $s_0 = n$.
Consider now any element $i \in \{1, 2, \ldots, n\}$ and construct the path $P = P(i)$ one level by one.

For $P$ to proceed from level $k$ to $k + 1$, the condition $s_k > 1$ is necessary.

How far could such a path $P$ possibly run until we have $s_k = 1$?

- We start with $s_0 = n$.
- **First Case**, *good* node: $s_{k+1} \leq \frac{2}{3} \cdot s_k$.
Consider now any element \( i \in \{1, 2, \ldots, n\} \) and construct the path \( P = P(i) \) one level by one.

For \( P \) to proceed from level \( k \) to \( k + 1 \), the condition \( s_k > 1 \) is necessary.

How far could such a path \( P \) possibly run until we have \( s_k = 1 \)?

We start with \( s_0 = n \).

**First Case, good node:** \( s_{k+1} \leq \frac{2}{3} \cdot s_k \).

**Second Case, bad node:** \( s_{k+1} \leq s_k \).
Consider now any element $i \in \{1, 2, \ldots, n\}$ and construct the path $P = P(i)$ one level by one.

For $P$ to proceed from level $k$ to $k + 1$, the condition $s_k > 1$ is necessary.

How far could such a path $P$ possibly run until we have $s_k = 1$?

- We start with $s_0 = n$.
- **First Case**, good node: $s_{k+1} \leq \frac{2}{3} \cdot s_k$.
- **Second Case**, bad node: $s_{k+1} \leq s_k$.

$\Rightarrow$ There are at most $T = \frac{\log n}{\log(3/2)} < 3 \log n$ many good nodes on any path $P$. 

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Randomised QuickSort: Analysis (3/4)
Consider now any element \( i \in \{1, 2, \ldots, n\} \) and construct the path \( P = P(i) \) one level by one.

For \( P \) to proceed from level \( k \) to \( k + 1 \), the condition \( s_k > 1 \) is necessary.

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- We start with \( s_0 = n \).
- **First Case, good node:** \( s_{k+1} \leq \frac{2}{3} \cdot s_k \).
- **Second Case, bad node:** \( s_{k+1} \leq s_k \).

\[ \Rightarrow \text{There are at most } T = \frac{\log n}{\log(3/2)} < 3 \log n \text{ many good nodes on any path } P. \]
Consider now any element \( i \in \{1, 2, \ldots, n\} \) and construct the path \( P = P(i) \) one level by one.

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\[ \Rightarrow \quad \text{There are at most } T = \frac{\log n}{\log(3/2)} < 3 \log n \text{ many good nodes on any path } P. \]

Assume \( |P| \geq C \log n \) for \( C := 24 \).
Consider now any element \( i \in \{1, 2, \ldots, n\} \) and construct the path \( P = P(i) \) one level by one.

For \( P \) to proceed from level \( k \) to \( k + 1 \), the condition \( s_k > 1 \) is necessary.

How far could such a path \( P \) possibly run until we have \( s_k = 1 \)?

- We start with \( s_0 = n \).
- **First Case**, good node: \( s_{k+1} \leq \frac{2}{3} \cdot s_k \).
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\[ T = \frac{\log n}{\log(3/2)} < 3 \log n \] many good nodes on any path \( P \).

Assume \( |P| \geq C \log n \) for \( C := 24 \).

\[ \Rightarrow \] number of bad vertices in the first \( 24 \log n \) levels is more than \( 21 \log n \).
Consider now any element \( i \in \{1, 2, \ldots, n\} \) and construct the path \( P = P(i) \) one level by one.

For \( P \) to proceed from level \( k \) to \( k + 1 \), the condition \( s_k > 1 \) is necessary.

How far could such a path \( P \) possibly run until we have \( s_k = 1 \)?

We start with \( s_0 = n \).

- **First Case**, **good** node: \( s_{k+1} \leq \frac{2}{3} \cdot s_k \).
- **Second Case**, **bad** node: \( s_{k+1} \leq s_k \).

\[ T = \frac{\log n}{\log(3/2)} < 3 \log n \] many good nodes on any path \( P \).

Assume \( |P| \geq C \log n \) for \( C := 24 \)

\[ \Rightarrow \text{number of bad vertices in the first } 24 \log n \text{ levels is more than } 21 \log n. \]

Let us now upper bound the probability that this “bad event” happens!
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the **deepest level** of element $i$. 

For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:

- $X_j = 1$ if the node at level $j$ is **bad**, 
- $X_j = 0$ if the node at level $j$ is **good**.

We can now apply the "nicer" Chernoff Bound. We have

$$E[X] \leq \left(\frac{2}{3}\right) \cdot 24 \log n = 16 \log n.$$ 

Then, by the "nicer" Chernoff Bounds

$$P[X \geq E[X] + t] \leq e^{-2t^2/n}.$$ 

This implies

$$P[X > 21 \log n] \leq P[X > E[X] + 5 \log n] \leq e^{-2 \left(\frac{5}{24} \log n\right)^2} \leq n^{-2}.$$ 

Hence $P$ has more than $24 \log n$ nodes with probability at most $n^{-2}$.

As there are in total $n$ paths, by the union bound, the probability that at least one of them has more than $24 \log n$ nodes is at most $n^{-1}$.

This implies $P[\bigcap_{i=1}^n \{H_i \leq 24 \log n\}] \geq 1 - n^{-1}$, as needed.
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
- For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$: 

$$X_j = 1 \text{ if the node at level } j \text{ is bad, }$$

$$X_j = 0 \text{ if the node at level } j \text{ is good}.$$

$$P[X_j = 1 | X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}.$$ 

Then, by the "nicer" Chernoff Bounds

$$P[X_j > 21 \log n] \leq e^{-2 \cdot 5 \log n / 24 \log n} = e^{-\frac{50}{24} \log n} \leq n^{-2}.$$ 

Hence, $P$ has more than $24 \log n$ nodes with probability at most $n^{-2}$. As there are in total $n$ paths, by the union bound, the probability that at least one of them has more than $24 \log n$ nodes is at most $n^{-1}$. This implies

$$P[\bigcap_{n=1}^{n} \{H_i \leq 24 \log n\}] \geq 1 - n^{-1},$$
as needed.
Consider the first \(24 \log n\) vertices of \(P\) to the deepest level of element \(i\).

For any level \(j \in \{0, 1, \ldots, 24 \log n - 1\}\), define an indicator variable \(X_j\):

- \(X_j = 1\) if the node at level \(j\) is bad,
- \(X_j = 0\) if the node at level \(j\) is good.
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
- $X_j = 1$ if the node at level $j$ is bad,
- $X_j = 0$ if the node at level $j$ is good.

$$P \left[ X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1} \right] \leq \frac{2}{3}$$
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
- $X_j = 1$ if the node at level $j$ is \textit{bad},
- $X_j = 0$ if the node at level $j$ is \textit{good}.

$\mathbb{P}[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}$

We can now apply the "nicer" Chernoff Bound! We have

$E[X] \leq \left(\frac{2}{3}\right) \cdot 24 \log n = 16 \log n$

Then, by the "nicer" Chernoff Bounds

$\mathbb{P}[X > E[X] + t] \leq e^{-\frac{2t^2}{n}}$

$\mathbb{P}[X > 21 \log n] \leq \mathbb{P}[X > E[X] + 5 \log n] \leq e^{-\frac{2(5 \log n)^2}{24 \log n}} = e^{-\frac{50}{24} \log n} \leq n^{-2}$.

Hence $\mathbb{P}$ has more than $24 \log n$ nodes with probability at most $n^{-2}$.

As there are in total $n$ paths, by the union bound, the probability that at least one of them has more than $24 \log n$ nodes is at most $n^{-1}$.

This implies $\mathbb{P}[\bigcap_{i=1}^n \{H_i \leq 24 \log n\}] \geq 1 - n^{-1}$, as needed.
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:

- $X_j = 1$ if the node at level $j$ is bad,
- $X_j = 0$ if the node at level $j$ is good.

$$\Pr\left[ X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1} \right] \leq \frac{2}{3}$$

We can now apply the “nicer” Chernoff Bound! We have

$$\mathbb{E}[X] \leq \left(\frac{2}{3}\right) \cdot 24 \log n = 16 \log n$$

Then, by the “nicer” Chernoff Bound

$$\Pr\left[ X \geq \mathbb{E}[X] + t \right] \leq e^{-\frac{2t^2}{n}}$$

Let $X > 24 \log n$

$$\Pr\left[ X > E[X] + 5 \log n \right] \leq e^{-\frac{2 \cdot 5 \log n}{24 \log n}} = e^{-\frac{50}{24} \log n} \leq n^{-2}.$$
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
- $X_j = 1$ if the node at level $j$ is bad,
- $X_j = 0$ if the node at level $j$ is good.

$P[ X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1} ] \leq \frac{2}{3}$

$X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

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$X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

**Question:** Edge Case: What if the path $P$ does not reach level $j$?
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
- For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
  - $X_j = 1$ if the node at level $j$ is **bad**, $X_j = 0$ if the node at level $j$ is **good**.
  
- $\Pr[ X_j = 1 \mid X_0 = x_0, \ldots, X_j = x_j ] \leq \frac{2}{3}$
  
- $X := \sum_{j=0}^{24 \log n-1} X_j$ satisfies relaxed independence assumption (Lecture 2)

**Question:** Edge Case: What if the path $P$ does not reach level $j$?

**Answer:** We can then simply define $X_j$ as 0 (deterministically).
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
- For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
  - $X_j = 1$ if the node at level $j$ is bad,
  - $X_j = 0$ if the node at level $j$ is good.
- $P[ X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1} ] \leq \frac{2}{3}$
- $X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

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$P \left[ X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1} \right] \leq \frac{2}{3}$

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We can now apply the “nicer” Chernoff Bound!
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
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  - $X_j = 0$ if the node at level $j$ is good.
- $P[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}$
- $X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

We can now apply the “nicer” Chernoff Bound!

- We have $E[X] \leq \frac{2}{3} \cdot 24 \log n = 16 \log n$
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- $X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

We can now apply the “nicer” Chernoff Bound!

- We have $E[ X ] \leq (2/3) \cdot 24 \log n = 16 \log n$
- Then, by the “nicer” Chernoff Bounds
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:

- $X_j = 1$ if the node at level $j$ is bad,
- $X_j = 0$ if the node at level $j$ is good.

$\mathbb{P}[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}$

$X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

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- We have $\mathbb{E}[X] \leq (2/3) \cdot 24 \log n = 16 \log n$

- Then, by the “nicer” Chernoff Bounds

$$\mathbb{P}[X \geq \mathbb{E}[X] + t] \leq e^{-\frac{2t^2}{n}}$$
We can now apply the “nicer” Chernoff Bound!

- We have $\mathbb{E}[X] \leq \frac{2}{3} \cdot 24 \log n = 16 \log n$
- Then, by the “nicer” Chernoff Bounds
  $$
P[X \geq \mathbb{E}[X] + t] \leq e^{-\frac{2t^2}{n}}$$

Thus,
$$
P[X > 21 \log n] \leq P[X > \mathbb{E}[X] + 5 \log n]$$
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
- For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
  - $X_j = 1$ if the node at level $j$ is bad,
  - $X_j = 0$ if the node at level $j$ is good.
- $\Pr[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}$
- $X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

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- We have $\mathbb{E}[X] \leq (2/3) \cdot 24 \log n = 16 \log n$
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  $$\Pr[X \geq \mathbb{E}[X] + t] \leq e^{-2t^2/n}$$
- $$\Pr[X > 21 \log n] \leq \Pr[X > \mathbb{E}[X] + 5 \log n] \leq e^{-2(5 \log n)^2/(24 \log n)}$$

We have more than $24 \log n$ nodes with probability at most $n^{-1}$, as needed.

3. Concentration © T. Sauerwald
Application 2: Randomised QuickSort
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
- $X_j = 1$ if the node at level $j$ is bad,
- $X_j = 0$ if the node at level $j$ is good.

$P[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}$

$X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

We can now apply the “nicer” Chernoff Bound!

We have $E[X] \leq (2/3) \cdot 24 \log n = 16 \log n$

Then, by the “nicer” Chernoff Bounds

$$P[X \geq E[X] + t] \leq e^{-2t^2/n}$$

$$P[X > 21 \log n] \leq P[X > E[X] + 5 \log n] \leq e^{-2(5 \log n)^2/(24 \log n)}$$

$$= e^{-(50/24) \log n}$$
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
- For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
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We can now apply the “nicer” Chernoff Bound!

- We have $E[X] \leq (2/3) \cdot 24 \log n = 16 \log n$
- Then, by the “nicer” Chernoff Bounds

\[
P[X \geq E[X] + t] \leq e^{-2t^2/n}
\]

\[
P[X > 21 \log n] \leq P[X > E[X] + 5 \log n] \leq e^{-2(5 \log n)^2/(24 \log n)}
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\[
= e^{-(50/24) \log n} \leq n^{-2}.
\]

Hence $P$ has more than $24 \log n$ nodes with probability at most $n^{-1}$. As there are in total $n$ paths, by the union bound, the probability that at least one of them has more than $24 \log n$ nodes is at most $n^{-1}$. This implies $P[\bigcap_{i=1}^{n} \{H_i \leq 24 \log n\}] \geq 1 - n^{-1}$, as needed.
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
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  - $X_j = 1$ if the node at level $j$ is bad,
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- $\mathbb{P}[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}$
- $X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

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- We have $\mathbb{E}[X] \leq (2/3) \cdot 24 \log n = 16 \log n$
- Then, by the “nicer” Chernoff Bounds $\mathbb{P}[X \geq \mathbb{E}[X] + t] \leq e^{-2t^2/n}$

$$\mathbb{P}[X > 21 \log n] \leq \mathbb{P}[X > \mathbb{E}[X] + 5 \log n] \leq e^{-2(5 \log n)^2/(24 \log n)} = e^{-(50/24) \log n} \leq n^{-2}.$$  

- Hence $P$ has more than $24 \log n$ nodes with probability at most $n^{-2}$. 

We have more than $24 \log n$ nodes with probability at most $n^{-2}$. 

3. Concentration © T. Sauerwald Application 2: Randomised QuickSort
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:

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$X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

We can now apply the “nicer” Chernoff Bound!

We have $E [ X ] \leq (2/3) \cdot 24 \log n = 16 \log n$

Then, by the “nicer” Chernoff Bounds

$P [ X \geq E [ X ] + t ] \leq e^{ -2t^2 / n }$

$P [ X > 21 \log n ] \leq P [ X > E [ X ] + 5 \log n ] \leq e^{ -2(5 \log n)^2 / (24 \log n) }$

$= e^{ -(50/24) \log n } \leq n^{-2}$.

Hence $P$ has more than $24 \log n$ nodes with probability at most $n^{-2}$.

As there are in total $n$ paths, by the union bound, the probability that at least one of them has more than $24 \log n$ nodes is at most $n^{-1}$. 
Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.

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We have $E[ X ] \leq (2/3) \cdot 24 \log n = 16 \log n$

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$$P[ X > 21 \log n ] \leq P[ X > E[ X ] + 5 \log n ] \leq e^{-2(5 \log n)^2/(24 \log n)}$$

$$= e^{-(50/24) \log n} \leq n^{-2}.$$

Hence $P$ has more than $24 \log n$ nodes with probability at most $n^{-2}$.

As there are in total $n$ paths, by the union bound, the probability that at least one of them has more than $24 \log n$ nodes is at most $n^{-1}$.

This implies $P[ \bigcap_{i=1}^{n} \{ H_i \leq 24 \log n \} ] \geq 1 - n^{-1}$, as needed.
Randomised QuickSort: Analysis (4/4)

- Consider the first $24 \log n$ vertices of $P$ to the deepest level of element $i$.
- For any level $j \in \{0, 1, \ldots, 24 \log n - 1\}$, define an indicator variable $X_j$:
  - $X_j = 1$ if the node at level $j$ is bad,
  - $X_j = 0$ if the node at level $j$ is good.

$$\mathbb{P}[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}$$

- $X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

We can now apply the “nicer” Chernoff Bound!

- We have $\mathbb{E}[X] \leq (2/3) \cdot 24 \log n = 16 \log n$
- Then, by the “nicer” Chernoff Bounds

$$\mathbb{P}[X \geq \mathbb{E}[X] + t] \leq e^{-2t^2/n}$$

$$\mathbb{P}[X > 21 \log n] \leq \mathbb{P}[X > \mathbb{E}[X] + 5 \log n] \leq e^{-2(5 \log n^2)/(24 \log n)} = e^{-(50/24) \log n} \leq n^{-2}.$$ 

- Hence $P$ has more than $24 \log n$ nodes with probability at most $n^{-2}$.
- As there are in total $n$ paths, by the union bound, the probability that at least one of them has more than $24 \log n$ nodes is at most $n^{-1}$.

- This implies $\mathbb{P}[\bigcap_{i=1}^{\infty} \{H_i \leq 24 \log n\}] \geq 1 - n^{-1}$, as needed.
Randomised QuickSort: Final Remarks

- Well-known: any comparison-based sorting algorithm needs $\Omega(n \log n)$
- A classical result: expected number of comparison of randomised QUICKSORT is $2n \log n + O(n)$ (see, e.g., book by Mitzenmacher & Upfal)
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Exercise: [Ex 2-3.6] Our upper bound of $O(n \log n)$ whp also immediately implies a $O(n \log n)$ bound on the expected number of comparisons!
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Exercise: [Ex 2-3.6] Our upper bound of $O(n \log n)$ whp also immediately implies a $O(n \log n)$ bound on the expected number of comparisons!

It is possible to deterministically find the best pivot element that divides the array into two subarrays of the same size.

The latter requires to compute the median of the array in linear time, which is not easy...

The presented randomised algorithm for QUICKSORT is much easier to implement!
Outline

Application 2: Randomised QuickSort

Extensions of Chernoff Bounds

Applications of Method of Bounded Differences

Appendix: More on Moment Generating Functions (non-examinable)
Hoeffding’s Extension

- Besides **sums of independent Bernoulli** random variables, **sums of independent and bounded** random variables are very frequent in applications.

\[ E\left[ e^{\lambda X}\right] \leq \exp \left( \frac{(b-a)^2\lambda^2}{8}\right) \]

Hoeffding’s Extension Lemma

You can always consider \( X' = X - E[X] \)

We omit the proof of this lemma!
Hoeffding’s Extension

- Besides **sums of independent Bernoulli** random variables, **sums of independent and bounded** random variables are very frequent in applications.
- Unfortunately the distribution of the $X_i$ may be unknown or hard to compute, thus it will be hard to compute the moment-generating function.
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\begin{center}
\textbf{Hoeffding’s Extension Lemma}
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Let $X$ be a random variable with mean 0 such that $a \leq X \leq b$. Then for all $\lambda \in \mathbb{R}$,

\[ \mathbb{E} \left[ e^{\lambda X} \right] \leq \exp \left( \frac{(b - a)^2 \lambda^2}{8} \right) \]
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We omit the proof of this lemma!
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Let $X_1, \ldots, X_n$ be independent random variable with mean $\mu_i$ such that $a_i \leq X_i \leq b_i$. Let $X = X_1 + \ldots + X_n$, and let $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} \mu_i$. Then for any $t > 0$

$$
P[X \geq \mu + t] \leq \exp \left( - \frac{2t^2}{\sum_{i=1}^{n} (b_i - a_i)^2} \right),$$

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P[X \leq \mu - t] \leq \exp \left( - \frac{2t^2}{\sum_{i=1}^{n} (b_i - a_i)^2} \right).$$
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Proof Outline (skipped):

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**Hoeffding Bounds**

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- Choose $\lambda = \frac{4t}{\sum_{i=1}^{n} (b_i - a_i)^2}$ to get the result.
Hoeffding Bounds

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- Choose \( \lambda = \frac{4t}{\sum_{i=1}^n (b_i - a_i)^2} \) to get the result.

This is not magic! you just need to optimise \( \lambda \)!
Method of Bounded Differences

Suppose, we have independent random variables $X_1, \ldots, X_n$. We want to study the random variable:

$$f(X_1, \ldots, X_n)$$
**Method of Bounded Differences**

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Some examples:

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Method of Bounded Differences

Framework

Suppose, we have independent random variables $X_1, \ldots, X_n$. We want to study the random variable:

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3. In a randomly generated graph, $X_i$ indicates if the $i$-th edge is present and $f(X_1, \ldots, X_m)$ represents the number of connected components of $G$
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3. In a randomly generated graph, $X_i$ indicates if the $i$-th edge is present and $f(X_1, \ldots, X_m)$ represents the number of connected components of $G$

In all those cases (and more) we can easily prove concentration of $f(X_1, \ldots, X_n)$ around its mean by the so-called Method of Bounded Differences.
Method of Bounded Differences

A function $f$ is called **Lipschitz with parameters** $c = (c_1, \ldots, c_n)$ if for all $i = 1, 2, \ldots, n$,

$$|f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_n)| \leq c_i,$$

where $x_i$ and $\tilde{x}_i$ are in the domain of the $i$-th coordinate.
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**McDiarmid’s inequality**

Let $X_1, \ldots, X_n$ be independent random variables. Let $f$ be **Lipschitz** with parameters $c = (c_1, \ldots, c_n)$. Let $X = f(X_1, \ldots, X_n)$. Then for any $t > 0$,

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- Notice the similarity with Hoeffding’s inequality! [Exercise 2/3.14]
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- Notice the similarity with Hoeffding’s inequality! [*Exercise 2/3.14*]
- The proof is omitted here (it requires the concept of martingales).
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Appendix: More on Moment Generating Functions (non-examinable)
Consider again $m$ balls assigned uniformly at random into $n$ bins.

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- Let $Z$ be the number of empty bins (after assigning the $m$ balls): $Z = Z(X_1, \ldots, X_m)$. 
- $Z$ is Lipschitz with $c = (1, \ldots, 1)$ (If we move one ball to another bin, number of empty bins changes by $\leq 1$).

By McDiarmid's inequality, for any $t \geq 0$, 

$$P\left[|Z - E[Z]| > t\right] \leq 2 \cdot e^{-2t^2/m}.$$ 

This is a decent bound, but for some values of $m$ it is far from tight and stronger bounds are possible through a refined analysis.
Consider again $m$ balls assigned uniformly at random into $n$ bins.

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We are given \( n \) items of sizes in the unit interval \([0, 1]\).
Application 4: Bin Packing

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- We want to pack those items into the fewest number of unit-capacity bins

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This is a typical example where proving concentration is much easier than calculating (or estimating) the expectation!
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- Let \( B = B(X_1, \ldots, X_n) \) be the optimal number of bins
- The Lipschitz conditions holds with \( c = (1, \ldots, 1) \). Why?
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Therefore

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\Pr\left[ \left| B - \mathbb{E}[B] \right| \geq t \right] \leq 2 \cdot e^{-2t^2/n}.
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The moment-generating function of a random variable $X$ is

$$M_X(t) = \mathbb{E}\left[e^{tX}\right], \quad \text{where } t \in \mathbb{R}.$$
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Using power series of $e$ and differentiating shows that $M_X(t)$ encapsulates all moments of $X$. 

1. If $X$ and $Y$ are two r.v.'s with $M_X(t) = M_Y(t)$ for all $t \in (-\delta, +\delta)$ for some $\delta > 0$, then the distributions $X$ and $Y$ are identical.

2. If $X$ and $Y$ are independent random variables, then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

3. Concentration © T. Sauerwald
The moment-generating function of a random variable $X$ is
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The moment-generating function of a random variable $X$ is

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$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

**Proof of 2:**

$$M_{X+Y}(t) = E\left[e^{t(X+Y)}\right] = E\left[e^{tX} \cdot e^{tY}\right] \overset{(1)}{=} E\left[e^{tX}\right] \cdot E\left[e^{tY}\right] = M_X(t)M_Y(t) \quad \square$$
Randomised Algorithms
Lecture 4: Markov Chains and Mixing Times

Thomas Sauerwald (tms41@cam.ac.uk)
Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Appendix: Remarks on Mixing Time (non-examin.)
Applications of Markov Chains in Computer Science

- Broadcasting
- Clustering
- Ranking Websites

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \ 
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
\]
Applications of Markov Chains in Computer Science

Broadcasting

\[ A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]
Applications of Markov Chains in Computer Science

Broadcasting

![Broadcasting Diagram]

Clustering

![Clustering Diagram]
Applications of Markov Chains in Computer Science

Broadcasting

Clustering

Ranking Websites

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

4. Markov Chains and Mixing Times © T. Sauerwald

Recap of Markov Chain Basics
Applications of Markov Chains in Computer Science

- Broadcasting
- Clustering
- Ranking Websites

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Applications of Markov Chains in Computer Science

- Broadcasting
- Clustering
- Ranking Websites
- Sampling and Optimisation
Applications of Markov Chains in Computer Science

Broadcasting

Ranking Websites

A =

Clustering

Sampling and Optimisation

4. Markov Chains and Mixing Times © T. Sauerwald

Recap of Markov Chain Basics
Applications of Markov Chains in Computer Science

- Broadcasting
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\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

- Sampling and Optimisation

- Load Balancing
- Particle Processes

4. Markov Chains and Mixing Times © T. Sauerwald

Recap of Markov Chain Basics
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- Ranking Websites
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- Particle Processes

Matrix A:

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
We say that $(X_t)_{t=0}^\infty$ is a Markov Chain on State Space $\Omega$ with Initial Distribution $\mu$ and Transition Matrix $P$ if:

1. For any $x \in \Omega$, $P[X_0 = x] = \mu(x)$.
2. The Markov Property holds: for all $t \geq 0$ and any $x_0, \ldots, x_t, x_{t+1} \in \Omega$,
   \[ P[X_{t+1} = x_{t+1} | X_t = x_t, \ldots, X_0 = x_0] = P[X_{t+1} = x_{t+1}]. \]

For all $0 \leq t_1 < t_2$, $x \in \Omega$,
\[ P[X_{t_2} = x] = \sum_{y \in \Omega} P[X_{t_2} = x | X_{t_1} = y] \cdot P[X_{t_1} = y]. \]
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   \[
P \left[ X_{t+1} = x_{t+1} \middle| X_t = x_t, \ldots, X_0 = x_0 \right] = P \left[ X_{t+1} = x_{t+1} \middle| X_t = x_t \right]
   := P(x_t, x_{t+1}).
\]
Markov Chains

We say that $(X_t)_{t=0}^\infty$ is a **Markov Chain** on State Space $\Omega$ with Initial Distribution $\mu$ and Transition Matrix $P$ if:

1. For any $x \in \Omega$, $P[X_0 = x] = \mu(x)$.
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   \[
P \left[ X_{t+1} = x_{t+1} \mid X_t = x_t, \ldots, X_0 = x_0 \right] = P \left[ X_{t+1} = x_{t+1} \mid X_t = x_t \right] = P(x_t, x_{t+1}).
   \]

From the definition one can deduce that (check!)
Markov Chains

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   \[
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   \]
   \[
   := P(x_t, x_{t+1}).
   \]

From the definition one can deduce that (check!)

- For all \(t, x_0, x_1, \ldots, x_t \in \Omega\),
  \[
P\left[X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0\right]
  = \mu(x_0) \cdot P(x_0, x_1) \cdot \ldots \cdot P(x_{t-2}, x_{t-1}) \cdot P(x_{t-1}, x_t).
  \]
We say that \((X_t)_{t=0}^{\infty}\) is a Markov Chain on State Space \(\Omega\) with Initial Distribution \(\mu\) and Transition Matrix \(P\) if:

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\]

\[
= P(x_t, x_{t+1}).
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\[
P \left[ X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0 \right]
\]

\[
= \mu(x_0) \cdot P(x_0, x_1) \cdot \ldots \cdot P(x_{t-2}, x_{t-1}) \cdot P(x_{t-1}, x_t).
\]

- For all \(0 \leq t_1 < t_2, x \in \Omega\),

\[
P \left[ X_{t_2} = x \right] = \sum_{y \in \Omega} P \left[ X_{t_2} = x \mid X_{t_1} = y \right] \cdot P \left[ X_{t_1} = y \right].
\]
What does a Markov Chain Look Like?

Example: the carbohydrate served with lunch in the college cafeteria.

This has transition matrix:

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 \\
1/4 & 0 & 3/4 \\
3/5 & 2/5 & 0
\end{bmatrix}
\]
Transition Matrices and Distributions

The Transition Matrix $P$ of a Markov chain $(\mu, P)$ on $\Omega = \{1, \ldots n\}$ is given by

$$P = \begin{pmatrix}
  P(1,1) & \cdots & P(1,n) \\
  \vdots & \ddots & \vdots \\
  P(n,1) & \cdots & P(n,n)
\end{pmatrix}.$$
Transition Matrices and Distributions

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\vdots & \ddots & \vdots \\
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\vdots & \ddots & \vdots \\
P(n, 1) & \ldots & P(n, n)
\end{pmatrix}.
$$

- $\rho^t = (\rho^t(1), \rho^t(2), \ldots, \rho^t(n))$: state vector at time $t$ (row vector).
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- Multiplying $\rho^t$ by $P$ corresponds to advancing the chain one step:

$$\rho^t(y) = \sum_{x \in \Omega} \rho^{t-1}(x) \cdot P(x, y) \quad \text{and thus} \quad \rho^t = \rho^{t-1} \cdot P.$$
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- The Markov Property and line above imply that for any $t \geq 0$
  $$\rho^t = \rho \cdot P^{t-1} \quad \text{and thus} \quad P^t(x, y) = \mathbb{P} [X_t = y \mid X_0 = x].$$
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$$\rho^t = \rho \cdot P^{t-1} \quad \text{and thus} \quad P^t(x, y) = \mathbb{P} [X_t = y \mid X_0 = x].$$

Thus $\rho^t(x) = (\mu P^t)(x)$ and so $\rho^t = \mu P^t = (\mu P^t(1), \mu P^t(2), \ldots, \mu P^t(n))$. 

Everything boils down to deterministic vector/matrix computations ⇒ can replace $\rho$ by any (load) vector and view $P$ as a balancing matrix!
Transition Matrices and Distributions

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\end{pmatrix}.
$$

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  $$

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  $$
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  $$

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- Everything boils down to deterministic vector/matrix computations

⇒ can replace $\rho$ by any (load) vector and view $P$ as a balancing matrix!
A non-negative integer random variable $\tau$ is a stopping time for $(X_t)_{t \geq 0}$ if for every $s \geq 0$ the event $\{\tau = s\}$ depends only on $X_0, \ldots, X_s$. 

Example - College Carbs Stopping times:

✓ "We had rice yesterday" leads to $\tau := \min\{t \geq 1 : X_t - 1 = \text{"rice"}\} \times \text{"We are having pasta next Thursday"}$. 

For two states $x, y \in \Omega$, we call $h(x, y)$ the hitting time of $y$ from $x$: 

$h(x, y) := \mathbb{E}_x[\tau_y] = \mathbb{E}_x[\tau_y | X_0 = x]$ 

where $\tau_y = \min\{t \geq 1 : X_t = y\}$. 

Some distinguish between $\tau^+_y = \min\{t \geq 1 : X_t = y\}$ and $\tau_y = \min\{t \geq 0 : X_t = y\}$. 

Hitting times are the solution to a set of linear equations: 

$h(x, y) = 1 + \sum_{z \in \Omega \setminus \{y\}} \mathbb{P}(x, z) \cdot h(z, y) \quad \forall x \neq y \in \Omega$. 

A Useful Identity
A non-negative integer random variable \( \tau \) is a stopping time for \((X_t)_{t \geq 0}\) if for every \( s \geq 0 \) the event \( \{ \tau = s \} \) depends only on \( X_0, \ldots, X_s \).

**Example - College Carbs Stopping times:**

✓ “We had rice yesterday”
Stopping and Hitting Times

A non-negative integer random variable $\tau$ is a stopping time for $(X_t)_{t \geq 0}$ if for every $s \geq 0$ the event \{\(\tau = s\)\} depends only on $X_0, \ldots, X_s$.

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✓ “We had rice yesterday” $\leadsto \tau := \min \{t \geq 1 : X_{t-1} = \text{“rice”}\}$
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× “We are having pasta next Thursday”
Stopping and Hitting Times

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**Example** - College Carbs Stopping times:

- “We had rice yesterday” $\sim \tau := \min \{t \geq 1 : X_{t-1} = \text{rice}\}$
- “We are having pasta next Thursday”

For two states $x, y \in \Omega$ we call $h(x, y)$ the hitting time of $y$ from $x$:

$$h(x, y) := E_x[\tau_y] = E[\tau_y | X_0 = x] \quad \text{where } \tau_y = \min\{t \geq 1 : X_t = y\}.$$
A non-negative integer random variable $\tau$ is a stopping time for $(X_t)_{t \geq 0}$ if for every $s \geq 0$ the event $\{\tau = s\}$ depends only on $X_0, \ldots, X_s$.

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× “We are having pasta next Thursday”

For two states $x, y \in \Omega$ we call $h(x, y)$ the hitting time of $y$ from $x$:

\[
h(x, y) := \mathbb{E}_x[\tau_y] = \mathbb{E}[\tau_y \mid X_0 = x] \quad \text{where } \tau_y = \min\{ t \geq 1 : X_t = y \}.
\]

Some distinguish between $\tau_y^+ = \min\{ t \geq 1 : X_t = y \}$ and $\tau_y = \min\{ t \geq 0 : X_t = y \}$.
Stopping and Hitting Times

A non-negative integer random variable $\tau$ is a stopping time for $(X_t)_{t \geq 0}$ if for every $s \geq 0$ the event $\{\tau = s\}$ depends only on $X_0, \ldots, X_s$.

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For two states $x, y \in \Omega$ we call $h(x, y)$ the hitting time of $y$ from $x$:

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**A Useful Identity**

Hitting times are the solution to a set of linear equations:

$$h(x, y) \overset{\text{Markov Prop.}}{=} 1 + \sum_{z \in \Omega \setminus \{y\}} P(x, z) \cdot h(z, y) \quad \forall x \neq y \in \Omega.$$
Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Appendix: Remarks on Mixing Time (non-examin.)
A Markov Chain is irreducible if for every pair of states \( x, y \in \Omega \) there is an integer \( k \geq 0 \) such that \( P^k(x, y) > 0 \).
A Markov Chain is **irreducible** if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$. 

![Graph of irreducible Markov Chain](image)
A Markov Chain is irreducible if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$.

Exercise: Which of the two chains (if any) are irreducible?
A Markov Chain is **irreducible** if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$.

**Exercise:** Which of the two chains (if any) are irreducible?
A Markov Chain is **irreducible** if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$.

**Finite Hitting Time Theorem**

For any states $x$ and $y$ of a **finite irreducible** Markov Chain $h(x, y) < \infty$. 
Stationary Distribution

A probability distribution $\pi = (\pi(1), \ldots, \pi(n))$ is the stationary distribution of a Markov Chain if $\pi P = \pi$ ($\pi$ is a left eigenvector with eigenvalue 1)
Stationary Distribution

A probability distribution $\pi = (\pi(1), \ldots, \pi(n))$ is the stationary distribution of a Markov Chain if $\pi P = \pi$ ($\pi$ is a left eigenvector with eigenvalue 1).

College carbs example:

$$\begin{pmatrix} 4/13, 4/13, 5/13 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 3/5 & 2/5 & 0 \end{pmatrix} = \begin{pmatrix} 4/13, 4/13, 5/13 \end{pmatrix}$$

Diagram:

- Rice to Pasta with probability $1/4$
- Pasta to Potato with probability $1/2$
- Potato to Rice with probability $3/5$
- Rice to Potato with probability $3/4$
- Pasta to Rice with probability $2/5$
A probability distribution $\pi = (\pi(1), \ldots, \pi(n))$ is the **stationary distribution** of a Markov Chain if $\pi P = \pi$ ($\pi$ is a left eigenvector with eigenvalue 1).

**College carbs example:**

$$
\begin{pmatrix}
\frac{4}{13} & \frac{4}{13} & \frac{5}{13} \\
\pi \\
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & 0 & \frac{3}{4} \\
\frac{3}{5} & \frac{2}{5} & 0 \\
\end{pmatrix}
= 
\begin{pmatrix}
\frac{4}{13} & \frac{4}{13} & \frac{5}{13} \\
\pi \\
\end{pmatrix}
$$

- A Markov Chain reaches **stationary distribution** if $\rho^t = \pi$ for some $t$. 
Stationary Distribution

A probability distribution $\pi = (\pi(1), \ldots, \pi(n))$ is the stationary distribution of a Markov Chain if $\pi P = \pi$ ($\pi$ is a left eigenvector with eigenvalue 1)

College carbs example:

$$
\begin{pmatrix}
\frac{4}{13}, \frac{4}{13}, \frac{5}{13} \\
\pi
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & 1/2 & 1/2 \\
1/4 & 0 & 3/4 \\
3/5 & 2/5 & 0
\end{pmatrix}
=
\begin{pmatrix}
\frac{4}{13}, \frac{4}{13}, \frac{5}{13} \\
\pi
\end{pmatrix}
$$

- A Markov Chain reaches stationary distribution if $\rho^t = \pi$ for some $t$.
- If reached, then it persists: If $\rho^t = \pi$ then $\rho^{t+k} = \pi$ for all $k \geq 0$. 
Stationary Distribution

A probability distribution \( \pi = (\pi(1), \ldots, \pi(n)) \) is the **stationary distribution** of a Markov Chain if \( \pi P = \pi \) (\( \pi \) is a left eigenvector with eigenvalue 1)

**College carbs example:**

\[
\left( \begin{array}{c} 4/13 \\ 4/13 \\ 5/13 \end{array} \right) \cdot \left( \begin{array}{ccc} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 3/5 & 2/5 & 0 \end{array} \right) = \left( \begin{array}{c} 4/13 \\ 4/13 \\ 5/13 \end{array} \right)
\]

- A Markov Chain reaches **stationary distribution** if \( \rho^t = \pi \) for some \( t \).
- If reached, then it **persists**: If \( \rho^t = \pi \) then \( \rho^{t+k} = \pi \) for all \( k \geq 0 \).

**Existence and Uniqueness** of a Positive Stationary Distribution

Let \( P \) be **finite, irreducible M.C.**, then there exists a unique probability distribution \( \pi \) on \( \Omega \) such that \( \pi = \pi P \) and \( \pi(x) = 1/h(x, x) > 0, \forall x \in \Omega \).
A Markov Chain is **aperiodic** if for all \( x \in \Omega \), \( \gcd\{ t \geq 1 : P^t(x, x) > 0 \} = 1 \). Otherwise we say it is periodic.

**Question:** Which of the two chains (if any) are aperiodic?
A Markov Chain is aperiodic if for all \( x \in \Omega \), \( \gcd\{ t \geq 1 : P^t(x, x) > 0 \} = 1 \).
Otherwise we say it is periodic.
Periodicity

- A Markov Chain is aperiodic if for all $x \in \Omega$, $\gcd\{t \geq 1 : P^t(x, x) > 0\} = 1$.
- Otherwise we say it is periodic.

Question: Which of the two chains (if any) are aperiodic?
Periodicity

- A Markov Chain is aperiodic if for all \( x \in \Omega \), \( \gcd\{ t \geq 1 : P^t(x, x) > 0 \} = 1 \).
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**Question:** Which of the two chains (if any) are aperiodic?
Convergence Theorem

Let $P$ be any finite, irreducible, aperiodic Markov Chain with stationary distribution $\pi$. Then for any $x, y \in \Omega$, 

$$\lim_{t \to \infty} P^t(x, y) = \pi(y).$$
Convergence Theorem

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- mentioned before: For finite irreducible M.C.’s $\pi$ exists, is unique and

$$\pi(y) = \frac{1}{h(y, y)} > 0.$$
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- mentioned before: For finite irreducible M.C.’s $\pi$ exists, is unique and

$$\pi(y) = \frac{1}{h(y, y)} > 0.$$

- We will prove a simpler version of the Convergence Theorem after introducing Spectral Graph Theory.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 0
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
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- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 3
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 4
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 6
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$.
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P_t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 8
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 9
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$.
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 10
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.  

Step: 11
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
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Convergence to Stationarity (Example)

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- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 14
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with \(1/2\) and moves left (or right) w.p. \(1/4\)
- At step \(t\) the value at vertex \(x \in \{1, 2, \ldots, 12\}\) is \(P^t(1, x)\).

Step: 15
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 16
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 17
### Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

![Diagram showing a Markov chain with probabilities at each vertex and step 18 highlighted.](image-url)
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 19
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

![Step: 20](image-url)
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 21
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with \(\frac{1}{2}\) and moves left (or right) w.p. \(\frac{1}{4}\)
- At step \(t\) the value at vertex \(x \in \{1, 2, \ldots, 12\}\) is \(P^t(1, x)\).
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 23
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 24
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4.
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 25
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
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Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $\frac{1}{2}$ and moves left (or right) w.p. $\frac{1}{4}$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

![Diagram showing the Markov chain with probabilities at each vertex at step 29]
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

**Step: 30**
Convergence to Stationarity (Example)

- Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 31
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 32
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 33
Convergence to Stationarity (Example)

- Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 34
Convergence to Stationarity (Example)

- Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 36
Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with \(\frac{1}{2}\) and moves left (or right) w.p. \(\frac{1}{4}\)
- At step \(t\) the value at vertex \(x \in \{1, 2, \ldots, 12\}\) is \(P^t(1, x)\).

Step: 39
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 41
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P_t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

```
<p>| | | | |</p>
<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<tr>
<td>0.091</td>
<td>0.090</td>
<td>0.090</td>
<td>0.091</td>
</tr>
</tbody>
</table>
```

Step: 44
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 46
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 47
Convergence to Stationarity (Example)

- Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 48
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

![Diagram showing the Markov chain process at step 50 with probabilities at each vertex]
Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Appendix: Remarks on Mixing Time (non-examin.)
How Similar are Two Probability Measures?

You are presented three loaded (unfair) dice $A, B, C$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[A = x]$</td>
<td>1/3</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>$P[B = x]$</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>$P[C = x]$</td>
<td>1/6</td>
<td>1/6</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>9/24</td>
</tr>
</tbody>
</table>

Loaded Dice

Question 1: Which dice is the least fair? Most choose $A$. Why?

Question 2: Which dice is the most fair? Dice $B$ and $C$ seem “fairer” than $A$ but which is fairest?
How Similar are Two Probability Measures?

**Loaded Dice**

- You are presented three loaded (unfair) dice $A$, $B$, $C$:

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<tr>
<td>$P[A=x]$</td>
<td>1/3</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>$P[B=x]$</td>
<td>1/4</td>
<td>1/8</td>
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<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>$P[C=x]$</td>
<td>1/6</td>
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**Question 1:** Which dice is the least fair?
How Similar are Two Probability Measures?

### Loaded Dice

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<tr>
<td>$P[A = x]$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{12}$</td>
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</tr>
<tr>
<td>$P[B = x]$</td>
<td>$\frac{1}{4}$</td>
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**Question 1:** Which dice is the least fair?

**Question 2:** Which dice is the most fair?
How Similar are Two Probability Measures?

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<td>1/8</td>
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Question 1: Which dice is the least fair?

Question 2: Which dice is the most fair?
How Similar are Two Probability Measures?

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#### Question 1: Which dice is the least fair?

#### Question 2: Which dice is the most fair?

![Probability bar graph for $A$, $B$, $C$.]
You are presented three loaded (unfair) dice $A, B, C$:

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<td>1/8</td>
<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>$P[C = x]$</td>
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**Question 1:** Which dice is the least fair?

**Question 2:** Which dice is the most fair?
How Similar are Two Probability Measures?

You are presented three loaded (unfair) dice $A$, $B$, $C$:

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<td>$P[B = x]$</td>
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<td>$P[C = x]$</td>
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<td>1/8</td>
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**Question 1:** Which dice is the least fair?

**Question 2:** Which dice is the most fair?
You are presented three loaded (unfair) dice $A$, $B$, $C$:

<table>
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<tr>
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**Question 1:** Which dice is the least fair? Most choose $A$.

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How Similar are Two Probability Measures?

You are presented three loaded (unfair) dice $A$, $B$, $C$:

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We need a **formal “fairness measure”** to compare probability distributions!
Total Variation Distance

The Total Variation Distance between two probability distributions $\mu$ and $\eta$ on a countable state space $\Omega$ is given by

$$
\|\mu - \eta\|_{tv} = \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \eta(\omega)|.
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**Loaded Dice:** Let $D = \text{Unif}\{1, 2, 3, 4, 5, 6\}$ be the law of a fair dice:

$$
\|D - A\|_{tv} = \frac{1}{2} \left( 2 \left| \frac{1}{6} - \frac{1}{3} \right| + 4 \left| \frac{1}{6} - \frac{1}{12} \right| \right) = \frac{1}{3}
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\end{align*}
\]

Thus \( \| D - B \|_{tv} = \| D - C \|_{tv} \) and \( \| D - A \|_{tv} \), \( \| D - B \|_{tv} \), \( \| D - C \|_{tv} \) are equally "fair" (in TV distance).
The **Total Variation Distance** between two probability distributions $\mu$ and $\eta$ on a countable state space $\Omega$ is given by

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\end{align*}
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Thus

$$\|D - B\|_{tv} = \|D - C\|_{tv} \quad \text{and} \quad \|D - B\|_{tv}, \|D - C\|_{tv} < \|D - A\|_{tv}.$$  

So $A$ is the least “fair”, however $B$ and $C$ are equally “fair” (in TV distance).
Let $P$ be a finite Markov Chain with stationary distribution $\pi$. 

**Exercise 4/5.5**

For any $\mu$, 
\[
\|P^t \mu - \pi\|_{tv} \leq \max_{x \in \Omega} \|P^t x - \pi\|_{tv},
\]

For any finite, irreducible, aperiodic Markov Chain 
\[
\lim_{t \to \infty} \max_{x \in \Omega} \|P^t x - \pi\|_{tv} = 0.
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- Let $\mu$ be a prob. vector on $\Omega$ (might be just one vertex) and $t \geq 0$. Then

$$P^t_\mu := \mathbb{P}[X_t = \cdot \mid X_0 \sim \mu],$$

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**Convergence Theorem (Implication for TV Distance)**

We will see a similar result later after introducing spectral techniques (Lecture 12)!
Let $P$ be a finite Markov Chain with stationary distribution $\pi$.

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Let $P$ be a finite Markov Chain with stationary distribution $\pi$.

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---

**Convergence Theorem** (Implication for TV Distance)

For any finite, irreducible, aperiodic Markov Chain

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Convergence Theorem: “Nice” Markov Chains converge to stationarity.
Mixing Time of a Markov Chain

Convergence Theorem: “Nice” Markov Chains converge to stationarity.

Question: How fast do they converge?
Convergence Theorem: “Nice” Markov Chains converge to stationarity.

**Question:** How fast do they converge?

**Mixing Time**

The mixing time $\tau_x(\epsilon)$ of a finite Markov Chain $P$ with stationary distribution $\pi$ is defined as

$$
\tau_x(\epsilon) = \min \left\{ t \geq 0 : \| P_x^t - \pi \|_{tv} \leq \epsilon \right\},
$$

where $P_x^t$ denotes the $t$-th step of the Markov chain starting at state $x$. The mixing time $\tau(\epsilon)$ is then defined as

$$
\tau(\epsilon) = \max_x \tau_x(\epsilon).
$$
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- This is how long we need to wait until we are “\( \epsilon \)-close” to stationarity
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See final slides for some comments on why we choose $1/4$. 
Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Appendix: Remarks on Mixing Time (non-examin.)
Thanks to Krzysztof Onak (pointer) and Eric Price (graph)

Source: Slides by Ronitt Rubinfeld
What is Card Shuffling?

Source: wikipedia

How long does it take to shuffle a deck of 52 cards?

Persi Diaconis (Professor of Statistics and former Magician)

One of the leading experts in the field who has related card shuffling to many other mathematical problems.

Here we will focus on one shuffling scheme which is easy to analyse. How quickly do we converge to the uniform distribution over all $n!$ permutations?

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The Card Shuffling Markov Chain

**TOPTORANDOMSHUFFLE** (Input: A pile of \( n \) cards)

1: For \( t = 1, 2, \ldots \)
2: Pick \( i \in \{1, 2, \ldots, n\} \) uniformly at random
3: Take the top card and insert it behind the \( i \)-th card
The Card Shuffling Markov Chain

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3: Take the top card and insert it behind the \( i \)-th card

This is a slightly informal definition, so let us look at a small example...
**The Card Shuffling Markov Chain**

**TOP_RANDOM_SHUFFLE** (Input: A pile of $n$ cards)

1. For $t = 1, 2, \ldots$
2. Pick $i \in \{1, 2, \ldots, n\}$ uniformly at random
3. Take the top card and insert it behind the $i$-th card

This is a slightly informal definition, so let us look at a small example...

We will focus on this “small” set of cards ($n = 8$)
Even if we know which set of cards come after 8, every permutation is equally likely! This leads to the deck of cards being perfectly mixed after the last card "8" reaches the top and is inserted to a random position!
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\[
\text{leadsto}
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Analysing the Mixing Time (Intuition)

A deck of cards is perfectly mixed after the last card “8” reaches the top and is inserted to a random position!

How long does it take for the last card “$n$” to become top card?

At the last position, card “$n$” moves up with probability $\frac{1}{n}$ at each step.

At the second last position, card “$n$” moves up with probability $\frac{2}{n}$.

... At the second position, card “$n$” moves up with probability $\frac{n-1}{n}$.

One final step to randomise card “$n$” (with probability 1).

This is a “reversed” coupon collector process with $n$ cards, which takes $n \log n$ in expectation.

Using the so-called coupling method, one could prove $t_{\text{mix}} \leq n \log n$. 

4. Markov Chains and Mixing Times © T. Sauerwald
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Using the so-called coupling method, one could prove $t_{\text{mix}} \leq n \log n$. 
Riffle Shuffle

1. Split a deck of \( n \) cards into two piles (thus the size of each portion will be Binomial)
2. Riffle the cards together so that the card drops from the left (or right) pile with probability proportional to the number of remaining cards.

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\| & P_t - \pi & \| \\
1.000 & 1.000 & 1.000 & 1.000 & 0.924 & 0.614 & 0.334 & 0.167 & 0.085 & 0.043 \\
\end{array}
\]

Figure: Total Variation Distance for \( t \) riffle shuffles of 52 cards.
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</tr>
<tr>
<td>2</td>
<td>1.000</td>
</tr>
<tr>
<td>3</td>
<td>1.000</td>
</tr>
<tr>
<td>4</td>
<td>1.000</td>
</tr>
<tr>
<td>5</td>
<td>0.924</td>
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<tr>
<td>6</td>
<td>0.614</td>
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<tr>
<td>7</td>
<td>0.334</td>
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<tr>
<td>8</td>
<td>0.167</td>
</tr>
<tr>
<td>9</td>
<td>0.085</td>
</tr>
<tr>
<td>10</td>
<td>0.043</td>
</tr>
</tbody>
</table>

Figure: Total Variation Distance for $t$ riffle shuffles of 52 cards.
Riffle Shuffle

1. Split a deck of $n$ cards into two piles (thus the size of each portion will be Binomial)

2. Riffle the cards together so that the card drops from the left (or right) pile with probability proportional to the number of remaining cards

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|P^t - \pi|_{tv}$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.924</td>
<td>0.614</td>
<td>0.334</td>
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<td>0.085</td>
<td>0.043</td>
</tr>
</tbody>
</table>

**Figure:** Total Variation Distance for $t$ riffle shuffles of 52 cards.

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TRAILING THE DOVETAIL SHUFFLE TO ITS LAIR

By Dave Bayer\(^1\) and Persi Diaconis\(^2\)

Columbia University and Harvard University

We analyze the most commonly used method for shuffling cards. The main result is a simple expression for the chance of any arrangement after any number of shuffles. This is used to give sharp bounds on the approach to randomness: \(\frac{3}{2} \log_2 n + \theta\) shuffles are necessary and sufficient to mix up $n$ cards.

Key ingredients are the analysis of a card trick and the determination of the idempotents of a natural commutative subalgebra in the symmetric group algebra.
Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Appendix: Remarks on Mixing Time (non-examin.)
Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$. 

---

Independent Set

Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$.
Given an undirected graph $G = (V, E)$, an **independent set** is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$. 

$S = \{1, 4\}$ is an independent set.
Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$. 

$S = \{2, 6, 8\}$ is an independent set.
Independent Set

Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$.

$S = \{1, 7, 8\}$ is not an independent set $\times$

Markov Chain for Sampling Independent Sets (1/2) (non-examin.)
Given an undirected graph \( G = (V, E) \), an independent set is a subset \( S \subseteq V \) such that there are no two vertices \( u, v \in S \) with \( \{u, v\} \in E(G) \).
Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$.

How can we take a sample from the space of all independent sets?
Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$.

How can we take a sample from the space of all independent sets?

Naive brute-force would take an insane amount of time (and space)!
Given an undirected graph $G = (V, E)$, an \textit{independent set} is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$.

How can we take a sample from the space of all independent sets?

Naive brute-force would take an insane amount of time (and space)!

We can use a \textit{generic Markov Chain Monte Carlo} approach to tackle this problem!
Markov Chain for Sampling Independent Sets (2/2) (non-examin.)

**INDEPENDENT SET SAMPLER**

1: Let $X_0$ be an arbitrary independent set in $G$

2: **For** $t = 0, 1, 2, \ldots$:

3: Pick a vertex $v \in V(G)$ uniformly at random

4: **If** $v \in X_t$ then $X_{t+1} \leftarrow X_t \setminus \{v\}$

5: **elif** $v \not\in X_t$ and $X_t \cup \{v\}$ is an independent set **then** $X_{t+1} \leftarrow X_t \cup \{v\}$

6: **else** $X_{t+1} \leftarrow X_t$

**Key Question:** What is the mixing time of this Markov Chain?

Not covered here, see the textbook by Mitzenmacher and Upfal.
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1: Let $X_0$ be an arbitrary independent set in $G$
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$X_0 = \{1, 4\}$
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2. For $t = 0, 1, 2, \ldots$
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$X_0 = \{1, 4\}$

$v = 1$
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$X_0 = \{1, 4\}$

$X_1 = \{4\}$

$v = 1$

$v = 8$
INDEPENDENT SET SAMPLER
1: Let $X_0$ be an arbitrary independent set in $G$
2: For $t = 0, 1, 2, \ldots$
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Key Question: What is the mixing time of this Markov Chain?

Not covered here, see the textbook by Mitzenmacher and Upfal

Markov Chains and Mixing Times © T. Sauerwald Application 2: Markov Chain Monte Carlo (non-examin.)
**INDEPENDENT_SET_SAMPLER**

1: Let $X_0$ be an arbitrary independent set in $G$
2: **For** $t = 0, 1, 2, \ldots$:
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6: **else** $X_{t+1} \leftarrow X_t$

$X_0 = \{1, 4\}$

**Diagram:**

- $v = 1$ leads to $X_1 = \{4\}$
- $v = 8$ leads to $X_1 = \{1, 4, 8\}$
- $v = 6$
INDEPENDENT SETSAMPLER
1: Let $X_0$ be an arbitrary independent set in $G$
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$X_1 = \{4\}$

$X_1 = \{1, 4, 8\}$

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INDEPENDENT_SET_SAMPLER
1: Let $X_0$ be an arbitrary independent set in $G$
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3: \begin{itemize}
   \item Pick a vertex $v \in V(G)$ uniformly at random
   \item If $v \in X_t$ then $X_{t+1} \leftarrow X_t \setminus \{v\}$
   \item elif $v \notin X_t$ and $X_t \cup \{v\}$ is an independent set then $X_{t+1} \leftarrow X_t \cup \{v\}$
   \item else $X_{t+1} \leftarrow X_t$
\end{itemize}

Remark
**INDEPENDENT SET SAMPLER**

1: Let $X_0$ be an arbitrary independent set in $G$
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**Remark**

- This is a *local* definition (no explicit definition of $P$!)


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- This is a *local* definition (no explicit definition of $P$!)
- This chain is *irreducible* (every independent set is reachable)
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Remark

- This is a local definition (no explicit definition of $P$!)
- This chain is irreducible (every independent set is reachable)
- This chain is aperiodic (Check!)
**Markov Chain for Sampling Independent Sets (2/2) (non-examin.)**

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6: **else** $X_{t+1} \leftarrow X_t$

---

**Remark**

- This is a **local** definition (no explicit definition of $P$!)
- This chain is **irreducible** (every independent set is reachable)
- This chain is **aperiodic** (Check!)
- The **stationary distribution** is uniform, since $P_{u,v} = P_{v,u}$ (Check!)
INDEPENDENT SET SAMPLER

1: Let $X_0$ be an arbitrary independent set in $G$
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- This is a local definition (no explicit definition of $P$!)
- This chain is irreducible (every independent set is reachable)
- This chain is aperiodic (Check!)
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Key Question: What is the mixing time of this Markov Chain?
Markov Chain for Sampling Independent Sets (2/2) (non-examin.)

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Application 1: Card Shuffling

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Further Remarks on the Mixing Time (non-examin.)

- One can prove $\max_x \| P_x^t - \pi \|_{tv}$ is non-increasing in $t$ (this means if the chain is $\epsilon$-mixed at step $t$, then this also holds in future steps) \[Mitzenmacher, Upfal, 12.3\]
Further Remarks on the Mixing Time (non-examin.)

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- We chose $t_{mix} := \tau(1/4)$, but other choices of $\epsilon$ are perfectly fine too (e.g, $t_{mix} := \tau(1/e)$ is often used); in fact, any constant $\epsilon \in (0, 1/2)$ is possible.

Remark: This freedom on how to pick $\epsilon$ relies on the sub-multiplicative property of a (version) of the variation distance. First, let $d(t) := \max_x \| P^t_x - \pi \|_{tv}$ be the variation distance after $t$ steps when starting from the worst state. Further, define $d(t) := \max_{\mu, \nu} \| P^t \mu - P^t \nu \|_{tv}$. These quantities are related by the following double inequality $d(t) \leq d(t) \leq 2d(t)$. Further, $d(t)$ is sub-multiplicative, that is for any $s, t \geq 1$, $d(s+t) \leq d(s) \cdot d(t)$. Hence for any fixed $0 < \epsilon < \delta < 1/2$ it follows from the above that $\tau(\epsilon) \leq \lceil \log \frac{\epsilon}{\ln(2\delta)} \rceil \tau(\delta)$. In particular, for any $\epsilon < 1/4$ $\tau(\epsilon) \leq \lceil \log \frac{2}{\epsilon} - 1 \rceil \tau(1/4)$. This 2 is the reason why we ultimately need $\epsilon < 1/2$ in this derivation. On the other hand, see Exercise (4/5).8 why $\epsilon < 1/2$ is also necessary. Hence smaller constants $\epsilon < 1/4$ only increase the mixing time by some constant factor.
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be the variation distance after $t$ steps when starting from the worst state. Further, define

$$\overline{d}(t) := \max_{\mu, \nu} ||P^t_\mu - P^t_\nu||_{tv}.$$ 

These quantities are related by the following double inequality

$$d(t) \leq \overline{d}(t) \leq 2d(t).$$

Further, $\overline{d}(t)$ is sub-multiplicative, that is for any $s, t \geq 1$,

$$\overline{d}(s + t) \leq \overline{d}(s) \cdot \overline{d}(t).$$

Hence for any fixed $0 < \epsilon < \delta < 1/2$ it follows from the above that

$$\tau(\epsilon) \leq \left\lceil \frac{\ln \epsilon}{\ln(2\delta)} \right\rceil \tau(\delta).$$

In particular, for any $\epsilon < 1/4$

$$\tau(\epsilon) \leq \left\lceil \log_2 \epsilon^{-1} \right\rceil \tau(1/4).$$
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$$\tau(\epsilon) \leq \left[ \log_2 \epsilon^{-1} \right] \tau(1/4).$$

Hence smaller constants $\epsilon < 1/4$ only increase the mixing time by some constant factor.
Randomised Algorithms
Lecture 5: Random Walks, Hitting Times and Application to 2-SAT

Thomas Sauerwald (tms41@cam.ac.uk)
Application 3: Ehrenfest Chain and Hypercubes

Random Walks on Graphs, Hitting Times and Cover Times

Random Walks on Paths and Grids

SAT and a Randomised Algorithm for 2-SAT
The Ehrenfest Markov Chain

Ehrenfest Model

- A simple model for the exchange of molecules between two boxes
The Ehrenfest Markov Chain

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Random Walk on the Hypercube

5. Hitting Times © T. Sauerwald Application 3: Ehrenfest Chain and Hypercubes
The Ehrenfest Markov Chain

Ehrenfest Model

- A simple model for the exchange of molecules between two boxes
- We have $d$ particles

\[
P(x, x-1) = \frac{x}{d}
\]
\[
P(x, x+1) = \frac{d-x}{d}
\]
The Ehrenfest Markov Chain

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- We have \( d \) particles
- At each step a particle is selected \textit{uniformly at random} and switches to the other box
The Ehrenfest Markov Chain

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\[
\begin{align*}
P(x, x - 1) &= \frac{x}{d} \\
P(x, x + 1) &= \frac{d - x}{d}
\end{align*}
\]

Let us now enlarge the state space by looking at each particle individually! For each particle an indicator variable \( \Omega = \{0, 1\}^d \)

At each step: pick a random coordinate in \( [d] \) and flip it

Random Walk on the Hypercube

5. Hitting Times © T. Sauerwald
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**Ehrenfest Model**

- A simple model for the exchange of molecules between two boxes
- We have \( d \) particles
- At each step a particle is selected uniformly at random and switches to the other box
- If \( \Omega = \{0, 1, \ldots, d\} \) denotes the number of particles in the red box, then:

\[
P_{x,x-1} = \frac{x}{d} \quad \text{and} \quad P_{x,x+1} = \frac{d - x}{d}.
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Random Walk on the Hypercube

5. Hitting Times © T. Sauerwald

Application 3: Ehrenfest Chain and Hypercubes
The Ehrenfest Markov Chain

A simple model for the exchange of molecules between two boxes

- We have \( d \) particles
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The Ehrenfest Markov Chain

**Ehrenfest Model**

- A simple model for the exchange of molecules between two boxes
- We have $d$ particles labelled 1, 2, ..., $d$
- At each step a particle is selected uniformly at random and switches to the other box
- If $\Omega = \{0, 1, \ldots, d\}$ denotes the number of particles in the red box, then:

  $$P_{x,x-1} = \frac{x}{d} \quad \text{and} \quad P_{x,x+1} = \frac{d-x}{d}.$$ 

Let us now enlarge the state space by looking at each particle individually!
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Random Walk on the Hypercube
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- For each particle an indicator variable $\Rightarrow \Omega = \{0, 1\}^d$
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Random Walk on the Hypercube

- For each particle an indicator variable $\Rightarrow \Omega = \{0, 1\}^d$
- At each step: pick a random coordinate in $[d]$ and flip it

$P_{7,6} = \frac{7}{10}$

$P_{7,8} = \frac{3}{10}$
(Non-Lazy) Random Walk on the Hypercube

- For each particle an indicator variable $\Omega = \{0, 1\}^d$
- At each step: pick a random coordinate in $[d]$ and flip it
Analysis of the Mixing Time

For each particle an indicator variable $\Omega = \{0, 1\}^d$

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**Problem:** This Markov Chain is periodic, as the number of ones always switches between odd to even!
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Lazy Random Walk (1st Version)
Analysis of the Mixing Time

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- For each particle an indicator variable \( \Omega = \{0, 1\}^d \)
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Lazy Random Walk (1st Version)

- At each step $t = 0, 1, 2 \ldots$
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  - With prob. 1/2 flip coordinate.
Analysis of the Mixing Time

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Lazy Random Walk (1st Version)

- At each step \( t = 0, 1, 2 \ldots \)
  - Pick a random coordinate in [d]
  - With prob. 1/2 flip coordinate.

Lazy Random Walk (2nd Version)

- At each step \( t = 0, 1, 2 \ldots \)
  - Pick a random coordinate in [d]
Analysis of the Mixing Time

(Non-Lazy) Random Walk on the Hypercube

- For each particle an indicator variable $\Omega = \{0, 1\}^d$
- At each step: pick a random coordinate in $[d]$ and flip it

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Solution: Add self-loops to break periodic behaviour!

Lazy Random Walk (1st Version)

- At each step $t = 0, 1, 2 \ldots$
  - Pick a random coordinate in $[d]$
  - With prob. $1/2$ flip coordinate.

Lazy Random Walk (2nd Version)

- At each step $t = 0, 1, 2 \ldots$
  - Pick a random coordinate in $[d]$
  - Set coordinate to $\{0, 1\}$ uniformly.
Analysis of the Mixing Time

For each particle an indicator variable $\Omega = \{0, 1\}^d$

At each step: pick a random coordinate in $[d]$ and flip it.

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  - With prob. $1/2$ flip coordinate.

**Lazy Random Walk (2nd Version)**
- At each step $t = 0, 1, 2, \ldots$
  - Pick a random coordinate in $[d]$
  - Set coordinate to $\{0, 1\}$ uniformly.

These two chains are equivalent!
Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \(\{0, 1\}^d\).

Coupon Collector leads to mixing time should be \(O(d \log d)\). We won’t formalise this argument here (see [Ex. 4/5.11]).
Example of a Random Walk on a 4-Dimensional Hypercube

<table>
<thead>
<tr>
<th>t</th>
<th>Coord.</th>
<th>$\chi_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0 0 0 0 0</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0 ? 0 0 0</td>
</tr>
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Coupon Collector $\Rightarrow$ mixing time should be $O(d \log d)$.

We won't formalise this argument here (see [Ex. 4/5.11]).
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Once all coordinates have been picked at least once, the state is uniformly at random in $\{0, 1\}^d$. Coupon Collector leads to mixing time should be $O(d \log d)$. We won’t formalise this argument here (see [Ex. 4/5.11]).

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<td>[0 0 0 0 0]</td>
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<tr>
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</tr>
</tbody>
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<tr>
<td>1</td>
<td>3</td>
<td>0010 0100 0010</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0010 0110 ? 0010</td>
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We won’t formalise this argument here (see [Ex. 4/5.11]).

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<tr>
<td>2</td>
<td></td>
<td>0 1 0 0 0</td>
</tr>
</tbody>
</table>

5. Hitting Times © T. Sauerwald Application 3: Ehrenfest Chain and Hypercubes
Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \{0, 1\}^d.

Coupon Collector \Rightarrow mixing time should be \(O(d \log d)\).

We won’t formalise this argument here (see [Ex. 4/5.11])

\[
\begin{array}{c|c|c}
\text{t} & \text{Coord.} & X_t \\
\hline
0 & 2 & 0 0 0 0 \text{ (grayed out)} \\
1 & 3 & 0 1 0 0 \text{ (green)} \\
2 & 3 & 0 1 0 0 \text{ (green)} \\
3 & & 0 1 ? 0 \text{ (grayed out)} \\
\end{array}
\]
Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).

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<td>[0\ 1\ 0\ 0]</td>
</tr>
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<td>4</td>
<td>[0\ 1\ 1\ 0]</td>
</tr>
<tr>
<td>4</td>
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<td>[0\ 1\ 1\ ?]</td>
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<td>0000 0000 0100</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0000 0000 0100</td>
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<tr>
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<td>4</td>
<td>0000 0000 1110</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0000 0000 1111</td>
</tr>
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Example of a Random Walk on a 4-Dimensional Hypercube

0000 0001
0010 0011
0110
0100 ...
8 0 0 1
9 0 0 0
10 done! 0 1 0

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<td>0 1 1 1</td>
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5. Hitting Times © T. Sauerwald Application 3: Ehrenfest Chain and Hypercubes
Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).

Several results about the behavior of random walks on hypercubes will be derived in the following sections. For now, consider the following example:

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As we saw in Example 4.5.3, \( \text{Mixing time} \) should be \( O(d \log d) \).
Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).

Coupon Collector/leads to mixing time should be \( O(d \log d) \).

We won’t formalise this argument here (see [Ex. 4/5.11]).

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</tr>
<tr>
<td>6</td>
<td></td>
<td>0 1 1 ?</td>
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Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \(\{0, 1\}^d\). 

Coupon Collector $\Rightarrow$ mixing time should be $O(d \log d)$.

We won't formalise this argument here (see [Ex. 4/5.11]).
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<td>0 1 1 0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0 ? 1 0</td>
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Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).

The coupon collector's problem/leadsto mixing time should be \( O(d \log d) \).

We won't formalise this argument here (see [Ex. 4/5.11]).

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\] |
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\] |
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0 1 1 0
\] |
| 4 | 2 | \[
0 1 1 1
\] |
| 5 | 4 | \[
0 1 1 1
\] |
| 6 | 2 | \[
0 1 1 0
\] |
| 7 |   | \[
0 0 1 0
\] |
Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).

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Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).

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We won't formalise this argument here (see [Ex. 4/5.11]).

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Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).

The coupon collector/mixing time should be \( O(d \log d) \).

We won’t formalise this argument here (see [Ex. 4/5.11]).

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Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).

**Coupon Collector** leads to a mixing time of \( O(d \log d) \).

We won't formalise this argument here (see [Ex. 4/5.11]).

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<td>( 0 \quad 0 \quad 0 \quad 0 )</td>
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<td>4</td>
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<td>( 0 \quad 1 \quad 1 \quad 1 )</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>( 0 \quad 1 \quad 1 \quad 1 )</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>( 0 \quad 1 \quad 1 \quad 0 )</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>( 0 \quad 0 \quad 1 \quad 0 )</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>( 0 \quad 0 \quad 1 \quad 0 )</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>( 0 \quad 0 \quad 1 \quad 0 )</td>
</tr>
</tbody>
</table>

---

5. Hitting Times © T. Sauerwald

Application 3: Ehrenfest Chain and Hypercubes
Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in $\{0, 1\}^d$. Coupon Collector leads to mixing time should be $O(d \log d)$. We won't formalise this argument here (see Ex. 4/5.11).

<table>
<thead>
<tr>
<th>$t$</th>
<th>Coord.</th>
<th>$X_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0 0 0 0 0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0 1 0 0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0 1 0 0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0 1 1 0</td>
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<td>4</td>
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<td>0 1 1 1</td>
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<td>7</td>
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<td>0 0 1 0</td>
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<tr>
<td>8</td>
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<td>0 0 1 0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0 0 1 0</td>
</tr>
<tr>
<td>10</td>
<td>done!</td>
<td>? 0 1 0</td>
</tr>
</tbody>
</table>
Example of a Random Walk on a 4-Dimensional Hypercube

Once all coordinates have been picked at least once, the state is uniformly at random in \( \{0, 1\}^d \).
Example of a Random Walk on a 4-Dimensional Hypercube

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Coupon Collector $\leadsto$ mixing time should be $O(d \log d)$.
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We won’t formalise this argument here (see [Ex. 4/5.11]).
Total Variation Distance of Random Walk on Hypercube \((d = 22)\)
Total Variation Distance of Random Walk on Hypercube ($d = 22$)

\[ d \log d \approx 68.00 \]
Fig. 1. The variation distance $V$ as a function of $N$, for $n = 10^{12}$.

**Theoretical Results (by Diaconis, Graham and Morrison)**

This is a numerical plot of a theoretical bound, where \( d = 10^{12} \)

(Minor Remark: This random walk is with a loop probability of \( 1/(d + 1) \))

The variation distance exhibits a so-called **cut-off** phenomena:

**Fig. 1.** The variation distance \( V \) as a function of \( N \), for \( n = 10^{12} \).

This is a numerical plot of a theoretical bound, where $d = 10^{12}$
(Minor Remark: This random walk is with a loop probability of $1/(d + 1)$)

- The variation distance exhibits a so-called cut-off phenomena:
  - Distance remains close to its maximum value 1 until step $\frac{1}{4} n \log n - \Theta(n)$
  - Then distance moves close to 0 before step $\frac{1}{4} n \log n + \Theta(n)$
Outline

Application 3: Ehrenfest Chain and Hypercubes

Random Walks on Graphs, Hitting Times and Cover Times

Random Walks on Paths and Grids

SAT and a Randomised Algorithm for 2-SAT
A Simple Random Walk (SRW) on a graph $G$ is a Markov chain on $V(G)$ with

$$P(u, v) = \begin{cases} \frac{1}{\deg(u)} & \text{if } \{u, v\} \in E, \\
0 & \text{if } \{u, v\} \notin E. \end{cases}$$

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$$\pi(u) = \frac{\deg(u)}{2|E|}.$$
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Recall: $h(u, v) = E_u[\min\{t \geq 1 : X_t = v\}]$ is the hitting time of $v$ from $u$. 
Lazy Random Walks and Periodicity

The Lazy Random Walk (LRW) on $G$ given by $\tilde{P} = (P + I) / 2$, 

$$
\tilde{P}_{u,v} = \begin{cases}
\frac{1}{2 \deg(u)} & \text{if } \{u, v\} \in E, \\
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0 & \text{otherwise}
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$P$ - SRW matrix
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SRW on $C_4$, Periodic
**Lazy Random Walks and Periodicity**

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SRW on $C_4$, *Periodic*  \hspace{1cm} LRW on $C_4$, *Aperiodic*
Outline

Application 3: Ehrenfest Chain and Hypercubes

Random Walks on Graphs, Hitting Times and Cover Times

Random Walks on Paths and Grids

SAT and a Randomised Algorithm for 2-SAT
Will a random walk always return to the origin?

In infinite 2D and 3D grids, a random walk may never return to the origin.

“A drunk man will find his way home, but a drunk bird may get lost forever.”

But for any regular (finite) graph, the expected return time to node $u$ is $1/\pi(u) = n$.
Will a random walk always return to the origin?

Infinite 2D Grid

“A drunk man will find his way home, but a drunk bird may get lost forever.”

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Will a random walk always return to the origin?

Infinite 2D Grid

A drunk man will find his way home, but a drunk bird may get lost forever.
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Infinite 2D Grid

Infinite 3D Grid
Will a random walk always return to the origin?

Infinite 2D Grid

Infinite 3D Grid

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Infinite 3D Grid

“A drunk man will find his way home, but a drunk bird may get lost forever.”
1921: The Birth of Random Walks on (Infinite) Graphs (Polyá)

Will a random walk always return to the origin?

Infinite 2D Grid

Infinite 3D Grid

“A drunk man will find his way home, but a drunk bird may get lost forever.”

But for any regular (finite) graph, the expected return time to $u$ is $1/\pi(u) = n$
Random Walk on a Path (1/2)

The \( n \)-path \( P_n \) is the graph with \( V(P_n) = [0, n] \), \( E(P_n) = \{\{i, j\} : j = i + 1\} \).

Exercise: [Exercise 4/5.15] What happens for the LRW on \( P_n \)?
The $n$-path $P_n$ is the graph with $V(P_n) = [0, n]$, $E(P_n) = \{\{i, j\} : j = i + 1\}$.

For the SRW on $P_n$ we have $h(k, n) = n^2 - k^2$, for any $0 \leq k < n$. 

Proposition

For the LRW on $P_n$ what happens?
The $n$-path $P_n$ is the graph with $V(P_n) = [0, n]$, $E(P_n) = \{\{i, j\} : j = i + 1\}$.

For the SRW on $P_n$ we have $h(k, n) = n^2 - k^2$, for any $0 \leq k < n$.
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For the SRW on $P_n$ we have $h(k, n) = n^2 - k^2$, for any $0 \leq k < n$. 

Proposition

For the SRW on $P_n$ we have $h(k, n) = n^2 - k^2$, for any $0 \leq k < n$. 

Diagram:

```
0 -- 1 -- 2 -- 3 -- 4
```
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**Proposition**

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**Exercise:** [Exercise 4/5.15] What happens for the LRW on $P_n$?
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For the SRW on $P_n$ we have $h(k, n) = n^2 - k^2$, for any $0 \leq k \leq n$. 
Random Walk on a Path (2/2)

Proposition

For the SRW on $P_n$ we have $h(k, n) = n^2 - k^2$, for any $0 \leq k \leq n$.

Recall: Hitting times are the solution to the set of linear equations:

$$h(x, y) = 1 + \sum_{z \in \Omega \setminus \{y\}} P(x, z) \cdot h(z, y) \quad \forall x \neq y \in V.$$
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Proof: Let $f(k) = h(k, n)$ and set $f(n) := 0$. 
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Proof: Let $f(k) = h(k, n)$ and set $f(n) := 0$. By the Markov property

$$f(0) = 1 + f(1)$$
For the SRW on $P_n$ we have $h(k, n) = n^2 - k^2$, for any $0 \leq k \leq n$.

Recall: Hitting times are the solution to the set of linear equations:

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$$f(0) = 1 + f(1) \quad \text{and} \quad f(k) = 1 + \frac{f(k - 1)}{2} + \frac{f(k + 1)}{2} \quad \text{for} \ 1 \leq k \leq n - 1.$$
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System of $n$ independent equations in $n$ unknowns, so has a unique solution.
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System of $n$ independent equations in $n$ unknowns, so has a unique solution. Thus it suffices to check that $f(k) = n^2 - k^2$ satisfies the above.
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Recall: Hitting times are the solution to the set of linear equations:

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System of $n$ independent equations in $n$ unknowns, so has a unique solution. Thus it suffices to check that $f(k) = n^2 - k^2$ satisfies the above. Indeed

$$f(0) = 1 + f(1) = 1 + n^2 - 1^2 = n^2,$$

and for any $1 \leq k \leq n - 1$ we have,

$$f(k) = 1 + \frac{n^2 - (k - 1)^2}{2} + \frac{n^2 - (k + 1)^2}{2} = n^2 - k^2.$$  


Outline

Application 3: Ehrenfest Chain and Hypercubes

Random Walks on Graphs, Hitting Times and Cover Times

Random Walks on Paths and Grids

SAT and a Randomised Algorithm for 2-SAT
A Satisfiability (SAT) formula is a logical expression that’s the conjunction (AND) of a set of Clauses, where a clause is the disjunction (OR) of Literals.
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A Solution to a SAT formula is an assignment of the variables to the values True and False so that all the clauses are satisfied.
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A Solution to a SAT formula is an assignment of the variables to the values True and False so that all the clauses are satisfied.

Example:

\[
\text{SAT: } (x_1 \lor x_2 \lor \overline{x_3}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2 \lor x_4) \land (x_4 \lor \overline{x_3}) \land (x_4 \lor \overline{x_1})
\]
A **Satisfiability (SAT)** formula is a logical expression that’s the conjunction (AND) of a set of **Clauses**, where a clause is the disjunction (OR) of **Literals**.

A **Solution** to a SAT formula is an assignment of the variables to the values **True** and **False** so that all the clauses are satisfied.

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\text{SAT: } (x_1 \lor \overline{x_2} \lor \overline{x_3}) \land (x_1 \lor \overline{x_3}) \land (x_1 \lor x_2 \lor x_4) \land (x_4 \lor \overline{x_3}) \land (x_4 \lor \overline{x_1})
\]

**Solution:** \( x_1 = \text{True}, \quad x_2 = \text{False}, \quad x_3 = \text{False} \quad \text{and} \quad x_4 = \text{True}. \)
A Satisfiability (SAT) formula is a logical expression that’s the conjunction (AND) of a set of Clauses, where a clause is the disjunction (OR) of Literals.

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Example:

SAT: \((x_1 \lor \overline{x_2} \lor x_3) \land (x_1 \lor x_3) \land (x_1 \lor x_2 \lor x_4) \land (x_4 \lor \overline{x_3}) \land (x_4 \lor \overline{x_1})\)

Solution: \(x_1 = \text{True}, \ x_2 = \text{False}, \ x_3 = \text{False} \quad \text{and} \quad x_4 = \text{True}.\)

- If each clause has \(k\) literals we call the problem \(k\)-SAT.
- In general, determining if a SAT formula has a solution is NP-hard
- In practice solvers are fast and used to great effect
- A huge amount of problems can be posed as a SAT:
A Satisfiability (SAT) formula is a logical expression that’s the conjunction (AND) of a set of Clauses, where a clause is the disjunction (OR) of Literals.

A Solution to a SAT formula is an assignment of the variables to the values True and False so that all the clauses are satisfied.

Example:

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\text{SAT: } (x_1 \lor \overline{x_2} \lor \overline{x_3}) \land (x_1 \lor \overline{x_3}) \land (x_1 \lor x_2 \lor x_4) \land (x_4 \lor \overline{x_3}) \land (x_4 \lor \overline{x_1})
\]

Solution: \(x_1 = \text{True}, \quad x_2 = \text{False}, \quad x_3 = \text{False}\) and \(x_4 = \text{True}\).

- If each clause has \(k\) literals we call the problem \(k\)-SAT.
- In general, determining if a SAT formula has a solution is NP-hard.
- In practice solvers are fast and used to great effect.
- A huge amount of problems can be posed as a SAT:
  - Model checking and hardware/software verification
  - Design of experiments
  - Classical planning
  - …
2-SAT

RANDOMISED-2-SAT (Input: a 2-SAT-Formula)

1: Start with an arbitrary truth assignment
2: Repeat up to $2n^2$ times
3: Pick an arbitrary unsatisfied clause
4: Choose a random literal and switch its value
5: If formula is satisfied then return "Satisfiable"
6: return "Unsatisfiable"

Call each loop of $(2)$ a step. Let $A_i$ be the variable assignment at step $i$.

Let $\alpha$ be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

Example 1: Solution Found

$(x_1 \lor x_2) \land (x_1 \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor x_1)$

$F T T F F T F$

$\alpha = (T, T, F, T)$.

t

x1  x2  x3  x4
0  F  F  F  F

5. Hitting Times © T. Sauerwald SAT and a Randomised Algorithm for 2-SAT 18
**RANDOMISED-2-SAT** (Input: a 2-SAT-Formula)

1. Start with an arbitrary truth assignment
### RANDOMISED-2-SAT (Input: a 2-SAT-Formula)

1. Start with an arbitrary truth assignment
2. **Repeat up to** $2n^2$ **times**
RANDOMISED-2-SAT (Input: a 2-SAT-Formula)

1: Start with an arbitrary truth assignment
2: Repeat up to $2n^2$ times
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**2-SAT**

**RANDOMISED-2-SAT (Input: a 2-SAT-Formula)**

1. Start with an arbitrary truth assignment
2. **Repeat up to** $2n^2$ times
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4. Choose a random literal and switch its value

Example 1: Solution Found

$$(x_1 \lor x_2) \land (x_1 \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor x_1)$$

<table>
<thead>
<tr>
<th>$x_1$</th>
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$\alpha = (T, T, F, T)$.
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$\alpha = (T, T, F, T)$. 

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5. Hitting Times © T. Sauerwald

SAT and a Randomised Algorithm for 2-SAT 18
2-SAT

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\[
( x_1 \lor \overline{x_2} ) \land ( \overline{x_1} \lor x_3 ) \land ( x_1 \lor x_2 ) \land ( x_4 \lor x_3 ) \land ( x_4 \lor \overline{x_1} )
\]

\[
\begin{array}{cccccc}
F & T & T & T & F & F \\
F & T & F & F & T & T \\
\end{array}
\]

$\alpha = (T, T, F, T)$.

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<tr>
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0 1 2 3 4
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F F T T T F F F T T F T

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\begin{array}{cccc}
F & F & T & T \\
F & T & F & F \\
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$\begin{array}{c}
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T \\
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F \\
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F \\
F
\end{array}$
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2-SAT

**Randomised-2-SAT** (Input: a 2-SAT-Formula)

1: Start with an arbitrary truth assignment
2: **Repeat up to** $2n^2$ times
3: Pick an arbitrary unsatisfied clause
4: Choose a random literal and switch its value
5: If formula is satisfied *then return* “Satisfiable”
6: *return* “Unsatisfiable”

- Call each loop of (2) a step. Let $A_i$ be the variable assignment at step $i$.
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**Example 1:**

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$$\begin{array}{cccc}
T & F & F & T \\
F & T & T & T \\
\end{array}$$

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5. Hitting Times © T. Sauerwald

SAT and a Randomised Algorithm for 2-SAT
2-SAT

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\[
\begin{array}{cccc}
T & F & F & T \\
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\[
\begin{array}{cccc}
| t | x_1 | x_2 | x_3 | x_4 |
\hline
0 | F | F | F | F \\
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![Diagram of hitting times](image)
**2-SAT**

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\[5. \text{Hitting Times © T. Sauerwald SAT and a Randomised Algorithm for 2-SAT}\]
### 2-SAT

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**2-SAT**

**RANDOMISED-2-SAT (Input: a 2-SAT-Formula)**

1. Start with an arbitrary truth assignment
2. **Repeat up to** $2n^2$ times
3. Pick an arbitrary unsatisfied clause
4. Choose a random literal and switch its value
5. If formula is satisfied then return “Satisfiable”
6. return “Unsatisfiable”

- Call each loop of (2) a step. Let $A_i$ be the variable assignment at step $i$.
- Let $\alpha$ be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$. 

**Example 2:**

\[
(x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor \overline{x_1})
\]

\[
\begin{array}{cccccccc}
F & T & T & T & F & F & F & F \\
\end{array}
\]

\[
\alpha = (T, F, F, T).
\]

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**2-SAT**

**RANDOMISED-2-SAT** (Input: a 2-SAT-Formula)

1. Start with an arbitrary truth assignment
2. Repeat up to $2n^2$ times
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4. Choose a random literal and switch its value
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**Example 2**:

$$(x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor \overline{x_1})$$

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$\alpha = (T, F, F, T)$. 

Call each loop of (2) a step. Let $A_i$ be the variable assignment at step $i$. Let $\alpha$ be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$. 

**Example 2**:

$$(x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor \overline{x_1})$$

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**2-SAT**

**RANDOMISED-2-SAT (Input: a 2-SAT-Formula)**

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- Let $\alpha$ be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

**Example 2**:

\[
(\overline{x_1} \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor \overline{x_3}) \land (x_4 \lor \overline{x_1})
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\end{array}
\]

$\alpha = (T, F, F, T)$.

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- Call each loop of (2) a **step**. Let $A_i$ be the variable assignment at step $i$.
- Let $\alpha$ be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

**Example 2 :**

\[
(x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor \overline{x_1})
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$\alpha = (T, F, F, T)$. 

5. Hitting Times © T. Sauerwald
**2-SAT**

**RANDOMISED-2-SAT** (Input: a 2-SAT-Formula)

1. Start with an arbitrary truth assignment
2. **Repeat up to** $2n^2$ **times**
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4. Choose a random literal and switch its value
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6. **return** “Unsatisfiable”

- Call each loop of (2) a step. Let $A_i$ be the variable assignment at step $i$.
- Let $\alpha$ be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

**Example 2:**

\[(x_1 \vee \overline{x_2}) \land (\overline{x_1} \vee x_3) \land (x_1 \vee x_2) \land (x_4 \vee x_3) \land (x_4 \vee \overline{x_1})\]

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$\alpha = (T, F, F, T)$. 

![Diagram](image)
RANDOMISED-2-SAT (Input: a 2-SAT-Formula)
1: Start with an arbitrary truth assignment
2: **Repeat up to** $2n^2$ **times**
3: Pick an arbitrary unsatisfied clause
4: Choose a random literal and switch its value
5: If formula is satisfied **then return** “Satisfiable”
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- Call each loop of (2) a step. Let $A_i$ be the variable assignment at step $i$.
- Let $\alpha$ be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

**Example 2:**

\[
(x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor \overline{x_1})
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$\alpha = (T, F, F, T)$. 

\[
\begin{array}{ccccc}
0 & 1 & 2 & 3 & 4 \\
\end{array}
\]
**2-SAT**

**RANDOMISED-2-SAT** (Input: a 2-SAT-Formula)

1. Start with an arbitrary truth assignment
2. **Repeat up to** $2n^2$ **times**
3. Pick an **arbitrary** unsatisfied clause
4. Choose a random literal and **switch** its value
5. If formula is satisfied then **return** “Satisfiable”
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- Call each loop of (2) a step. Let $A_i$ be the variable assignment at step $i$.
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Example 2:

\[
(x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor \overline{x_1})
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$\alpha = (T, F, F, T)$.  

\[
\begin{array}{c|cccc}
0 & F & F & F & F \\
1 & F & F & F & T \\
2 & F & T & F & T \\
\end{array}
\]
2-SAT

**RANDOMISED-2-SAT (Input: a 2-SAT-Formula)**

1: Start with an arbitrary truth assignment
2: **Repeat up to** \(2n^2\) **times**
3: Pick an arbitrary unsatisfied clause
4: Choose a random literal and switch its value
5: **If** formula is satisfied **then return** “Satisfiable”
6: **return** “Unsatisfiable”

- Call each loop of (2) a step. Let \(A_i\) be the variable assignment at step \(i\).
- Let \(\alpha\) be any solution and \(X_i = \) |variable values shared by \(A_i\) and \(\alpha\)|.

**Example 2**:

\[
(x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor \overline{x_1})
\]

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\begin{array}{cccc}
T & F & F & T \\
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\end{array}
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\(\alpha = (T, F, F, T)\).

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**2-SAT**

**RANDOMISED-2-SAT** (Input: a 2-SAT Formula)

1. Start with an arbitrary truth assignment
2. **Repeat up to** $2n^2$ **times**
3. Pick an arbitrary unsatisfied clause
4. Choose a random literal and switch its value
5. If formula is satisfied then return “Satisfiable”
6. return “Unsatisfiable”

- Call each loop of (2) a step. Let $A_i$ be the variable assignment at step $i$.
- Let $\alpha$ be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

**Example 2**: (Another) Solution Found

$(x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3) \land (x_1 \lor x_2) \land (x_4 \lor x_3) \land (x_4 \lor \overline{x_1})$

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$\alpha = (T, F, F, T)$. 

![Diagram](image)
If the formula is satisfiable, then the expected number of steps before \textsc{Randomised-2-SAT} outputs a valid solution is at most $n^2$. 

Expected iterations of (2) in \textsc{Randomised-2-SAT}

If the formula is \textit{satisfiable}, then the \textit{expected number of steps} before \textsc{Randomised-2-SAT} outputs a valid solution is at most $n^2$. 

Running for $2n^2$ steps and using Markov's inequality yields:
If the formula is satisfiable, then the expected number of steps before \textsc{Randomised-2-SAT} outputs a valid solution is at most \( n^2 \).

**Proof:** Fix any solution \( \alpha \), then for any \( i \geq 0 \) and \( 1 \leq k \leq n - 1 \),
2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED-2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED-2-SAT outputs a valid solution is at most \( n^2 \).

Proof: Fix any solution \( \alpha \), then for any \( i \geq 0 \) and \( 1 \leq k \leq n - 1 \),

(i) \( \mathbb{P}[X_{i+1} = 1 \mid X_i = 0] = 1 \)
2-SAT and the SRW on the Path

**Expected iterations of (2) in RANDOMISED-2-SAT**

If the formula is **satisfiable**, then the **expected number of steps** before **RANDOMISED-2-SAT** outputs a valid solution is at most $n^2$.

**Proof:** Fix any solution $\alpha$, then for any $i \geq 0$ and $1 \leq k \leq n - 1$,

1. $P [ X_{i+1} = 1 \mid X_i = 0 ] = 1$
2. $P [ X_{i+1} = k + 1 \mid X_i = k ] \geq 1/2$

Running for $2n^2$ steps and using Markov's inequality yields:

5. Hitting Times © T. Sauerwald
If the formula is satisfiable, then the expected number of steps before $\text{RANDOMISED-2-SAT}$ outputs a valid solution is at most $n^2$.

Proof: Fix any solution $\alpha$, then for any $i \geq 0$ and $1 \leq k \leq n - 1$,

(i) $P[X_{i+1} = 1 | X_i = 0] = 1$
(ii) $P[X_{i+1} = k + 1 | X_i = k] \geq 1/2$
(iii) $P[X_{i+1} = k - 1 | X_i = k] \leq 1/2$. 

Running for $2n^2$ steps and using Markov’s inequality yields:
If the formula is satisfiable, then the expected number of steps before 
\textsc{Randomised-2-SAT} outputs a valid solution is at most \( n^2 \).

\textbf{Proof:} Fix any solution \( \alpha \), then for any \( i \geq 0 \) and \( 1 \leq k \leq n - 1 \),
\begin{enumerate}[(i)]
  \item \( P[ X_{i+1} = 1 \mid X_i = 0 ] = 1 \)
  \item \( P[ X_{i+1} = k + 1 \mid X_i = k ] \geq 1/2 \)
  \item \( P[ X_{i+1} = k - 1 \mid X_i = k ] \leq 1/2. \)
\end{enumerate}

Notice that if \( X_i = n \) then \( A_i = \alpha \) thus solution found (may find another first).
If the formula is satisfiable, then the expected number of steps before \textsc{Randomised-2-SAT} outputs a valid solution is at most $n^2$.

**Proof:** Fix any solution $\alpha$, then for any $i \geq 0$ and $1 \leq k \leq n - 1$,

(i) $\Pr[X_{i+1} = 1 \mid X_i = 0] = 1$

(ii) $\Pr[X_{i+1} = k + 1 \mid X_i = k] \geq \frac{1}{2}$

(iii) $\Pr[X_{i+1} = k - 1 \mid X_i = k] \leq \frac{1}{2}$.

Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

Assume (pessimistically) that $X_0 = 0$ (none of our initial guesses is right).
2-SAT and the SRW on the Path

If the formula is satisfiable, then the expected number of steps before RANDOMISED-2-SAT outputs a valid solution is at most $n^2$.

Proof: Fix any solution $\alpha$, then for any $i \geq 0$ and $1 \leq k \leq n - 1$,

(i) $P[X_{i+1} = 1 | X_i = 0] = 1$

(ii) $P[X_{i+1} = k + 1 | X_i = k] \geq 1/2$

(iii) $P[X_{i+1} = k - 1 | X_i = k] \leq 1/2$.

Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

Assume (pessimistically) that $X_0 = 0$ (none of our initial guesses is right).

The process $X_i$ is complicated to describe in full; however by (i) – (iii) we can bound it by $Y_i$ (SRW on the $n$-path from 0).
If the formula is satisfiable, then the expected number of steps before RANDOMISED-2-SAT outputs a valid solution is at most $n^2$.

Proof: Fix any solution $\alpha$, then for any $i \geq 0$ and $1 \leq k \leq n - 1$,

(i) $P[X_{i+1} = 1 | X_i = 0] = 1$
(ii) $P[X_{i+1} = k + 1 | X_i = k] \geq 1/2$
(iii) $P[X_{i+1} = k - 1 | X_i = k] \leq 1/2$.

Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

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The process $X_i$ is complicated to describe in full; however by (i) – (iii) we can bound it by $Y_i$ (SRW on the $n$-path from 0). This gives (see also [Ex 4/5.16])

$$E[\text{time to find sol}] \leq E_0[\min\{t : X_t = n\}] \leq E_0[\min\{t : Y_t = n\}] = h(0, n) = n^2.$$
2-SAT and the SRW on the Path

Expected iterations of (2) in \textsc{Randomised-2-SAT}

If the formula is \textit{satisfiable}, then the \textit{expected number of steps} before \textsc{Randomised-2-SAT} outputs a valid solution is at most \(n^2\).

\textbf{Proof:} Fix any solution \(\alpha\), then for any \(i \geq 0\) and \(1 \leq k \leq n - 1\),

(i) \(P[X_{i+1} = 1 \mid X_i = 0] = 1\)

(ii) \(P[X_{i+1} = k + 1 \mid X_i = k] \geq 1/2\)

(iii) \(P[X_{i+1} = k - 1 \mid X_i = k] \leq 1/2\).

Notice that if \(X_i = n\) then \(A_i = \alpha\) thus solution found (may find another first).

Assume (pessimistically) that \(X_0 = 0\) (none of our initial guesses is right).

The process \(X_i\) is complicated to describe in full; however by (i) – (iii) we can \textbf{bound} it by \(Y_i\) (SRW on the \(n\)-path from 0). This gives (see also \[Ex 4/5.16\])

\[
E[\text{time to find sol}] \leq E_0[\min\{t : X_t = n\}] \leq E_0[\min\{t : Y_t = n\}] = h(0, n) = n^2.
\]

Running for \(2n^2\) \textbf{steps} and using Markov’s inequality yields:
2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED-2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED-2-SAT outputs a valid solution is at most $n^2$.

Proof: Fix any solution $\alpha$, then for any $i \geq 0$ and $1 \leq k \leq n - 1$,

(i) $\Pr[X_{i+1} = 1 \mid X_i = 0] = 1$
(ii) $\Pr[X_{i+1} = k + 1 \mid X_i = k] \geq 1/2$
(iii) $\Pr[X_{i+1} = k - 1 \mid X_i = k] \leq 1/2$.

Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

Assume (pessimistically) that $X_0 = 0$ (none of our initial guesses is right).

The process $X_i$ is complicated to describe in full; however by (i) – (iii) we can bound it by $Y_i$ (SRW on the $n$-path from 0). This gives (see also [Ex 4/5.16])

$$E[\text{time to find sol}] \leq E_0[\min\{t : X_t = n\}] \leq E_0[\min\{t : Y_t = n\}] = h(0, n) = n^2.$$

Running for $2n^2$ steps and using Markov's inequality yields:

Provided a solution exists, RANDOMISED-2-SAT will return a valid solution in $O(n^2)$ steps with probability at least $1/2$. 

Proposition
2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED-2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED-2-SAT outputs a valid solution is at most $n^2$.

Proof: Fix any solution $\alpha$, then for any $i \geq 0$ and $1 \leq k \leq n - 1$,

(i) $P [ X_{i+1} = 1 \mid X_i = 0 ] = 1$
(ii) $P [ X_{i+1} = k + 1 \mid X_i = k ] \geq 1/2$
(iii) $P [ X_{i+1} = k - 1 \mid X_i = k ] \leq 1/2$.

Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

Assume (pessimistically) that $X_0 = 0$ (none of our initial guesses is right).

The process $X_i$ is complicated to describe in full; however by (i) – (iii) we can bound it by $Y_i$ (SRW on the $n$-path from 0). This gives (see also [Ex 4/5.16])

$E [ \text{time to find sol} ] \leq E_0[\min\{t : X_t = n\}] \leq E_0[\min\{t : Y_t = n\}] = h(0, n) = n^2$.

Exercise: (difficult, beyond this course) What happens to the above analysis if we apply the same algorithm to 3-SAT?
Boosting Success Probabilities

Boosting Lemma

Suppose a randomised algorithm succeeds with probability (at least) $p$. Then for any $C \geq 1$, $\lceil \frac{C}{p} \cdot \log n \rceil$ repetitions are sufficient to succeed (in at least one repetition) with probability at least $1 - n^{-C}$.
Boosting Success Probabilities

Boosting Lemma

Suppose a randomised algorithm succeeds with probability (at least) $p$. Then for any $C \geq 1$, $\lceil \frac{C}{p} \log n \rceil$ repetitions are sufficient to succeed (in at least one repetition) with probability at least $1 - n^{-C}$.

Proof: Recall that $1 - p \leq e^{-p}$ for all real $p$. Let $t = \lceil \frac{C}{p} \log n \rceil$ and observe

\[
P[\text{t runs all fail}] \leq (1 - p)^t \leq e^{-pt} \leq n^{-C},
\]

thus the probability one of the runs succeeds is at least $1 - n^{-C}$. \qed
Boosting Success Probabilities

Boosting Lemma

Suppose a randomised algorithm succeeds with probability (at least) $p$. Then for any $C \geq 1$, $\lceil \frac{Cp}{p} \cdot \log n \rceil$ repetitions are sufficient to succeed (in at least one repetition) with probability at least $1 - n^{-C}$.

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RANDOMISED-2-SAT

There is a $O(n^2 \log n)$-step algorithm for 2-SAT which succeeds w.h.p.
Randomised Algorithms
Lecture 6: Linear Programming: Introduction

Thomas Sauerwald (tms41@cam.ac.uk)
Outline

Introduction

A Simple Example of a Linear Program

Formulating Problems as Linear Programs

Standard and Slack Forms
- **linear programming** is a powerful tool in optimisation
- inspired more sophisticated techniques such as **quadratic optimisation**, **convex optimisation**, **integer programming** and **semi-definite programming**
- we will later use the connection between **linear** and **integer programming** to tackle several problems (Vertex-Cover, Set-Cover, TSP, satisfiability)
Introduction

A Simple Example of a Linear Program

Formulating Problems as Linear Programs

Standard and Slack Forms
What are Linear Programs?

Linear Programming (informal definition)

- maximise or minimise an objective, given limited resources (competing constraint)
- constraints are specified as (in)equalitys
- objective function and constraints are linear
A Simple Example of a Linear Optimisation Problem

- Laptop

    Laptop

    selling price to retailer: 1,000 GBP
    glass: 4 units
    copper: 2 units
    rare-earth elements: 1 unit

    Smartphone

    selling price to retailer: 1,000 GBP
    glass: 1 unit
    copper: 1 unit
    rare-earth elements: 2 units

    You have a daily supply of:
    glass: 20 units
    copper: 10 units
    rare-earth elements: 14 units
    (and enough of everything else...)

    How to maximise your daily earnings?
A Simple Example of a Linear Optimisation Problem

- **Laptop**
  - selling price to retailer: 1,000 GBP
A Simple Example of a Linear Optimisation Problem

- Laptop
  - selling price to retailer: 1,000 GBP
  - glass: 4 units
A Simple Example of a Linear Optimisation Problem

- **Laptop**
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How to maximise your daily earnings?
The Linear Program

Linear Program for the Production Problem

\[
\begin{align*}
\text{maximise} & \quad x_1 + x_2 \\
\text{subject to} & \\
4x_1 + x_2 & \leq 20 \\
2x_1 + x_2 & \leq 10 \\
x_1 + 2x_2 & \leq 14 \\
x_1, x_2 & \geq 0
\end{align*}
\]
The Linear Program

Linear Program for the Production Problem

maximise \( x_1 + x_2 \)

subject to

\[
4x_1 + x_2 \leq 20 \\
2x_1 + x_2 \leq 10 \\
x_1 + 2x_2 \leq 14 \\
x_1, x_2 \geq 0
\]

The solution of this linear program yields the optimal production schedule.
The Linear Program

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**Formal Definition of Linear Program**
The Linear Program

Linear Program for the Production Problem

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& \quad x_1, x_2 \geq 0
\end{align*}
\]

The solution of this linear program yields the optimal production schedule.

Formal Definition of Linear Program

- Given \(a_1, a_2, \ldots, a_n\) and a set of variables \(x_1, x_2, \ldots, x_n\), a linear function \(f\) is defined by

\[
f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.
\]
The Linear Program

Linear Program for the Production Problem

maximise \( x_1 + x_2 \)
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- Linear Equality: \( f(x_1, x_2, \ldots, x_n) = b \)
- Linear Inequality: \( f(x_1, x_2, \ldots, x_n) \geq b \)
The Linear Program

Linear Program for the Production Problem

maximise $x_1 + x_2$
subject to

$4x_1 + x_2 \leq 20$
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The solution of this linear program yields the optimal production schedule.

Formal Definition of Linear Program

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$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$  

- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
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Linear Constraints
The Linear Program

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- Linear Programming Problem: either minimise or maximise a linear function subject to a set of linear constraints
Finding the Optimal Production Schedule

maximise \[ x_1 + x_2 \]
subject to
\[ 4x_1 + x_2 \leq 20 \]
\[ 2x_1 + x_2 \leq 10 \]
\[ x_1 + 2x_2 \leq 14 \]
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Finding the Optimal Production Schedule

maximise $x_1 + x_2$
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Any setting of $x_1$ and $x_2$ satisfying all constraints is a feasible solution.
Finding the Optimal Production Schedule

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Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.
Finding the Optimal Production Schedule

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Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.

Question: Which aspect did we ignore in the formulation of the linear program?
Finding the Optimal Production Schedule

\[ \text{maximise} \quad x_1 + x_2 \]

subject to

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Graphical Procedure: Move the line \( x_1 + x_2 = z \) as far up as possible.

While the same approach also works for higher-dimensions, we need to take a more systematic and algebraic procedure.
Outline

Introduction

A Simple Example of a Linear Program

Formulating Problems as Linear Programs

Standard and Slack Forms
Shortest Paths

Single-Pair Shortest Path Problem

- **Given:** directed graph \( G = (V, E) \) with edge weights \( w : E \rightarrow \mathbb{R} \), pair of vertices \( s, t \in V \)

![Graph with edge weights](image)

Maximise \( d_t \) subject to
\[
d_v \leq d_u + w(u, v)
\]
for each edge \((u, v) \in E\),
\[d_s = 0\]
Shortest Paths

Single-Pair Shortest Path Problem

- **Given**: directed graph $G = (V, E)$ with edge weights $w : E \rightarrow \mathbb{R}$, pair of vertices $s, t \in V$
- **Goal**: Find a path of minimum weight from $s$ to $t$ in $G$
Shortest Paths

Single-Pair Shortest Path Problem

- **Given**: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$
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$p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimised.
Shortest Paths

**Single-Pair Shortest Path Problem**

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Let $p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimised.

---

| Shortest Paths | 6. Linear Programming © T. Sauerwald Formulating Problems as Linear Programs | 10 |
Shortest Paths

Single-Pair Shortest Path Problem

- **Given:** directed graph \( G = (V, E) \) with edge weights \( w : E \to \mathbb{R} \), pair of vertices \( s, t \in V \)
- **Goal:** Find a path of minimum weight from \( s \) to \( t \) in \( G \)

\[ p = (v_0 = s, v_1, \ldots, v_k = t) \] such that \( w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k) \) is minimised.

**Exercise:** Translate the SPSP problem into a linear program!
Shortest Paths

Single-Pair Shortest Path Problem

- **Given**: directed graph \( G = (V, E) \) with edge weights \( w : E \rightarrow \mathbb{R} \), pair of vertices \( s, t \in V \)
- **Goal**: Find a path of minimum weight from \( s \) to \( t \) in \( G \)

\[ p = (v_0 = s, v_1, \ldots, v_k = t) \] such that \( w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k) \) is minimised.

Shortest Paths as LP

subject to
**Shortest Paths**

---

**Single-Pair Shortest Path Problem**

- **Given**: directed graph \( G = (V, E) \) with edge weights \( w : E \rightarrow \mathbb{R} \), pair of vertices \( s, t \in V \)
- **Goal**: Find a path of minimum weight from \( s \) to \( t \) in \( G \)

\[
p = (v_0 = s, v_1, \ldots, v_k = t) \text{ such that } w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k) \text{ is minimised.}
\]

---

**Shortest Paths as LP**

subject to

\[
\begin{align*}
d_v & \leq d_u + w(u, v) \quad \text{for each edge } (u, v) \in E, \\
d_s & = 0.
\end{align*}
\]
Shortest Paths

Given: directed graph $G = (V, E)$ with edge weights $w : E \rightarrow \mathbb{R}$, pair of vertices $s, t \in V$

Goal: Find a path of minimum weight from $s$ to $t$ in $G$

$p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimised.

Shortest Paths as LP

maximise $d_t$

subject to

$\begin{align*}
  d_v &\leq d_u + w(u, v) & \text{for each edge } (u, v) \in E, \\
  d_s &= 0.
\end{align*}$
**Shortest Paths**

- **Single-Pair Shortest Path Problem**
  - **Given:** directed graph \( G = (V, E) \) with edge weights \( w : E \rightarrow \mathbb{R} \), pair of vertices \( s, t \in V \)
  - **Goal:** Find a path of minimum weight from \( s \) to \( t \) in \( G \)

\[
p = (v_0 = s, v_1, \ldots, v_k = t) \text{ such that } w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k) \text{ is minimised.}
\]

**Shortest Paths as LP**

\[
\begin{align*}
\text{maximise} & \quad d_t \\
\text{subject to} & \quad d_v \leq d_u + w(u, v) \quad \text{for each edge } (u, v) \in E, \\
& \quad d_s = 0.
\end{align*}
\]

this is a maximisation problem!
Shortest Paths

**Single-Pair Shortest Path Problem**

- **Given:** directed graph \( G = (V, E) \) with edge weights \( w : E \to \mathbb{R} \), pair of vertices \( s, t \in V \)
- **Goal:** Find a path of minimum weight from \( s \) to \( t \) in \( G \)

\[ p = (v_0 = s, v_1, \ldots, v_k = t) \] such that \( w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k) \) is minimised.

**Shortest Paths as LP**

maximise \( d_t \)

subject to

\[ d_v \leq d_u + w(u, v) \quad \text{for each edge } (u, v) \in E, \]

\[ d_s = 0. \]

Recall: When **BELLMAN-FORD** terminates, all these inequalities are satisfied.

this is a maximisation problem!
Shortest Paths

Single-Pair Shortest Path Problem

- **Given:** directed graph \( G = (V, E) \) with edge weights \( w : E \to \mathbb{R} \), pair of vertices \( s, t \in V \)
- **Goal:** Find a path of minimum weight from \( s \) to \( t \) in \( G \)

\[ p = (v_0 = s, v_1, \ldots, v_k = t) \] such that
\[ w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k) \] is minimised.

Shortest Paths as LP

maximise \( d_t \)
subject to \( d_v \leq d_u + w(u, v) \) for each edge \( (u, v) \in E \),
\( d_s = 0 \).

this is a maximisation problem!

Recall: When BELLMAN-FORD terminates, all these inequalities are satisfied.

Solution \( \overline{d} \) satisfies
\[ \overline{d}_v = \min_{(u,v) \in E} \{ \overline{d}_u + w(u, v) \} \]
Maximum Flow

Maximum Flow Problem

- **Given:** directed graph $G = (V, E)$ with edge capacities $c : E \rightarrow \mathbb{R}^+$
  (recall $c(u, v) = 0$ if $(u, v) \notin E$), pair of vertices $s, t \in V$
**Maximum Flow**

**Maximum Flow Problem**

- **Given:** directed graph $G = (V, E)$ with edge capacities $c : E \rightarrow \mathbb{R}^+$ (recall $c(u, v) = 0$ if $(u, v) \notin E$), pair of vertices $s, t \in V$

```

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```
Maximum Flow

- **Given**: directed graph $G = (V, E)$ with edge capacities $c : E \rightarrow \mathbb{R}^+$ (recall $c(u, v) = 0$ if $(u, v) \notin E$), pair of vertices $s, t \in V$

- **Goal**: Find a maximum flow $f : V \times V \rightarrow \mathbb{R}$ from $s$ to $t$ which satisfies the capacity constraints and flow conservation
**Maximum Flow**

- **Maximum Flow Problem**
  - **Given**: directed graph $G = (V, E)$ with edge capacities $c : E \rightarrow \mathbb{R}^+$ (recall $c(u, v) = 0$ if $(u, v) \notin E$), pair of vertices $s, t \in V$
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$|f| = 19$
Maximum Flow

**Maximum Flow Problem**

- **Given**: directed graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}^+$ (recall $c(u, v) = 0$ if $(u, v) \not\in E$), pair of vertices $s, t \in V$
- **Goal**: Find a maximum flow $f : V \times V \to \mathbb{R}$ from $s$ to $t$ which satisfies the capacity constraints and flow conservation

**Maximum Flow as LP**

maximise $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$
subject to

- $f_{uv} \leq c(u, v)$ for each $u, v \in V$,
- $\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$ for each $u \in V \setminus \{s, t\}$,
- $f_{uv} \geq 0$ for each $u, v \in V$. 

![Graph with flow values and maximum flow](image)
Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem
Minimum-Cost Flow

- **Given:** directed graph $G = (V, E)$ with capacities $c : E \rightarrow \mathbb{R}^+$, pair of vertices $s, t \in V$, cost function $a : E \rightarrow \mathbb{R}^+$, flow demand of $d$ units

---

Optimal Solution with total cost:

\[
\sum_{(u,v) \in E} a(u,v) f(u,v) = (2 \cdot 2) + (5 \cdot 2) + (3 \cdot 1) + (7 \cdot 1) + (1 \cdot 3) = 27
\]
Minimum-Cost Flow

Minimum-Cost-Flow Problem

- **Given**: directed graph \( G = (V, E) \) with capacities \( c : E \to \mathbb{R}^+ \), pair of vertices \( s, t \in V \), cost function \( a : E \to \mathbb{R}^+ \), flow demand of \( d \) units
- **Goal**: Find a flow \( f : V \times V \to \mathbb{R} \) from \( s \) to \( t \) with \( |f| = d \) while minimising the total cost \( \sum_{(u,v) \in E} a(u,v)f_{uv} \) incurred by the flow.

Extension of the Maximum Flow Problem

Optimal Solution with total cost: \( (u, v) \in E a(u, v)f_{uv} = 27 \)
Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

- **Given:** directed graph \( G = (V, E) \) with capacities \( c : E \to \mathbb{R}^+ \), pair of vertices \( s, t \in V \), cost function \( a : E \to \mathbb{R}^+ \), flow demand of \( d \) units

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---

**Figure 29.3** (a) An example of a minimum-cost-flow problem. We denote the capacities by \( c \) and the costs by \( a \). Vertex \( s \) is the source and vertex \( t \) is the sink, and we wish to send 4 units of flow from \( s \) to \( t \). (b) A solution to the minimum-cost flow problem in which 4 units of flow are sent from \( s \) to \( t \). For each edge, the flow and capacity are written as flow/capacity.
Minimum-Cost Flow

**Extension of the Maximum Flow Problem**

**Minimum-Cost-Flow Problem**

- **Given:** directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{R}^+$, pair of vertices $s, t \in V$, cost function $a : E \to \mathbb{R}^+$, flow demand of $d$ units

- **Goal:** Find a flow $f : V \times V \to \mathbb{R}$ from $s$ to $t$ with $|f| = d$ while minimising the total cost $\sum_{(u,v) \in E} a(u, v)f_{uv}$ incurred by the flow.

**Optimal Solution** with total cost:

$\sum_{(u,v) \in E} a(u, v)f_{uv} = (2 \cdot 2) + (5 \cdot 2) + (3 \cdot 1) + (7 \cdot 1) + (1 \cdot 3) = 27$

Figure 29.3  (a) An example of a minimum-cost-flow problem. We denote the capacities by $c$ and the costs by $a$. Vertex $s$ is the source and vertex $t$ is the sink, and we wish to send 4 units of flow from $s$ to $t$. (b) A solution to the minimum-cost flow problem in which 4 units of flow are sent from $s$ to $t$. For each edge, the flow and capacity are written as flow/capacity.
Minimum Cost Flow as a LP

\[
\begin{align*}
\text{minimise} & \quad \sum_{(u,v) \in E} a(u, v)f_{uv} \\
\text{subject to} & \quad f_{uv} \leq c(u, v) \quad \text{for } u, v \in V, \\
& \quad \sum_{v \in V} f_{vu} - \sum_{v \in V} f_{uv} = 0 \quad \text{for } u \in V \setminus \{s, t\}, \\
& \quad \sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs} = d, \\
& \quad f_{uv} \geq 0 \quad \text{for } u, v \in V.
\end{align*}
\]
Minimum Cost Flow as a LP

minimise \( \sum_{(u,v) \in E} a(u, v)f_{uv} \)
subject to
\[ f_{uv} \leq c(u, v) \quad \text{for } u, v \in V, \]
\[ \sum_{v \in V} f_{vu} - \sum_{v \in V} f_{uv} = 0 \quad \text{for } u \in V \setminus \{s, t\}, \]
\[ \sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs} = d, \]
\[ f_{uv} \geq 0 \quad \text{for } u, v \in V. \]

Real power of Linear Programming comes from the ability to solve new problems!
Outline

Introduction

A Simple Example of a Linear Program

Formulating Problems as Linear Programs

Standard and Slack Forms
Standard and Slack Forms

Standard Form

maximise \[ \sum_{j=1}^{n} c_j x_j \]

subject to

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m \]

\[ x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n \]
Standard and Slack Forms

Standard Form

\[
\begin{align*}
\text{maximise} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m \\
& x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n
\end{align*}
\]
Standard and Slack Forms

**Standard Form**

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\begin{align*}
\text{maximise} & \quad \sum_{j=1}^{n} c_j x_j \\
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& \quad x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n
\end{align*}
\]
Standard and Slack Forms

**Standard Form**

- **Objective Function**
  \[ \text{maximise} \sum_{j=1}^{n} c_j x_j \]

- **Subject to**
  \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, \ldots, m \]
  \[ x_j \geq 0 \text{ for } j = 1, 2, \ldots, n \]

- **Non-Negativity Constraints**

**n + m constraints**
Standard and Slack Forms

**Standard Form**

maximise \[ \sum_{j=1}^{n} c_j x_j \]  \hspace{1cm} \text{Objective Function}

subject to

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \]  \hspace{1cm} \text{for } i = 1, 2, \ldots, m

\[ x_j \geq 0 \]  \hspace{1cm} \text{for } j = 1, 2, \ldots, n

\( n + m \) constraints

**Standard Form (Matrix-Vector-Notation)**

maximise \[ c^T x \]  \hspace{1cm} \text{Inner product of two vectors}

subject to

\[ Ax \leq b \]  \hspace{1cm} \text{Matrix-vector product}

\[ x \geq 0 \]
Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.
2. There might be variables without nonnegativity constraints.
3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).
Reasons for a LP not being in standard form:
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Goal: Convert linear program into an equivalent program which is in standard form
Converting Linear Programs into Standard Form

Reasons for a LP not being in standard form:
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3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).

**Goal:** Convert linear program into an equivalent program which is in standard form

**Equivalence:** a correspondence (not necessarily a bijection) between solutions.
Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.
Reasons for a LP not being in standard form:
1. The objective might be a \textit{minimisation} rather than \textit{maximisation}.

\begin{align*}
\text{minimise} & \quad -2x_1 + 3x_2 \\
\text{subject to} & \\
& \quad x_1 + x_2 = 7 \\
& \quad x_1 - 2x_2 \leq 4 \\
& \quad x_1 \geq 0
\end{align*}
Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.

\[
\begin{align*}
\text{minimise} & \quad -2x_1 + 3x_2 \\
\text{subject to} & \quad x_1 + x_2 = 7 \\
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& \quad x_1 \geq 0 \\
\end{align*}
\]

Negate objective function
Reasons for a LP not being in standard form:

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\[
\begin{align*}
\text{minimise} & \quad -2x_1 + 3x_2 \\
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Negate objective function

\[
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\text{maximise} & \quad 2x_1 - 3x_2 \\
\text{subject to} & \quad x_1 + x_2 = 7 \\
& \quad x_1 - 2x_2 \leq 4 \\
& \quad x_1 \geq 0 \\
\end{align*}
\]
Reasons for a LP not being in standard form:
2. There might be variables without nonnegativity constraints.
Reasons for a LP not being in standard form:

2. There might be variables without **nonnegativity constraints**.

\[
\begin{align*}
\text{maximise} & \quad 2x_1 - 3x_2 \\
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\end{align*}
\]
Reasons for a LP not being in standard form:
2. There might be variables without nonnegativity constraints.

\[
\begin{align*}
\text{maximise} & \quad 2x_1 - 3x_2 \\
\text{subject to} & \quad x_1 + x_2 = 7 \\
& \quad x_1 - 2x_2 \leq 4 \\
& \quad x_1 \geq 0
\end{align*}
\]

Replace \(x_2\) by two non-negative variables \(x'_2\) and \(x''_2\).
Converting into Standard Form (2/5)

Reasons for a LP not being in standard form:
2. There might be variables without nonnegativity constraints.

maximise \[ 2x_1 - 3x_2 \]
subject to
\[
\begin{align*}
x_1 + x_2 &= 7 \\
x_1 - 2x_2 &\leq 4 \\
x_1 &\geq 0
\end{align*}
\]
Replace \( x_2 \) by two non-negative variables \( x'_2 \) and \( x''_2 \)

maximise \[ 2x_1 - 3x'_2 + 3x''_2 \]
subject to
\[
\begin{align*}
x_1 + \begin{pmatrix} x'_2 - x''_2 \end{pmatrix} &= 7 \\
x_1 - \begin{pmatrix} 2x'_2 + 2x''_2 \end{pmatrix} &\leq 4 \\
x_1, x'_2, x''_2 &\geq 0
\end{align*}
\]
Reasons for a LP not being in standard form:

3. There might be equality constraints.
Reasons for a LP not being in standard form:

3. There might be equality constraints.

maximise $2x_1 - 3x_2' + 3x_2''$

subject to

$x_1 + x_2' - x_2'' = 7$
$x_1 - 2x_2' + 2x_2'' \leq 4$
$x_1, x_2', x_2'' \geq 0$

Replace each equality by two inequalities.
Reasons for a LP not being in standard form:

3. There might be equality constraints.

maximise \[ 2x_1 - 3x_2' + 3x_2'' \]
subject to

\[
\begin{align*}
  x_1 + x_2' - x_2'' &= 7 \\
  x_1 - 2x_2' + 2x_2'' &\leq 4 \\
  x_1, x_2', x_2'' &\geq 0
\end{align*}
\]

Replace each equality by two inequalities.
Converting into Standard Form (3/5)

Reasons for a LP not being in standard form:
3. There might be equality constraints.

maximise \(2x_1 - 3x' + 3x''\)
subject to
\[
\begin{align*}
x_1 + x' - x'' &= 7 \\
x_1 - 2x' + 2x'' &\leq 4 \\
x_1, x', x'' &\geq 0
\end{align*}
\]

maximise \(2x_1 - 3x' + 3x''\)
subject to
\[
\begin{align*}
x_1 + x' - x'' &\leq 7 \\
x_1 + x' - x'' &\geq 7 \\
x_1 - 2x' + 2x'' &\leq 4 \\
x_1, x', x'' &\geq 0
\end{align*}
\]

Replace each equality by two inequalities.
Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

Negate respective inequalities.
Reasons for a LP not being in standard form:

4. There might be inequality constraints (with \( \geq \) instead of \( \leq \)).

maximise \( 2x_1 - 3x'_2 + 3x''_2 \)
subject to

\[
\begin{align*}
x_1 + x'_2 - x''_2 & \leq 7 \\
x_1 + x'_2 - x''_2 & \geq 7 \\
x_1 - 2x'_2 + 2x''_2 & \leq 4 \\
x_1, x'_2, x''_2 & \geq 0
\end{align*}
\]
Converting into Standard Form (4/5)

Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

maximise $2x_1 - 3x'_2 + 3x''_2$

subject to

$x_1 + x'_2 - x''_2 \leq 7$

$x_1 + x'_2 - x''_2 \geq 7$

$x_1 - 2x'_2 + 2x''_2 \leq 4$

$x_1, x'_2, x''_2 \geq 0$

Negate respective inequalities.
Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

\[
\begin{align*}
\text{maximise } & \quad 2x_1 - 3x'_2 + 3x''_2 \\
\text{subject to } & \quad x_1 + x'_2 - x''_2 \leq 7 \\
& \quad x_1 + x'_2 - x''_2 \geq 7 \\
& \quad x_1 - 2x'_2 + 2x''_2 \leq 4 \\
& \quad x_1, x'_2, x''_2 \geq 0
\end{align*}
\]

Negate respective inequalities.

\[
\begin{align*}
\text{maximise } & \quad 2x_1 - 3x'_2 + 3x''_2 \\
\text{subject to } & \quad -x_1 - x'_2 + x''_2 \leq -7 \\
& \quad x_1 - 2x'_2 + 2x''_2 \leq 4 \\
& \quad x_1, x'_2, x''_2 \geq 0
\end{align*}
\]
Converting into Standard Form (5/5)

maximise \[2x_1 - 3x_2 + 3x_3\]

subject to

\[\begin{align*}
  x_1 + x_2 - x_3 & \leq 7 \\
  -x_1 - x_2 + x_3 & \leq -7 \\
  x_1 - 2x_2 + 2x_3 & \leq 4 \\
  x_1, x_2, x_3 & \geq 0
\end{align*}\]
maximise \[ 2x_1 - 3x_2 + 3x_3 \]
subject to
\[ x_1 + x_2 - x_3 \leq 7 \]
\[ -x_1 - x_2 + x_3 \leq -7 \]
\[ x_1 - 2x_2 + 2x_3 \leq 4 \]
\[ x_1, x_2, x_3 \geq 0 \]
Renaming variable names (for consistency).

maximise \[ 2x_1 - 3x_2 + 3x_3 \]
subject to
\[ x_1 + x_2 - x_3 \leq 7 \]
\[ -x_1 - x_2 + x_3 \leq -7 \]
\[ x_1 - 2x_2 + 2x_3 \leq 4 \]
\[ x_1, x_2, x_3 \geq 0 \]

It is always possible to convert a linear program into standard form.
Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.
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For the simplex algorithm, it is more convenient to work with equality constraints.
Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables
Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ be an inequality constraint
Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable $s$ by
Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let \( \sum_{j=1}^n a_{ij}x_j \leq b_i \) be an inequality constraint
- Introduce a slack variable \( s \) by
  \[
  s = b_i - \sum_{j=1}^n a_{ij}x_j
  \]
**Goal:** Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable $s$ by

\[
s = b_i - \sum_{j=1}^{n} a_{ij} x_j
\]

$s \geq 0$. 

---

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Standard and Slack Forms 22
Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

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Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable $s$ by

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$s$ measures the slack between the two sides of the inequality.

$s \geq 0$. 
**Goal:** Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

---

**Introducing Slack Variables**

- Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable $s$ by

$$s = b_i - \sum_{j=1}^{n} a_{ij}x_j$$

$s$ measures the slack between the two sides of the inequality.

- Denote slack variable of the $i$-th inequality by $x_{n+i}$
maximise 2\,x_1 - 3\,x_2 + 3\,x_3

subject to

\begin{align*}
x_1 &+ x_2 - x_3 \leq 7 \\
-x_1 &- x_2 + x_3 \leq -7 \\
x_1 &- 2\,x_2 + 2\,x_3 \leq 4 \\
x_1,\,x_2,\,x_3 \geq 0
\end{align*}
Converting Standard Form into Slack Form (2/3)

maximise \( 2x_1 - 3x_2 + 3x_3 \)

subject to
\[
\begin{align*}
  x_1 & + & x_2 & - & x_3 & \leq & 7 \\
-x_1 & - & x_2 & + & x_3 & \leq & -7 \\
  x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
  x_1, x_2, x_3 & \geq & 0
\end{align*}
\]

Introduce slack variables
Converting Standard Form into Slack Form (2/3)

maximise \[ 2x_1 - 3x_2 + 3x_3 \]
subject to
\[ x_1 + x_2 - x_3 \leq 7 \]
\[ -x_1 - x_2 + x_3 \leq -7 \]
\[ x_1 - 2x_2 + 2x_3 \leq 4 \]
\[ x_1, x_2, x_3 \geq 0 \]

Introduce slack variables

subject to
\[ x_4 = 7 - x_1 - x_2 + x_3 \]
Converting Standard Form into Slack Form (2/3)

maximise \(2x_1 - 3x_2 + 3x_3\)

subject to

\[
\begin{align*}
    x_1 + x_2 - x_3 & \leq 7 \\
    -x_1 - x_2 + x_3 & \leq -7 \\
    x_1 - 2x_2 + 2x_3 & \leq 4 \\
    x_1, x_2, x_3 & \geq 0
\end{align*}
\]

Introduce slack variables

subject to

\[
\begin{align*}
    x_4 &= 7 - x_1 - x_2 + x_3 \\
    x_5 &= -7 + x_1 + x_2 - x_3
\end{align*}
\]
Converting Standard Form into Slack Form (2/3)

maximise \[ 2x_1 - 3x_2 + 3x_3 \]
subject to
\[ x_1 + x_2 - x_3 \leq 7 \]
\[ -x_1 - x_2 + x_3 \leq -7 \]
\[ x_1 - 2x_2 + 2x_3 \leq 4 \]
\[ x_1, x_2, x_3 \geq 0 \]

Introduce slack variables

subject to
\[ x_4 = 7 - x_1 - x_2 + x_3 \]
\[ x_5 = -7 + x_1 + x_2 - x_3 \]
\[ x_6 = 4 - x_1 + 2x_2 - 2x_3 \]
Converting Standard Form into Slack Form (2/3)

maximise $2x_1 - 3x_2 + 3x_3$

subject to

$x_1 + x_2 - x_3 \leq 7$
$-x_1 - x_2 + x_3 \leq -7$
$x_1 - 2x_2 + 2x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

Introduce slack variables

subject to

$x_4 = 7 - x_1 - x_2 + x_3$
$x_5 = -7 + x_1 + x_2 - x_3$
$x_6 = 4 - x_1 + 2x_2 - 2x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$
maximise  \[ 2x_1 - 3x_2 + 3x_3 \]

subject to

\[
\begin{align*}
x_1 + x_2 - x_3 &\leq 7 \\
-x_1 - x_2 + x_3 &\leq -7 \\
x_1 - 2x_2 + 2x_3 &\leq 4 \\
\end{align*}
\]

\[ x_1, x_2, x_3 \geq 0 \]

Introduce slack variables

maximise  \[ 2x_1 - 3x_2 + 3x_3 \]

subject to

\[
\begin{align*}
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3 \\
\end{align*}
\]

\[ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \]
maximise \(2x_1 - 3x_2 + 3x_3\)

subject to

\[
\begin{align*}
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3 \\
\end{align*}
\]

\(x_1, x_2, x_3, x_4, x_5, x_6 \geq 0\)
Converting Standard Form into Slack Form (3/3)

maximise \(2x_1 - 3x_2 + 3x_3\)
subject to
\[
\begin{align*}
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3
\end{align*}
\]
\(x_1, x_2, x_3, x_4, x_5, x_6 \geq 0\)

Use variable \(z\) to denote objective function and omit the nonnegativity constraints.
Converting Standard Form into Slack Form (3/3)

maximise $2x_1 - 3x_2 + 3x_3$

subject to

$x_4 = 7 - x_1 - x_2 + x_3$
$x_5 = -7 + x_1 + x_2 - x_3$
$x_6 = 4 - x_1 + 2x_2 - 2x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Use variable $z$ to denote objective function and omit the nonnegativity constraints.

$z = 2x_1 - 3x_2 + 3x_3$

$x_4 = 7 - x_1 - x_2 + x_3$
$x_5 = -7 + x_1 + x_2 - x_3$
$x_6 = 4 - x_1 + 2x_2 - 2x_3$
Converting Standard Form into Slack Form (3/3)

maximise \( 2x_1 - 3x_2 + 3x_3 \)

subject to

\[
\begin{align*}
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3
\end{align*}
\]

\( x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \)

Use variable \( z \) to denote objective function and omit the nonnegativity constraints.

\[
\begin{align*}
z &= 2x_1 - 3x_2 + 3x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3
\end{align*}
\]

This is called slack form.
Basic and Non-Basic Variables

\[
\begin{align*}
  z &= 2x_1 - 3x_2 + 3x_3 \\
  x_4 &= 7 - x_1 - x_2 + x_3 \\
  x_5 &= -7 + x_1 + x_2 - x_3 \\
  x_6 &= 4 - x_1 + 2x_2 - 2x_3
\end{align*}
\]
Basic and Non-Basic Variables

\[ z = 2x_1 - 3x_2 + 3x_3 \]
\[ x_4 = 7 - x_1 - x_2 + x_3 \]
\[ x_5 = -7 + x_1 + x_2 - x_3 \]
\[ x_6 = 4 - x_1 + 2x_2 - 2x_3 \]

Basic Variables: \( B = \{4, 5, 6\} \)
Basic and Non-Basic Variables

\[ z = 2x_1 - 3x_2 + 3x_3 \]
\[ x_4 = 7 - x_1 - x_2 + x_3 \]
\[ x_5 = -7 + x_1 + x_2 - x_3 \]
\[ x_6 = 4 - x_1 + 2x_2 - 2x_3 \]

**Basic Variables:** \( B = \{4, 5, 6\} \)

**Non-Basic Variables:** \( N = \{1, 2, 3\} \)
Basic and Non-Basic Variables

\[
\begin{align*}
  z &= 2x_1 - 3x_2 + 3x_3 \\
  x_4 &= 7 - x_1 - x_2 + x_3 \\
  x_5 &= -7 + x_1 + x_2 - x_3 \\
  x_6 &= 4 - x_1 + 2x_2 - 2x_3
\end{align*}
\]

Basic Variables: \( B = \{4, 5, 6\} \)
Non-Basic Variables: \( N = \{1, 2, 3\} \)

Slack Form (Formal Definition)

Slack form is given by a tuple \((N, B, A, b, c, v)\) so that

\[

d. Linear Programming © T. Sauerwald Standard and Slack Forms 25
Basic and Non-Basic Variables

\[ z = 2x_1 - 3x_2 + 3x_3 \]
\[ x_4 = 7 - x_1 - x_2 + x_3 \]
\[ x_5 = -7 + x_1 + x_2 - x_3 \]
\[ x_6 = 4 - x_1 + 2x_2 - 2x_3 \]

Basic Variables: \( B = \{4, 5, 6\} \)

Non-Basic Variables: \( N = \{1, 2, 3\} \)

Slack Form (Formal Definition)

Slack form is given by a tuple \((N, B, A, b, c, v)\) so that

\[ z = v + \sum_{j \in N} c_j x_j \]
\[ x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \]

and all variables are non-negative.

Variables/Coefficients on the right hand side are indexed by \( B \) and \( N \).
Slack Form (Example)

\[ \begin{align*}
  z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
  x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
  x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
  x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{align*} \]
Slack Form (Example)

\[ z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \]
\[ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \]
\[ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \]
\[ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \]
Slack Form (Example)

\[ z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \]

\[ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \]

\[ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \]

\[ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \]

---

Slack Form Notation

- \( B = \{1, 2, 4\}, \ N = \{3, 5, 6\} \)
Slack Form (Example)

\[
z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}
\]

\[
x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}
\]

\[
x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}
\]

\[
x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}
\]

Slack Form Notation

- \(B = \{1, 2, 4\}, \ N = \{3, 5, 6\}\)
- \(A = \begin{pmatrix}
a_{13} & a_{15} & a_{16} \\
a_{23} & a_{25} & a_{26} \\
a_{43} & a_{45} & a_{46}
\end{pmatrix} = \begin{pmatrix}
-1/6 & -1/6 & 1/3 \\
8/3 & 2/3 & -1/3 \\
1/2 & -1/2 & 0
\end{pmatrix}\)
Slack Form (Example)

\[
z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}
\]

\[
x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}
\]

\[
x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}
\]

\[
x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}
\]

---

**Slack Form Notation**

- \( B = \{1, 2, 4\}, \ N = \{3, 5, 6\} \)

- \[
A = \begin{pmatrix}
a_{13} & a_{15} & a_{16} \\
 a_{23} & a_{25} & a_{26} \\
a_{43} & a_{45} & a_{46}
\end{pmatrix} = \begin{pmatrix}
-1/6 & -1/6 & 1/3 \\
 8/3 & 2/3 & -1/3 \\
 1/2 & -1/2 & 0
\end{pmatrix}
\]

- \[
b = \begin{pmatrix}
b_1 \\
b_2 \\
b_4
\end{pmatrix} = \begin{pmatrix}
8 \\
4 \\
18
\end{pmatrix}
\]
Slack Form (Example)

\[ z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \]
\[ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \]
\[ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \]
\[ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \]

---

Slack Form Notation

- \( B = \{1, 2, 4\}, \ N = \{3, 5, 6\} \)
- \( A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix} \)
- \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \ c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix} \)
Slack Form (Example)

\[ z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \]
\[ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \]
\[ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \]
\[ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \]

Slack Form Notation

- \( B = \{1, 2, 4\}, \ N = \{3, 5, 6\} \)
- \[
A = \begin{pmatrix}
a_{13} & a_{15} & a_{16} \\
a_{23} & a_{25} & a_{26} \\
a_{43} & a_{45} & a_{46}
\end{pmatrix}
= \begin{pmatrix}
-1/6 & -1/6 & 1/3 \\
8/3 & 2/3 & -1/3 \\
1/2 & -1/2 & 0
\end{pmatrix}
\]
- \[
b = \begin{pmatrix}
b_1 \\
b_2 \\
b_4
\end{pmatrix} = \begin{pmatrix}
8 \\
4 \\
18
\end{pmatrix}, \quad c = \begin{pmatrix}
c_3 \\
c_5 \\
c_6
\end{pmatrix} = \begin{pmatrix}
-1/6 \\
-1/6 \\
-2/3
\end{pmatrix}
\]
- \( v = 28 \)
Randomised Algorithms
Lecture 7: Linear Programming: Simplex Algorithm

Thomas Sauerwald (tms41@cam.ac.uk)
Outline

Simplex Algorithm by Example

Details of the Simplex Algorithm

Finding an Initial Solution

Appendix: Cycling and Termination (non-examinable)
Simplex Algorithm: Introduction

Simplex Algorithm

- classical method for solving linear programs (Dantzig, 1947)
- usually fast in practice although worst-case runtime not polynomial
- iterative procedure somewhat similar to Gaussian elimination
Simplex Algorithm: Introduction

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- usually fast in practice although worst-case runtime not polynomial
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Basic Idea:
- Each iteration corresponds to a “basic solution” of the slack form
- All non-basic variables are 0, and the basic variables are determined from the equality constraints
- Each iteration converts one slack form into an equivalent one while the objective value will not decrease
- Conversion (“pivoting”) is achieved by switching the roles of one basic and one non-basic variable
Simplex Algorithm: Introduction

- classical method for solving linear programs (Dantzig, 1947)
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Basic Idea:
- Each iteration corresponds to a “basic solution” of the slack form
- All non-basic variables are 0, and the basic variables are determined from the equality constraints
- Each iteration converts one slack form into an equivalent one while the objective value will not decrease
- Conversion (“pivoting”) is achieved by switching the roles of one basic and one non-basic variable

In that sense, it is a greedy algorithm.
Extended Example: Conversion into Slack Form

maximise $3x_1 + x_2 + 2x_3$

subject to

$x_1 + x_2 + 3x_3 \leq 30$
$2x_1 + 2x_2 + 5x_3 \leq 24$
$4x_1 + x_2 + 2x_3 \leq 36$
$x_1, x_2, x_3 \geq 0$
Extended Example: Conversion into Slack Form

<table>
<thead>
<tr>
<th>Maximise</th>
<th>$3x_1 + x_2 + 2x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject to</td>
<td></td>
</tr>
<tr>
<td>$x_1 + x_2 + 3x_3 \leq 30$</td>
<td></td>
</tr>
<tr>
<td>$2x_1 + 2x_2 + 5x_3 \leq 24$</td>
<td></td>
</tr>
<tr>
<td>$4x_1 + x_2 + 2x_3 \leq 36$</td>
<td></td>
</tr>
<tr>
<td>$x_1, x_2, x_3 \geq 0$</td>
<td></td>
</tr>
</tbody>
</table>

Conversion into slack form
Extended Example: Conversion into Slack Form

maximise $3x_1 + x_2 + 2x_3$

subject to

$x_1 + x_2 + 3x_3 ≤ 30$
$2x_1 + 2x_2 + 5x_3 ≤ 24$
$4x_1 + x_2 + 2x_3 ≤ 36$

$x_1, x_2, x_3 ≥ 0$

Conversion into slack form

$z = 3x_1 + x_2 + 2x_3$
$x_4 = 30 - x_1 - x_2 - 3x_3$
$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$
$x_6 = 36 - 4x_1 - x_2 - 2x_3$
Extended Example: Iteration 1

\[
\begin{align*}
  z &= 3x_1 + x_2 + 2x_3 \\
  x_4 &= 30 - x_1 - x_2 - 3x_3 \\
  x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{align*}
\]

Basic solution: \((x_1, x_2, \ldots, x_6) = (0, 0, 0, 30, 24, 36)\) with objective value 27.

Basic solution: \((x_1, x_2, \ldots, x_6) = (9, 0, 0, 21, 6, 0)\) with objective value 27.

Basic solution: \((x_1, x_2, \ldots, x_6) = (33, 0, 3, 69, 0, 0)\) with objective value 111.

Basic solution: \((x_1, x_2, \ldots, x_6) = (8, 4, 0, 18, 0, 0)\) with objective value 28.

This basic solution is feasible.

Objective value is 0.

Increasing the value of \(x_1\) would increase the objective value.

Increasing the value of \(x_3\) would increase the objective value.

Increasing the value of \(x_2\) would increase the objective value.

All coefficients are negative, and hence this basic solution is optimal.

The third constraint is the tightest and limits how much we can increase \(x_1\).

The second constraint is the tightest and limits how much we can increase \(x_3\).

Switch roles of \(x_1\) and \(x_6\):

Solving for \(x_1\) yields:
\[
  x_1 = 9 - 4x_2 - 2x_3 - x_6
\]

Substitute this into \(x_1\) in the other three equations.

Switch roles of \(x_3\) and \(x_5\):

Solving for \(x_3\) yields:
\[
  x_3 = 3 - 8x_2 - 2x_5 - 3x_6
\]

Substitute this into \(x_3\) in the other three equations.

Switch roles of \(x_2\) and \(x_3\):

Solving for \(x_2\) yields:
\[
  x_2 = 4 - 3x_3 - 2x_5 - 4x_6
\]

Substitute this into \(x_2\) in the other three equations.
Extended Example: Iteration 1

\[

ez = 3x_1 + x_2 + 2x_3 \\
x_4 = 30 - x_1 - x_2 - 3x_3 \\
x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 = 36 - 4x_1 - x_2 - 2x_3
\]

Basic solution: \((x_1, x_2, \ldots, x_6) = (0, 0, 0, 30, 24, 36)\)
Extended Example: Iteration 1

\[ z = 3x_1 + x_2 + 2x_3 \]
\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

Basic solution: \((x_1, x_2, \ldots, x_6) = (0, 0, 0, 30, 24, 36)\)

This basic solution is feasible.
Extended Example: Iteration 1

\[ z = 3x_1 + x_2 + 2x_3 \]

\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]

\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]

\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

Basic solution: \((x_1, x_2, \ldots, x_6) = (0, 0, 0, 30, 24, 36)\)

This basic solution is feasible.

Objective value is 0.
Extended Example: Iteration 1

Increasing the value of $x_1$ would increase the objective value.

\[
\begin{align*}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{align*}
\]

Basic solution: \((x_1, x_2, \ldots, x_6) = (0, 0, 0, 30, 24, 36)\)

This basic solution is **feasible**

Objective value is 0.
Extended Example: Iteration 1

Increasing the value of $x_1$ would increase the objective value.

\[
\begin{align*}
    z &= 3x_1 + x_2 + 2x_3 \\
    x_4 &= 30 - x_1 - x_2 - 3x_3 \\
    x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
    x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{align*}
\]

The third constraint is the tightest and limits how much we can increase $x_1$. 
Extended Example: Iteration 1

Increasing the value of $x_1$ would increase the objective value.

\[ z = 3x_1 + x_2 + 2x_3 \]

\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]

\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]

\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

The third constraint is the tightest and limits how much we can increase $x_1$.

Switch roles of $x_1$ and $x_6$: 
Extended Example: Iteration 1

Increasing the value of $x_1$ would increase the objective value.

\[ z = 3x_1 + x_2 + 2x_3 \]
\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

The third constraint is the tightest and limits how much we can increase $x_1$.

Switch roles of $x_1$ and $x_6$:
- Solving for $x_1$ yields:
  \[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}. \]
Extended Example: Iteration 1

Increasing the value of $x_1$ would increase the objective value.

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

The third constraint is the tightest and limits how much we can increase $x_1$.

Switch roles of $x_1$ and $x_6$:
- Solving for $x_1$ yields:
  $$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}.$$  
- Substitute this into $x_1$ in the other three equations
Extended Example: Iteration 2

\[ z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \]

\[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \]

\[ x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \]

\[ x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \]
Extended Example: Iteration 2

\[
\begin{align*}
    z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
    x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
    x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
    x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{align*}
\]

Basic solution: \((x_1, x_2, \ldots, x_6) = (9, 0, 0, 21, 6, 0)\) with objective value 27
Extended Example: Iteration 2

Increasing the value of $x_3$ would increase the objective value.

$$
z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}
$$

$$
x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}
$$

$$
x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}
$$

$$
x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
$$

Basic solution: $(x_1, x_2, \ldots, x_6) = (9, 0, 0, 21, 6, 0)$ with objective value 27
Extended Example: Iteration 2

Increasing the value of \( x_3 \) would increase the objective value.

\[
    z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}
\]

\[
    x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}
\]

\[
    x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}
\]

\[
    x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\]

The third constraint is the tightest and limits how much we can increase \( x_3 \).
Extended Example: Iteration 2

Increasing the value of $x_3$ would increase the objective value.

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$

The third constraint is the tightest and limits how much we can increase $x_3$.

Switch roles of $x_3$ and $x_5$: 

...
Extended Example: Iteration 2

Increasing the value of $x_3$ would increase the objective value.

\[
\begin{align*}
    z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
    x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
    x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
    x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{align*}
\]

The third constraint is the tightest and limits how much we can increase $x_3$.

Switch roles of $x_3$ and $x_5$:
- Solving for $x_3$ yields:

\[
x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} - \frac{x_6}{8}.
\]
Extended Example: Iteration 2

\[ z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \]
\[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \]
\[ x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \]
\[ x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \]

Increasing the value of \( x_3 \) would increase the objective value.

The third constraint is the tightest and limits how much we can increase \( x_3 \).

**Switch roles of \( x_3 \) and \( x_5 \):**
- Solving for \( x_3 \) yields:
  \[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} - \frac{x_6}{8} \]
- Substitute this into \( x_3 \) in the other three equations
Extended Example: Iteration 3

\[
\begin{align*}
  z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
  x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
  x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
  x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{align*}
\]
Extended Example: Iteration 3

\[
\begin{align*}
    z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
    x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
    x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
    x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{align*}
\]

Basic solution: \((x_1, x_2, \ldots, x_6) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)\) with objective value \(\frac{111}{4} = 27.75\).
Extended Example: Iteration 3

Increasing the value of $x_2$ would increase the objective value.

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$ with objective value $\frac{111}{4} = 27.75$
Extended Example: Iteration 3

Increasing the value of $x_2$ would increase the objective value.

\[
\begin{align*}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{align*}
\]

The second constraint is the tightest and limits how much we can increase $x_2$. 

Objective value is 0.

Increasing the value of $x_1$ would increase the objective value.

Increasing the value of $x_3$ would increase the objective value.

Switch roles of $x_1$ and $x_6$:

Solving for $x_1$ yields:
\[
x_1 = 9 - 4x_2 - 3x_3 - 18x_6
\]

Substitute this into $x_1$ in the other three equations.

Switch roles of $x_3$ and $x_5$:

Solving for $x_3$ yields:
\[
x_3 = 3 - 8x_2 - 4x_5 + 8x_6
\]

Substitute this into $x_3$ in the other three equations.

Switch roles of $x_2$ and $x_3$:

Solving for $x_2$ yields:
\[
x_2 = 4 - 8x_3 - 3x_5 + 8x_6
\]

Substitute this into $x_2$ in the other three equations.
Extended Example: Iteration 3

Increasing the value of $x_2$ would increase the objective value.

\[
\begin{align*}
    z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
    x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
    x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
    x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{align*}
\]

The second constraint is the tightest and limits how much we can increase $x_2$.

Switch roles of $x_2$ and $x_3$:
Extended Example: Iteration 3

Increasing the value of $x_2$ would increase the objective value.

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

The second constraint is the tightest and limits how much we can increase $x_2$.

Switch roles of $x_2$ and $x_3$:

- Solving for $x_2$ yields:

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}.$$
Extended Example: Iteration 3

Increasing the value of $x_2$ would increase the objective value.

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

The second constraint is the tightest and limits how much we can increase $x_2$.

Switch roles of $x_2$ and $x_3$:

- Solving for $x_2$ yields:
  $$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}.$$
- Substitute this into $x_2$ in the other three equations
Extended Example: Iteration 4

\[ z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \]

\[ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \]

\[ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \]

\[ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \]
Extended Example: Iteration 4

\[
\begin{align*}
  z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
  x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
  x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
  x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{align*}
\]

Basic solution: \((x_1, x_2, \ldots, x_6) = (8, 4, 0, 18, 0, 0)\) with objective value 28
All coefficients are negative, and hence this basic solution is optimal!

\[
\begin{align*}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{align*}
\]

Basic solution: \((x_1, x_2, \ldots, x_6) = (8, 4, 0, 18, 0, 0)\) with objective value 28
Extended Example: Visualization of SIMPLEX

Exercise: [Ex. 6/7.6] How many basic solutions (including non-feasible ones) are there?
Exercise: How many basic solutions (including non-feasible ones) are there?
Exercise: [Ex. 6/7.6] How many basic solutions (including non-feasible ones) are there?
Exercise: [Ex. 6/7.6] How many basic solutions (including non-feasible ones) are there?
Exercise: [Ex. 6/7.6] How many basic solutions (including non-feasible ones) are there?
Extended Example: Alternative Runs (1/2)

\[ z = 3x_1 + x_2 + 2x_3 \]
\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]
Extended Example: Alternative Runs (1/2)

\[
\begin{align*}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
\end{align*}
\]

Switch roles of \(x_2\) and \(x_5\)
Extended Example: Alternative Runs (1/2)

\[
z = 3x_1 + x_2 + 2x_3
\]

\[
x_4 = 30 - x_1 - x_2 - 3x_3
\]

\[
x_5 = 24 - 2x_1 - 2x_2 - 5x_3
\]

\[
x_6 = 36 - 4x_1 - x_2 - 2x_3
\]

Switch roles of \(x_2\) and \(x_5\)

\[
z = 12 + 2x_1 - \frac{x_3}{2} - \frac{x_5}{2}
\]

\[
x_2 = 12 - x_1 - \frac{5x_3}{2} - \frac{x_5}{2}
\]

\[
x_4 = 18 - x_2 - \frac{x_3}{2} + \frac{x_5}{2}
\]

\[
x_6 = 24 - 3x_1 + \frac{x_3}{2} + \frac{x_5}{2}
\]
Extended Example: Alternative Runs (1/2)

\[
\begin{align*}
  z &= 3x_1 + x_2 + 2x_3 \\
  x_4 &= 30 - x_1 - x_2 - 3x_3 \\
  x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{align*}
\]

Switch roles of \(x_2\) and \(x_5\)

\[
\begin{align*}
  z &= 12 + 2x_1 - \frac{x_3}{2} - \frac{x_5}{2} \\
  x_2 &= 12 - x_1 - \frac{5x_3}{2} - \frac{x_5}{2} \\
  x_4 &= 18 - x_2 - \frac{x_3}{2} + \frac{x_5}{2} \\
  x_6 &= 24 - 3x_1 + \frac{x_3}{2} + \frac{x_5}{2}
\end{align*}
\]

Switch roles of \(x_1\) and \(x_6\)
Extended Example: Alternative Runs (1/2)

\[
z = 3x_1 + x_2 + 2x_3 \\
x_4 = 30 - x_1 - x_2 - 3x_3 \\
x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 = 36 - 4x_1 - x_2 - 2x_3 \\
\]

Switch roles of \(x_2\) and \(x_5\)

\[
z = 12 + 2x_1 - \frac{x_3}{2} - \frac{x_5}{2} \\
x_2 = 12 - x_1 - \frac{5x_3}{2} - \frac{x_5}{2} \\
x_4 = 18 - x_2 - \frac{x_3}{2} + \frac{x_5}{2} \\
x_6 = 24 - 3x_1 + \frac{x_3}{2} + \frac{x_5}{2} \\
\]

Switch roles of \(x_1\) and \(x_6\)

\[
z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \\
\]
Extended Example: Alternative Runs (2/2)

\[
\begin{align*}
  z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{align*}
\]
Extended Example: Alternative Runs (2/2)

\[ z = 3x_1 + x_2 + 2x_3 \]
\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

Switch roles of \( x_3 \) and \( x_5 \)
Extended Example: Alternative Runs (2/2)

\[
\begin{align*}
  z &= 3x_1 + x_2 + 2x_3 \\
  x_4 &= 30 - x_1 - x_2 - 3x_3 \\
  x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{align*}
\]

Switch roles of \( x_3 \) and \( x_5 \)

\[
\begin{align*}
  z &= \frac{48}{5} + \frac{11x_1}{5} + \frac{x_2}{5} - \frac{2x_5}{5} \\
  x_4 &= \frac{78}{5} + \frac{x_1}{5} + \frac{x_2}{5} + \frac{3x_5}{5} \\
  x_3 &= \frac{24}{5} - \frac{2x_1}{5} - \frac{2x_2}{5} - \frac{x_5}{5} \\
  x_6 &= \frac{132}{5} - \frac{16x_1}{5} - \frac{x_2}{5} + \frac{2x_3}{5}
\end{align*}
\]
Extended Example: Alternative Runs (2/2)

\[
\begin{align*}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{align*}
\]

Switch roles of \(x_3\) and \(x_5\)

\[
\begin{align*}
z &= \frac{48}{5} + \frac{11x_1}{5} + \frac{x_2}{5} - \frac{2x_5}{5} \\
x_4 &= \frac{78}{5} + \frac{x_1}{5} + \frac{x_2}{5} + \frac{3x_5}{5} \\
x_3 &= \frac{24}{5} - \frac{2x_1}{5} - \frac{2x_2}{5} - \frac{x_5}{5} \\
x_6 &= \frac{132}{5} - \frac{16x_1}{5} - \frac{x_2}{5} + \frac{2x_3}{5}
\end{align*}
\]

Switch roles of \(x_1\) and \(x_6\)
Extended Example: Alternative Runs (2/2)

\[ z = 3x_1 + x_2 + 2x_3 \]
\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \]

Switch roles of \( x_3 \) and \( x_5 \)

\[ z = \frac{48}{5} + \frac{11x_1}{5} + \frac{x_2}{5} - \frac{2x_5}{5} \]
\[ x_4 = \frac{78}{5} + \frac{x_1}{5} + \frac{x_2}{5} + \frac{3x_5}{5} \]
\[ x_3 = \frac{24}{5} - \frac{2x_1}{5} - \frac{2x_2}{5} - \frac{x_5}{5} \]
\[ x_6 = \frac{132}{5} - \frac{16x_1}{5} - \frac{x_2}{5} + \frac{2x_3}{5} \]

Switch roles of \( x_1 \) and \( x_6 \)

\[ z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \]
\[ x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \]
\[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \]
\[ x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \]
Extended Example: Alternative Runs (2/2)

\[
\begin{align*}
    z &= 3x_1 + x_2 + 2x_3 \\
    x_4 &= 30 - x_1 - x_2 - 3x_3 \\
    x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
    x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
    x_4 &= 30 - x_1 - x_2 - 3x_3 \\
    x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
    x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
\end{align*}
\]

Switch roles of \( x_3 \) and \( x_5 \)

\[
\begin{align*}
    z &= \frac{48}{5} + \frac{11x_1}{5} + \frac{x_2}{5} - \frac{2x_5}{5} \\
    x_4 &= \frac{78}{5} + \frac{x_1}{5} + \frac{x_2}{5} + \frac{3x_5}{5} \\
    x_3 &= \frac{24}{5} - \frac{2x_1}{5} - \frac{2x_2}{5} - \frac{x_5}{5} \\
    x_6 &= \frac{132}{5} - \frac{16x_1}{5} - \frac{x_2}{5} + \frac{2x_3}{5} \\
\end{align*}
\]

Switch roles of \( x_1 \) and \( x_6 \)

Switch roles of \( x_2 \) and \( x_3 \)

\[
\begin{align*}
    z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
    x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
    x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
    x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \\
\end{align*}
\]
Extended Example: Alternative Runs (2/2)

\[
\begin{align*}
  z &= 3x_1 + x_2 + 2x_3 \\
  x_4 &= 30 - x_1 - x_2 - 3x_3 \\
  x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
  \text{Switch roles of } x_3 \text{ and } x_5 \\

  z &= \frac{48}{5} + \frac{11x_1}{5} + \frac{x_2}{5} - \frac{2x_5}{5} \\
  x_4 &= \frac{78}{5} + \frac{x_1}{5} + \frac{x_2}{5} + \frac{3x_5}{5} \\
  x_3 &= \frac{24}{5} - \frac{2x_1}{5} - \frac{2x_2}{5} - \frac{x_5}{5} \\
  x_6 &= \frac{132}{5} - \frac{16x_1}{5} - \frac{x_2}{5} + \frac{2x_3}{5} \\
  \text{Switch roles of } x_1 \text{ and } x_6 \\

  z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
  x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
  x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
  x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_6}{8} - \frac{x_6}{16} \\
  \text{Switch roles of } x_2 \text{ and } x_3 \\

  z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
  x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
  x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
  x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{align*}
\]
Outline

Simplex Algorithm by Example

Details of the Simplex Algorithm

Finding an Initial Solution

Appendix: Cycling and Termination (non-examinable)
The Pivot Step Formally

PIVOT($N, B, A, b, c, v, l, e$)

1. // Compute the coefficients of the equation for new basic variable $x_e$.
2. let $\hat{A}$ be a new $m \times n$ matrix
3. $\hat{b}_e = b_l/a_{le}$
4. for each $j \in N - \{e\}$
5. $\hat{a}_{ej} = a_{lj}/a_{le}$
6. $\hat{a}_{el} = 1/a_{le}$
7. // Compute the coefficients of the remaining constraints.
8. for each $i \in B - \{l\}$
9. $\hat{b}_i = b_i - a_{ie}\hat{b}_e$
10. for each $j \in N - \{e\}$
11. $\hat{a}_{ij} = a_{ij} - a_{ie}\hat{a}_{ej}$
12. $\hat{a}_{il} = -a_{ie}\hat{a}_{el}$
13. // Compute the objective function.
14. $\hat{v} = v + c_e\hat{b}_e$
15. for each $j \in N - \{e\}$
16. $\hat{c}_j = c_j - c_e\hat{a}_{ej}$
17. $\hat{c}_l = -c_e\hat{a}_{el}$
18. // Compute new sets of basic and nonbasic variables.
19. $\hat{N} = N - \{e\} \cup \{l\}$
20. $\hat{B} = B - \{l\} \cup \{e\}$
21. return ($\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v}$)
The Pivot Step Formally

\[
P\text{IVOT}(N, B, A, b, c, v, l, e)
\]

1 // Compute the coefficients of the equation for new basic variable \(x_e\).
2 let \(\hat{A}\) be a new \(m \times n\) matrix
3 \(\hat{b}_e = b_l / a_{le}\)
4 for each \(j \in N - \{e\}\)
5 \(\hat{a}_{ej} = a_{lj} / a_{le}\)
6 \(\hat{a}_{el} = 1 / a_{le}\)
7 // Compute the coefficients of the remaining constraints.
8 for each \(i \in B - \{l\}\)
9 \(\hat{b}_i = b_i - a_{ie}\hat{b}_e\)
10 for each \(j \in N - \{e\}\)
11 \(\hat{a}_{ij} = a_{ij} - a_{ie}\hat{a}_{ej}\)
12 \(\hat{a}_{il} = -a_{ie}\hat{a}_{el}\)
13 // Compute the objective function.
14 \(\hat{v} = v + c_e\hat{b}_e\)
15 for each \(j \in N - \{e\}\)
16 \(\hat{c}_j = c_j - c_e\hat{a}_{ej}\)
17 \(\hat{c}_l = -c_e\hat{a}_{el}\)
18 // Compute new sets of basic and nonbasic variables.
19 \(\hat{N} = N - \{e\} \cup \{l\}\)
20 \(\hat{B} = B - \{l\} \cup \{e\}\)
21 return \((\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})\)

Rewrite “tight” equation for entering variable \(x_e\).
The Pivot Step Formally

Pivot($N, B, A, b, c, v, l, e$)

1 // Compute the coefficients of the equation for new basic variable $x_e$.
2 let $\hat{A}$ be a new $m \times n$ matrix
3 $\hat{b}_e = b_l / a_{le}$
4 for each $j \in N - \{e\}$
5 $\hat{a}_{ej} = a_{lj} / a_{le}$
6 $\hat{a}_{el} = 1 / a_{le}$
7 // Compute the coefficients of the remaining constraints.
8 for each $i \in B - \{l\}$
9 $\hat{b}_i = b_i - a_{ie} \hat{b}_e$
10 for each $j \in N - \{e\}$
11 $\hat{a}_{ij} = a_{ij} - a_{ie} \hat{a}_{ej}$
12 $\hat{a}_{il} = -a_{ie} \hat{a}_{el}$
13 // Compute the objective function.
14 $\hat{v} = v + c_e \hat{b}_e$
15 for each $j \in N - \{e\}$
16 $\hat{c}_j = c_j - c_e \hat{a}_{ej}$
17 $\hat{c}_l = -c_e \hat{a}_{el}$
18 // Compute new sets of basic and nonbasic variables.
19 $\hat{N} = N - \{e\} \cup \{l\}$
20 $\hat{B} = B - \{l\} \cup \{e\}$
21 return $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})$
The Pivot Step Formally

PIVOT \((N, B, A, b, c, v, l, e)\)

1. // Compute the coefficients of the equation for new basic variable \(x_e\).
2. let \(\hat{A}\) be a new \(m \times n\) matrix
3. \(\hat{b}_e = b_l/a_{le}\)
4. for each \(j \in N - \{e\}\)
5. \(\hat{a}_{ej} = a_{lj}/a_{le}\)
6. \(\hat{a}_{el} = 1/a_{le}\)
7. // Compute the coefficients of the remaining constraints.
8. for each \(i \in B - \{l\}\)
9. \(\hat{b}_i = b_i - a_{ie}\hat{b}_e\)
10. for each \(j \in N - \{e\}\)
11. \(\hat{a}_{ij} = a_{ij} - a_{ie}\hat{a}_{ej}\)
12. \(\hat{a}_{il} = -a_{ie}\hat{a}_{el}\)
13. // Compute the objective function.
14. \(\hat{v} = v + c_e\hat{b}_e\)
15. for each \(j \in N - \{e\}\)
16. \(\hat{c}_j = c_j - c_e\hat{a}_{ej}\)
17. \(\hat{c}_l = -c_e\hat{a}_{el}\)
18. // Compute new sets of basic and nonbasic variables.
19. \(\hat{N} = N - \{e\} \cup \{l\}\)
20. \(\hat{B} = B - \{l\} \cup \{e\}\)
21. return \((\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})\)
The Pivot Step Formally

\[
P\text{IVOT}(N, B, A, b, c, v, l, e)
\]

1. \text{// Compute the coefficients of the equation for new basic variable } x_e. \\
2. \text{let } \hat{A} \text{ be a new } m \times n \text{ matrix} \\
3. \hat{b}_e = b_l/a_{le} \\
4. \text{for each } j \in N - \{e\} \\
5. \phantom{4.} \hat{a}_{ej} = a_{lj}/a_{le} \\
6. \phantom{4.} \hat{a}_{el} = 1/a_{le} \\
7. \text{// Compute the coefficients of the remaining constraints.} \\
8. \text{for each } i \in B - \{l\} \\
9. \phantom{8.} \hat{b}_i = b_i - a_{ie}\hat{b}_e \\
10. \phantom{8.} \text{for each } j \in N - \{e\} \\
11. \phantom{10.} \hat{a}_{ij} = a_{ij} - a_{ie}\hat{a}_{ej} \\
12. \phantom{10.} \hat{a}_{il} = -a_{ie}\hat{a}_{el} \\
13. \text{// Compute the objective function.} \\
14. \hat{v} = v + c_e\hat{b}_e \\
15. \text{for each } j \in N - \{e\} \\
16. \phantom{15.} \hat{c}_j = c_j - c_e\hat{a}_{ej} \\
17. \phantom{15.} \hat{c}_l = -c_e\hat{a}_{el} \\
18. \text{// Compute new sets of basic and nonbasic variables.} \\
19. \hat{N} = N - \{e\} \cup \{l\} \\
20. \hat{B} = B - \{l\} \cup \{e\} \\
21. \text{return (}\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})\text{)
The Pivot Step Formally

PIVOT \((N, B, A, b, c, v, l, e)\)

1. // Compute the coefficients of the equation for new basic variable \(x_e\).
2. let \(\hat{A}\) be a new \(m \times n\) matrix
3. \(\hat{b}_e = b_l/a_le\)
4. for each \(j \in N - \{e\}\)
5. \(\hat{a}_{ej} = a_{lj}/a_le\)
6. \(\hat{a}_{el} = 1/a_le\)
7. // Compute the coefficients of the remaining constraints.
8. for each \(i \in B - \{l\}\)
9. \(\hat{b}_i = b_i - a_{ie}\hat{b}_e\)
10. for each \(j \in N - \{e\}\)
11. \(\hat{a}_{ij} = a_{ij} - a_{ie}\hat{a}_{ej}\)
12. \(\hat{a}_{il} = -a_{ie}\hat{a}_{el}\)
13. // Compute the objective function.
14. \(\hat{v} = v + c_e\hat{b}_e\)
15. for each \(j \in N - \{e\}\)
16. \(\hat{c}_j = c_j - c_e\hat{a}_{ej}\)
17. \(\hat{c}_l = -c_e\hat{a}_{el}\)
18. // Compute new sets of basic and nonbasic variables.
19. \(\hat{N} = N - \{e\} \cup \{l\}\)
20. \(\hat{B} = B - \{l\} \cup \{e\}\)
21. return \((\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})\)
Lemma 29.1

Consider a call to \( \text{PIVOT}(N, B, A, b, c, v, l, e) \) in which \( a_{le} \neq 0 \). Let the values returned from the call be \( (\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v}) \), and let \( \bar{x} \) denote the basic solution after the call. Then

1. \( x_j = 0 \) for each \( j \in \hat{N} \).
2. \( x_e = \frac{b_l}{a_{le}} \).
3. \( x_i = \hat{b}_i - a_{ie} \hat{b}_e \) for each \( i \in \hat{B} \setminus \{e\} \).
Consider a call to \textsc{Pivot}(N, B, A, b, c, v, l, e) in which $a_{le} \neq 0$. Let the values returned from the call be $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})$, and let $\bar{x}$ denote the basic solution after the call. Then

1. $\bar{x}_j = 0$ for each $j \in \hat{N}$.
2. $\bar{x}_e = b_l / a_{le}$.
3. $\bar{x}_i = b_i - a_{ie} \hat{b}_e$ for each $i \in \hat{B} \setminus \{e\}$. 

Lemma 29.1
Consider a call to \( \text{PIVOT}(N, B, A, b, c, v, l, e) \) in which \( a_{le} \neq 0 \). Let the values returned from the call be \( (\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v}) \), and let \( \bar{x} \) denote the basic solution after the call. Then

1. \( \bar{x}_j = 0 \) for each \( j \in \hat{N} \).
2. \( \bar{x}_e = b_l / a_{le} \).
3. \( \bar{x}_i = b_i - a_{ie} \hat{b}_e \) for each \( i \in \hat{B} \setminus \{e\} \).

Proof:
Effect of the Pivot Step (extra material, non-examinable)

Lemma 29.1

Consider a call to \textsc{Pivot}(N, B, A, b, c, v, l, e) in which \(a_{le} \neq 0\). Let the values returned from the call be \((\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})\), and let \(\bar{x}\) denote the basic solution after the call. Then

1. \(\bar{x}_j = 0\) for each \(j \in \hat{N}\).
2. \(\bar{x}_e = b_l / a_{le}\).
3. \(\bar{x}_i = b_i - a_{ie}\hat{b}_e\) for each \(i \in \hat{B} \setminus \{e\}\).

Proof:

1. Holds since the basic solution always sets all non-basic variables to zero.
2. When we set each non-basic variable to 0 in a constraint
   \[ x_i = \hat{b}_i - \sum_{j \in \hat{N}} \hat{a}_{ij} x_j, \]
   we have \(\bar{x}_i = \hat{b}_i\) for each \(i \in \hat{B}\). Hence \(\bar{x}_e = \hat{b}_e = b_l / a_{le}\).
3. After substituting into the other constraints, we have
   \[ \bar{x}_i = \hat{b}_i = b_i - a_{ie}\hat{b}_e. \]
Consider a call to `PIVOT(N, B, A, b, c, v, l, e)` in which $a_{le} \neq 0$. Let the values returned from the call be $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})$, and let $\bar{x}$ denote the basic solution after the call. Then

1. $\bar{x}_j = 0$ for each $j \in \hat{N}$.
2. $\bar{x}_e = b_l / a_{le}$.
3. $\bar{x}_i = b_i - a_{ie} \hat{b}_e$ for each $i \in \hat{B} \setminus \{e\}$.

Proof:

1. holds since the basic solution always sets all non-basic variables to zero.
2. When we set each non-basic variable to 0 in a constraint
   \[ x_i = \hat{b}_i - \sum_{j \in \hat{N}} \hat{a}_{ij} x_j, \]
   we have $\bar{x}_i = \hat{b}_i$ for each $i \in \hat{B}$. Hence $\bar{x}_e = \hat{b}_e = b_l / a_{le}$.
3. After substituting into the other constraints, we have
   \[ \bar{x}_i = \hat{b}_i = b_i - a_{ie} \hat{b}_e. \]
Questions:

- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?
Formalizing the Simplex Algorithm: Questions

Questions:
- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

Example before was a particularly nice one!
The formal procedure SIMPLEX

SIMPLEX(A, b, c)
1 (N, B, A, b, c, v) = INITIALIZE-SIMPLEX(A, b, c)
2 let Δ be a new vector of length \( m \)
3 while some index \( j \in N \) has \( c_j > 0 \)
4 choose an index \( e \in N \) for which \( c_e > 0 \)
5 for each index \( i \in B \)
6 if \( a_{ie} > 0 \)
7 \( \Delta_i = b_i/a_{ie} \)
8 else \( \Delta_i = \infty \)
9 choose an index \( l \in B \) that minimizes \( \Delta_i \)
10 if \( \Delta_l = \infty \)
11 return “unbounded”
12 else \( (N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, e) \)
13 for \( i = 1 \) to \( n \)
14 if \( i \in B \)
15 \( \bar{x}_i = b_i \)
16 else \( \bar{x}_i = 0 \)
17 return \( (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \)
The formal procedure SIMPLEX

SIMPLEX\((A, b, c)\)

1. \((N, B, A, b, c, v) = \text{INITIALIZE-SIMPLEX}(A, b, c)\)
2. let \(\Delta\) be a new vector of length \(m\)
3. while some index \(j \in N\) has \(c_j > 0\)
4. choose an index \(e \in N\) for which \(c_e > 0\)
5. for each index \(i \in B\)
6. if \(a_{ie} > 0\)
7. \(\Delta_i = b_i/a_{ie}\)
8. else \(\Delta_i = \infty\)
9. choose an index \(l \in B\) that minimizes \(\Delta_i\)
10. if \(\Delta_l == \infty\)
11. return “unbounded”
12. else \((N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, e)\)
13. for \(i = 1\) to \(n\)
14. if \(i \in B\)
15. \(\tilde{x}_i = b_i\)
16. else \(\tilde{x}_i = 0\)
17. return \((\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)\)
The formal procedure **SIMPLEX**

\[
\text{SIMPLEX}(A, b, c)
\]

1. \((N, B, A, b, c, v) = \text{INITIALIZE-SIMPLEX}(A, b, c)\)
2. let \(\Delta\) be a new vector of length \(m\)
3. **while** some index \(j \in N\) has \(c_j > 0\)
4. \hspace{1em} choose an index \(e \in N\) for which \(c_e > 0\)
5. \hspace{1em} **for** each index \(i \in B\)
6. \hspace{2em} **if** \(a_{ie} > 0\)
7. \hspace{3em} \(\Delta_i = b_i / a_{ie}\)
8. \hspace{2em} **else** \(\Delta_i = \infty\)
9. \hspace{1em} choose an index \(l \in B\) that minimizes \(\Delta_i\)
10. \hspace{1em} **if** \(\Delta_l = \infty\)
11. \hspace{2em} **return** “unbounded”
12. \hspace{1em} **else** \((N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, e)\)
13. \hspace{1em} **for** \(i = 1\) to \(n\)
14. \hspace{2em} **if** \(i \in B\)
15. \hspace{3em} \(\bar{x}_i = b_i\)
16. \hspace{2em} **else** \(\bar{x}_i = 0\)
17. **return** \((\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\)
The formal procedure SIMPLEX

\text{SIMPLEX}(A, b, c)

1. \((N, B, A, b, c, v) = \text{INITIALIZE-SIMPLEX}(A, b, c)\)
2. let \(\Delta\) be a new vector of length \(m\)
3. while some index \(j \in N\) has \(c_j > 0\)
4. \quad choose an index \(e \in N\) for which \(c_e > 0\)
5. \quad for each index \(i \in B\)
6. \quad \quad if \(a_{ie} > 0\)
7. \quad \quad \quad \(\Delta_i = b_i/a_{ie}\)
8. \quad \quad \quad else \(\Delta_i = \infty\)
9. \quad \quad choose an index \(l \in B\) that minimizes \(\Delta_i\)
10. \quad \quad if \(\Delta_l = \infty\)
11. \quad \quad \quad return “unbounded”
12. \quad \quad else \((N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, e)\)
13. \quad for \(i = 1\) to \(n\)
14. \quad \quad if \(i \in B\)
15. \quad \quad \quad \(\tilde{x}_i = b_i\)
16. \quad \quad else \(\tilde{x}_i = 0\)
17. return \((\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)\)

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:
The formal procedure SIMPLEX

SIMPLEX\([A, b, c]\)

1. \((N, B, A, b, c, v) = \text{INITIALIZE-SIMPLEX}(A, b, c)\)
2. let \(\Delta\) be a new vector of length \(m\)
3. while some index \(j \in N\) has \(c_j > 0\)
4. choose an index \(e \in N\) for which \(c_e > 0\)
5. for each index \(i \in B\)
   6. if \(a_{ie} > 0\)
   7. \(\Delta_i = b_i / a_{ie}\)
   8. else \(\Delta_i = \infty\)
9. choose an index \(l \in B\) that minimizes \(\Delta_i\)
10. if \(\Delta_l = \infty\)
   11. return "unbounded"
12. else \((N, B, A, b, c, v) = \text{Pivot}(N, B, A, b, c, v, l, e)\)
13. for \(i = 1\) to \(n\)
   14. if \(i \in B\)
   15. \(\bar{x}_i = b_i\)
   16. else \(\bar{x}_i = 0\)
17. return \((\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\)

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:
- terminates if all coefficients in objective function are non-positive
- Line 4 picks entering variable \(x_e\) with positive coefficient
- Lines 6 – 9 pick the tightest constraint, associated with \(x_l\)
- Line 11 returns "unbounded" if there are no constraints
- Line 12 calls PIVOT, switching roles of \(x_l\) and \(x_e\)
The formal procedure SIMPLEX

\texttt{SIMPLEX}(A, b, c)

1. \((N, B, A, b, c, v) = \text{INITIALIZE-SIMPLEX}(A, b, c)\)
2. let \(\Delta\) be a new vector of length \(m\)
3. \textbf{while} some index \(j \in N\) has \(c_j > 0\)
4. \hspace{1em} choose an index \(e \in N\) for which \(c_e > 0\)
5. \hspace{1em} for each index \(i \in B\)
6. \hspace{2em} if \(a_{ie} > 0\)
7. \hspace{3em} \(\Delta_i = b_i/a_{ie}\)
8. \hspace{2em} else \(\Delta_i = \infty\)
9. \hspace{1em} choose an index \(l \in B\) that minimizes \(\Delta_i\)
10. \hspace{1em} if \(\Delta_l = \infty\)
11. \hspace{2em} return “unbounded”
12. \hspace{1em} else \((N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, e)\)
13. \textbf{for} \(i = 1\) \textbf{to} \(n\)
14. \hspace{1em} if \(i \in B\)
15. \hspace{2em} \(\bar{x}_i = b_i\)
16. \hspace{1em} else \(\bar{x}_i = 0\)
17. return \((\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\)

\textbf{Main Loop:}

- terminates if all coefficients in objective function are \textbf{non-positive}
- Line 4 picks entering variable \(x_e\) with \textbf{positive} coefficient
- Lines 6 — 9 pick the tightest constraint, associated with \(x_j\)
- Line 11 returns “unbounded” if there are no constraints
- Line 12 calls \text{PIVOT}, switching roles of \(x_i\) and \(x_e\)

Returns a slack form with a feasible basic solution (if it exists)

Return corresponding solution.
The formal procedure SIMPLEX

SIMPLEX\((A, b, c)\)

1. \((N, B, A, b, c, v) = \text{INITIALIZE-SIMPLEX}(A, b, c)\)
2. let \(\Delta\) be a new vector of length \(m\)
3. while some index \(j \in N\) has \(c_j > 0\)
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. 
15. 
16. 
17. 

Lemma 29.2

Suppose the call to \(\text{INITIALIZE-SIMPLEX}\) in line 1 returns a slack form for which the basic solution is feasible. Then if \(\text{SIMPLEX}\) returns a solution, it is a feasible solution. If \(\text{SIMPLEX}\) returns “unbounded”, the linear program is unbounded.
The formal procedure SIMPLEX

SIMPLEX\((A, b, c)\)

1. \((N, B, A, b, c, v) = \text{INITIALIZE-SIMPLEX}(A, b, c)\)
2. let \(\Delta\) be a new vector of length \(m\)
3. while some index \(j \in N\) has \(c_j > 0\)
4. choose an index \(e \in N\) for which \(c_e > 0\)
5. for each index \(i \in B\)
6. if \(a_{ie} > 0\)
7. \(\Delta_i = b_i/a_{ie}\)
8. else \(\Delta_i = \infty\)
9. choose an index \(l \in B\) that minimizes \(\Delta_i\)
10. if \(\Delta_l = \infty\)
11. return “unbounded”

Returns a slack form with a feasible basic solution (if it exists)

Proof is based on the following three-part loop invariant:

Lemma 29.2

Suppose the call to \text{INITIALIZE-SIMPLEX} in line 1 returns a slack form for which the basic solution is feasible. Then if \text{SIMPLEX} returns a solution, it is a feasible solution. If \text{SIMPLEX} returns “unbounded”, the linear program is unbounded.
The formal procedure **SIMPLEX**

\[
\text{SIMPLEX}(A, b, c)
\]

1. \((N, B, A, b, c, v) = \text{INITIALIZE-SIMPLEX}(A, b, c)\)
2. let \(\Delta\) be a new vector of length \(m\)
3. \textbf{while} some index \(j \in N\) has \(c_j > 0\)
4. \textbf{choose} an index \(e \in N\) for which \(c_e > 0\)
5. \textbf{for} each index \(i \in B\)
6. \textbf{if} \(a_{ie} > 0\)
7. \(\Delta_i = b_i / a_{ie}\)
8. \textbf{else} \(\Delta_i = \infty\)
9. \textbf{choose} an index \(l \in B\) that minimizes \(\Delta_i\)
10. \textbf{if} \(\Delta_l = \infty\)
11. \textbf{return} “unbounded”

**Proof** is based on the following three-part loop invariant:

1. the slack form is always equivalent to the one returned by \text{INITIALIZE-SIMPLEX},
2. for each \(i \in B\), we have \(b_i \geq 0\),
3. the basic solution associated with the (current) slack form is feasible.

---

**Lemma 29.2**

Suppose the call to \text{INITIALIZE-SIMPLEX} in line 1 returns a slack form for which the basic solution is feasible. Then if \text{SIMPLEX} returns a solution, it is a feasible solution. If \text{SIMPLEX} returns “unbounded”, the linear program is unbounded.
maximise $2x_1 - x_2$
subject to

\[
\begin{align*}
2x_1 - x_2 & \leq 2 \\
x_1 - 5x_2 & \leq -4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Basic solution $(x_1,x_2,x_3,x_4) = (0,0,2,-4)$ is not feasible!
maximise \( 2x_1 - x_2 \)
subject to
\[
\begin{align*}
2x_1 - x_2 & \leq 2 \\
x_1 - 5x_2 & \leq -4 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Conversion into slack form

Basic solution \((x_1, x_2, x_3, x_4) = (0, 0, 2, -4)\) is not feasible!
maximise $2x_1 - x_2$
subject to

\[
\begin{align*}
2x_1 - x_2 & \leq 2 \\
x_1 - 5x_2 & \leq -4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Conversion into slack form

\[
\begin{align*}
z & = 2x_1 - x_2 \\
x_3 & = 2 - 2x_1 + x_2 \\
x_4 & = -4 - x_1 + 5x_2
\end{align*}
\]

Basic solution $(x_1, x_2, x_3, x_4) = (0, 0, 2, -4)$ is not feasible!
maximise $2x_1 - x_2$
subject to

$$2x_1 - x_2 \leq 2$$
$$x_1 - 5x_2 \leq -4$$
$$x_1, x_2 \geq 0$$
maximise $2x_1 - x_2$
subject to

\[
\begin{align*}
2x_1 &- x_2 \leq 2 \\
x_1 &- 5x_2 \leq -4 \\
x_1, x_2 &\geq 0
\end{align*}
\]
Geometric Illustration

maximise \[ 2x_1 - x_2 \]
subject to \[
\begin{align*}
2x_1 - x_2 & \leq 2 \\
x_1 - 5x_2 & \leq -4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Questions:
- How to determine whether there is any feasible solution?
- If there is one, how to determine an initial basic solution?
Formulating an Auxiliary Linear Program

\[
\begin{align*}
\text{maximise} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m, \\
& \quad x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n
\end{align*}
\]

Lemma 29.11
Proof. Exercise!
Formulating an Auxiliary Linear Program

Let $L_{aux}$ be the auxiliary LP of a linear program $L$ in standard form. Then $L$ is feasible if and only if the optimal objective value of $L_{aux}$ is 0.

Lemma 29.11
Proof. Exercise!
Formulating an Auxiliary Linear Program

maximise \[ \sum_{j=1}^{n} c_j x_j \]
subject to
\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for} \quad i = 1, 2, \ldots, m, \]
\[ x_j \geq 0 \quad \text{for} \quad j = 1, 2, \ldots, n \]

maximise \(-x_0\)
subject to
\[ \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \quad \text{for} \quad i = 1, 2, \ldots, m, \]
\[ x_j \geq 0 \quad \text{for} \quad j = 0, 1, \ldots, n \]
Formulating an Auxiliary Linear Program

maximise \[ \sum_{j=1}^{n} c_j x_j \]
subject to
\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \] for \( i = 1, 2, \ldots, m \),
\[ x_j \geq 0 \] for \( j = 1, 2, \ldots, n \)

maximise \( -x_0 \)
subject to
\[ \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \] for \( i = 1, 2, \ldots, m \),
\[ x_j \geq 0 \] for \( j = 1, 2, \ldots, n \)

Lemma 29.11

Let \( L_{aux} \) be the auxiliary LP of a linear program \( L \) in standard form. Then \( L \) is feasible if and only if the optimal objective value of \( L_{aux} \) is 0.
Formulating an Auxiliary Linear Program

maximise $\sum_{j=1}^{n} c_j x_j$
subject to
$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for $i = 1, 2, \ldots, m,$
$x_j \geq 0$ for $j = 1, 2, \ldots, n$

maximise $-x_0$
subject to
$\sum_{j=1}^{n} a_{i j} x_j - x_0 \leq b_i$ for $i = 1, 2, \ldots, m,$
$x_j \geq 0$ for $j = 0, 1, \ldots, n$

Lemma 29.11

Let $L_{aux}$ be the auxiliary LP of a linear program $L$ in standard form. Then $L$ is feasible if and only if the optimal objective value of $L_{aux}$ is 0.

Proof. Exercise!
Let us illustrate the role of \( x_0 \) as “distance from feasibility”
- Let us illustrate the role of $x_0$ as “distance from feasibility”
- We’ll also see that increasing $x_0$ enlarges the feasible region
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

\(x_0 = 0\)
maximise $-x_0$

subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

$x_0 = 1 \cdot 10^{-1}$
maximise $-x_0$
subject to
$2x_1 - x_2 - x_0 \leq 2$
$x_1 - 5x_2 - x_0 \leq -4$
$x_0, x_1, x_2 \geq 0$
$x_0 = 0.2$
maximise $-x_0$
subject to
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\[x_0 = 0.3\]
Geometric Illustration

maximise \(-x_0\)
subject to
\[\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}\]

\(x_0 = 0.4\)
Geometric Illustration

maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq 2 \\
x_1 - 5x_2 - x_0 &\leq -4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

\[x_0 = 0.5\]
maximise $-x_0$
subject to
\[2x_1 - x_2 - x_0 \leq 2\]
\[x_1 - 5x_2 - x_0 \leq -4\]
\[x_0, x_1, x_2 \geq 0\]

$x_0 = 0.6$
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 &- x_2 - x_0 \leq 2 \\
x_1 &- 5x_2 - x_0 \leq -4 \\
\end{align*}
\]
\[
\begin{align*}
x_0, x_1, x_2 &\geq 0 \\
\end{align*}
\]

\[x_0 = 0.7\]
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq 2$

$x_1 - 5x_2 - x_0 \leq -4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 0.8$
Geometric Illustration

maximise $-x_0$
subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
-x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

$x_0 = 0.9$
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 &- x_2 &- x_0 &\leq 2 \\
x_1 &- 5x_2 &- x_0 &\leq -4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

\(x_0 = 1\)
Geometric Illustration

maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq 2$
$x_1 - 5x_2 - x_0 \leq -4$
$x_0 \geq 0$

$x_0, x_1, x_2 \geq 0$

$x_0 = 1.1$
Geometric Illustration

maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq 2$

$x_1 - 5x_2 - x_0 \leq -4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 1.2$

7. Linear Programming © T. Sauerwald
Finding an Initial Solution
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq 2$

$x_1 - 5x_2 - x_0 \leq -4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 1.3$
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 & - x_2 - x_0 & \leq & 2 \\
-x_1 & - 5x_2 - x_0 & \leq & -4 \\
x_0, x_1, x_2 & \geq & 0
\end{align*}
\]
\(x_0 = 1.4\)
maximise $-x_0$
subject to
\[\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}\]
\[x_0 = 1.5\]
Geometric Illustration

maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

\[
x_0 = 1.6
\]
maximise $-x_0$
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

$x_0 = 1.7$
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq 2$

$x_1 - 5x_2 - x_0 \leq -4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 1.8$
Geometric Illustration

maximise $-x_0$
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

$x_0 = 1.9$
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]
\(x_0 = 2\)
Geometric Illustration

maximise $-x_0$
subject to

$2x_1 - x_2 - x_0 \leq 2$
$x_1 - 5x_2 - x_0 \leq -4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 2.1$
Geometric Illustration

maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq 2 \\
x_1 - 5x_2 - x_0 &\leq -4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

\(x_0 = 2.2\)

\(x_1 - 5x_2 - x_0 \leq -4\)
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 & - x_2 - x_0 \leq 2 \\
x_1 & - 5x_2 - x_0 \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]
\(x_0 = 2.3\)
maximise \(-x_0\)
subject to
\[\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}\]

\[x_0 = 2.4\]
Geometric Illustration

maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq 2$
$x_1 - 5x_2 - x_0 \leq -4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 2.5$
maximise $-x_0$
subject to
2$x_1$ − $x_2$ − $x_0$ ≤ 2
$x_1$ − 5$x_2$ − $x_0$ ≤ −4
$x_0$, $x_1$, $x_2$ ≥ 0

$x_0 = 2.6$
maximise $-x_0$

subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq 2 \\
x_1 - 5x_2 - x_0 &\leq -4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

$x_0 = 2.7$
Geometric Illustration

maximise \(-x_0\)
subject to

\[
\begin{align*}
2x_1 &- x_2 - x_0 \leq 2 \\
x_1 &- 5x_2 - x_0 \leq -4 \\
x_0, x_1, x_2 \geq 0
\end{align*}
\]

\[x_0 = 2.8\]
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq 2 \\
x_1 - 5x_2 - x_0 &\leq -4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

\(x_0 = 2.9\)
Geometric Illustration

maximise $-x_0$

subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0 & \geq 0
\end{align*}
\]

$x_0, x_1, x_2$

$x_0 = 3$
Geometric Illustration

maximise $-x_0$
subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

$x_0 = 3.1$
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 & - x_2 & - x_0 & \leq 2 \\
x_1 & - 5x_2 & - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]
\(x_0 = 3.2\)
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]
\(x_0 = 3.3\)
maximise $-x_0$
subject to
\[ \begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*} \]

$x_0 = 3.4$
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq 2$
$x_1 - 5x_2 - x_0 \leq -4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 3.5$

$x_0 = 4$
maximise $-x_0$
subject to

\[\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}\]

$x_0 = 3.6$
Geometric Illustration

maximise $-x_0$

subject to

$$2x_1 - x_2 - x_0 \leq 2$$

$$x_1 - 5x_2 - x_0 \leq -4$$

$$x_0, x_1, x_2 \geq 0$$

$x_0 = 3.7$
maximise $-x_0$

subject to

\[\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_0 & \geq 0
\end{align*}\]

$x_0, x_1, x_2 \geq 0$

$x_0 = 3.8$
maximise $-x_0$

subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
0 & \leq x_0, x_1, x_2
\end{align*}
\]

$x_0 = 3.9$
maximise $-x_0$
subject to
\[
\begin{align*}
2x_1 & - x_2 - x_0 & \leq & 2 \\
x_1 & - 5x_2 - x_0 & \leq & -4 \\
x_0, x_1, x_2 & \geq & 0
\end{align*}
\]

$x_0 = 4$
Let us now modify the original linear program so that it is not feasible.
Let us now modify the original linear program so that it is not feasible.

⇒ Hence the auxiliary linear program has only a solution for a sufficiently large $x_0 > 0$!
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 0$
Geometric Illustration

maximise $-x_0$
subject to

$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 1 \cdot 10^{-1}$
maximise \(-x_0\)
subject to
\[
2x_1 - x_2 - x_0 \leq -2
\]
\[
-x_1 + 5x_2 - x_0 \leq 4
\]
\[
x_0, x_1, x_2 \geq 0
\]

\(x_0 = 0.2\)
Geometric Illustration

maximise \(-x_0\)

subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

\(x_0 = 0.3\)
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

\(x_0 = 0.4\)
Geometric Illustration

maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 0.5$
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 0.6$
**Geometric Illustration**

maximise $-x_0$

subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

$x_0 = 0.7$
maximise $-x_0$
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]
$x_0 = 0.8$
Geometric Illustration

maximise $-x_0$
subject to
$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 0.9$
Geometric Illustration

maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 1$
Geometric Illustration

maximise \(-x_0\)

subject to

\begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}

x_0 = 1.1
Geometric Illustration

maximise $-x_0$
subject to
$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 1.2$
maximise $-x_0$
subject to
\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
$x_0 = 1.3$
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 1.4$
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]
\(x_0 = 1.5\)
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

\(x_0 = 1.6\)
Geometric Illustration

maximise  \(-x_0\)
subject to
\[
egin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

\[x_0 = 1.7\]
maximise $-x_0$

subject to

\[ \begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
x_0, x_1, x_2 & \geq 0
\end{align*} \]

$x_0 = 1.8$
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]
\(x_0 = 1.9\)
maximise \( -x_0 \)
subject to
\[
2x_1 - x_2 - x_0 \leq -2 \\
-x_1 + 5x_2 - x_0 \leq 4 \\
x_0, x_1, x_2 \geq 0
\]

\( x_0 = 2 \)
maximise $-x_0$
subject to
\[\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}\]
maximise \(-x_0\)
subject to
\[\begin{align*}
2x_1 - x_2 &- x_0 \leq -2 \\
-x_1 + 5x_2 &- x_0 \leq 4 \\
\end{align*}\]
\[x_0, x_1, x_2 \geq 0\]

\[x_0 = 2.2\]
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 &- x_2 - x_0 &\leq & -2 \\
-x_1 &+ 5x_2 &- x_0 &\leq 4 \\
& & &\geq 0
\end{align*}
\]
\(x_0, x_1, x_2 \geq 0\)

\(x_0 = 2.3\)
maximise $-x_0$

subject to

\[
\begin{align*}
2x_1 & - x_2 & - x_0 & \leq -2 \\
-x_1 & + 5x_2 & - x_0 & \leq 4 \\
& & x_0, x_1, x_2 & \geq 0
\end{align*}
\]

$x_0 = 2.4$
maximise $-x_0$

subject to

\begin{align*}
  2x_1 - x_2 - x_0 & \leq -2 \\
  -x_1 + 5x_2 - x_0 & \leq 4 \\
  x_0, x_1, x_2 & \geq 0
\end{align*}

$x_0 = 2.5$
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
\end{align*}
\]
x_0, x_1, x_2 \geq 0
\[
\begin{align*}
x_0 & = 2.6
\end{align*}
\]
**Geometric Illustration**

maximise \(-x_0\)

subject to

\[
2x_1 - x_2 - x_0 \leq -2 \\
-x_1 + 5x_2 - x_0 \leq 4 \\
x_0, x_1, x_2 \geq 0
\]

\[x_0 = 2.7\]
maximise $-x_0$
subject to
$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 2.8$
maximise $-x_0$
subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

$x_0 = 2.9$
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$

$-x_1 + 5x_2 - x_0 \leq 4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 3$
maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$

$-x_1 + 5x_2 - x_0 \leq 4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 3.1$
maximise $-x_0$
subject to
\[\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
\end{align*}\]
\[\begin{align*}
x_0, x_1, x_2 &\geq 0
\end{align*}\]
$x_0 = 3.2$
maximise $-x_0$
subject to
\[\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}\]

$x_0 = 3.3$
Geometric Illustration

maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
x_0, x_1, x_2 & \geq 0
\end{align*}
\]

\(x_0 = 3.4\)

\(x_1 + 5x_2 - x_0 \leq 4\)
Geometric Illustration

maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$

$-x_1 + 5x_2 - x_0 \leq 4$

$x_0, x_1, x_2 \geq 0$

$x_0 = 3.5$
Geometric Illustration

maximise $-x_0$

subject to

$2x_1 - x_2 - x_0 \leq -2$
$-x_1 + 5x_2 - x_0 \leq 4$
$x_0, x_1, x_2 \geq 0$

$x_0 = 3.6$
Geometric Illustration

maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 &\leq -2 \\
-x_1 + 5x_2 - x_0 &\leq 4 \\
x_0, x_1, x_2 &\geq 0
\end{align*}
\]

\(x_0 = 3.7\)
Geometric Illustration

maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \quad (i)
\end{align*}
\]
\[x_0, x_1, x_2 \geq 0\]

\(x_0 = 3.8\)
maximise  

subject to  

\begin{align*}
2x_1 &- x_2 - x_0 &\leq & -2 \\
-x_1 + 5x_2 &- x_0 &\leq & 4 \\
\end{align*}

\begin{align*}
x_0, x_1, x_2 &\geq 0
\end{align*}

\[x_0 = 3.9\]
maximise \(-x_0\)
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq -2 \\
-x_1 + 5x_2 - x_0 & \leq 4 \\
\end{align*}
\]
\[
\begin{align*}
x_0, x_1, x_2 & \geq 0 \\
x_0 & = 4
\end{align*}
\]
**Initialize-Simplex**

**Initialize-Simplex** \((A, b, c)\)

1. let \(k\) be the index of the minimum \(b_i\)
2. if \(b_k \geq 0\) // is the initial basic solution feasible?
3. return \({1, 2, \ldots, n}, \{n + 1, n + 2, \ldots, n + m\}, A, b, c, 0\)
4. form \(L_{aux}\) by adding \(-x_0\) to the left-hand side of each constraint and setting the objective function to \(-x_0\)
5. let \((N, B, A, b, c, v)\) be the resulting slack form for \(L_{aux}\)
6. \(l = n + k\)
7. // \(L_{aux}\) has \(n + 1\) nonbasic variables and \(m\) basic variables.
8. \((N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, 0)\)
9. // The basic solution is now feasible for \(L_{aux}\).
10. iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution to \(L_{aux}\) is found
11. if the optimal solution to \(L_{aux}\) sets \(x_0\) to 0
12. if \(x_0\) is basic
13. perform one (degenerate) pivot to make it nonbasic
14. from the final slack form of \(L_{aux}\), remove \(x_0\) from the constraints and restore the original objective function of \(L\), but replace each basic variable in this objective function by the right-hand side of its associated constraint
15. return the modified final slack form
16. else return “infeasible”
**Initialize-Simplex**

**Initialize-Simplex** \((A, b, c)\)

1. let \(k\) be the index of the minimum \(b_i\)
2. if \(b_k \geq 0\)  
   // is the initial basic solution feasible?
3. return \((\{1, 2, \ldots, n\}, \{n + 1, n + 2, \ldots, n + m\}, A, b, c, 0)\)
4. form \(L_{aux}\) by adding \(-x_0\) to the left-hand side of each constraint and setting the objective function to \(-x_0\)
5. let \((N, B, A, b, c, v)\) be the resulting slack form for \(L_{aux}\)
6. \(l = n + k\)
7. // \(L_{aux}\) has \(n + 1\) nonbasic variables and \(m\) basic variables.
8. \((N, B, A, b, c, v) = \text{Pivot}(N, B, A, b, c, v, l, 0)\)
9. // The basic solution is now feasible for \(L_{aux}\).
10. iterate the **while** loop of lines 3–12 of **Simplex** until an optimal solution to \(L_{aux}\) is found
11. if the optimal solution to \(L_{aux}\) sets \(\bar{x}_0\) to 0
12. if \(\bar{x}_0\) is basic
13. perform one (degenerate) pivot to make it nonbasic
14. from the final slack form of \(L_{aux}\), remove \(x_0\) from the constraints and restore the original objective function of \(L\), but replace each basic variable in this objective function by the right-hand side of its associated constraint
15. return the modified final slack form
16. else return “infeasible”

Test solution with \(N = \{1, 2, \ldots, n\}, B = \{n + 1, n + 2, \ldots, n + m\}, \bar{x}_i = b_i\) for \(i \in B\), \(\bar{x}_i = 0\) otherwise.
**Initialize-Simplex**

**Initialize-Simplex** ($A, b, c$)
1. let $k$ be the index of the minimum $b_i$
2. if $b_k \geq 0$  // is the initial basic solution feasible?
   - return $\{1, 2, \ldots, n\}, \{n + 1, n + 2, \ldots, n + m\}, A, b, c, 0$
3. form $L_{aux}$ by adding $-x_0$ to the left-hand side of each constraint
   - and setting the objective function to $-x_0$
4. let $(N, B, A, b, c, \nu)$ be the resulting slack form for $L_{aux}$
5. $l = n + k$
6. // $L_{aux}$ has $n + 1$ nonbasic variables and $m$ basic variables.
7. $(N, B, A, b, c, \nu) =$ PIVOT$(N, B, A, b, c, \nu, l, 0)$
8. // The basic solution is now feasible for $L_{aux}$.
9. iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
to $L_{aux}$ is found
10. if the optimal solution to $L_{aux}$ sets $x_0$ to 0
11. if $x_0$ is basic
12. perform one (degenerate) pivot to make it nonbasic
13. from the final slack form of $L_{aux}$, remove $x_0$ from the constraints and
   - restore the original objective function of $L$, but replace each basic
   - variable in this objective function by the right-hand side of its
   - associated constraint
14. return the modified final slack form
15. else return “infeasible”

Test solution with $N = \{1, 2, \ldots, n\}, B = \{n + 1, n + 2, \ldots, n + m\}, \bar{x}_i = b_i$ for $i \in B$, $\bar{x}_i = 0$ otherwise.

ℓ will be the leaving variable so that $x_\ell$ has the most negative value.
**Initialize-Simplex**

**Initialize-Simplex** $(A, b, c)$

1. let $k$ be the index of the minimum $b_i$
2. if $b_k \geq 0$  // is the initial basic solution feasible?
   3. return $(\{1, 2, \ldots, n\}, \{n + 1, n + 2, \ldots, n + m\}, A, b, c, 0)$
4. form $L_{aux}$ by adding $-x_0$ to the left-hand side of each constraint and setting the objective function to $-x_0$
5. let $(N, B, A, b, c, v)$ be the resulting slack form for $L_{aux}$
6. $l = n + k$
7. // $L_{aux}$ has $n + 1$ nonbasic variables and $m$ basic variables.
8. $(N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, 0)$
9. // The basic solution is now feasible for $L_{aux}$.
10. iterate the while loop of lines 3–12 of Simplex until an optimal solution to $L_{aux}$ is found
11. if the optimal solution to $L_{aux}$ sets $\bar{x}_0$ to 0
   12. if $\bar{x}_0$ is basic
   13. perform one (degenerate) pivot to make it nonbasic
   14. from the final slack form of $L_{aux}$, remove $x_0$ from the constraints and restore the original objective function of $L$, but replace each basic variable in this objective function by the right-hand side of its associated constraint
   15. return the modified final slack form
16. else return “infeasible”

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n + 1, n + 2, \ldots, n + m\}$, $\bar{x}_i = b_i$ for $i \in B$, $\bar{x}_i = 0$ otherwise.

- Pivot step with $x_0$ leaving and $x_0$ entering.
- $\ell$ will be the leaving variable so that $x_\ell$ has the most negative value.
**INITIALIZE-SIMPLEX**

**INITIALIZE-SIMPLEX** \((A, b, c)\)

1. Let \(k\) be the index of the minimum \(b_i\).
2. If \(b_k \geq 0\)  
   // Is the initial basic solution feasible?
   3. Return \(([1, 2, \ldots, n], \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)\)
4. Form \(L_{aux}\) by adding \(-x_0\) to the left-hand side of each constraint  
   and setting the objective function to \(-x_0\).
5. Let \((N, B, A, b, c, \nu)\) be the resulting slack form for \(L_{aux}\)
6. \(l = n + k\)
7. // \(L_{aux}\) has \(n+1\) nonbasic variables and \(m\) basic variables.
8. \((N, B, A, b, c, \nu) = PIVOT(N, B, A, b, c, \nu, l, 0)\)
9. // The basic solution is now feasible for \(L_{aux}\).
10. Iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution to \(L_{aux}\) is found
11. If the optimal solution to \(L_{aux}\) sets \(x_0\) to 0
12. If \(x_0\) is basic
   13. Perform one (degenerate) pivot to make it nonbasic
   14. From the final slack form of \(L_{aux}\), remove \(x_0\) from the constraints and restore the original objective function of \(L\), but replace each basic variable in this objective function by the right-hand side of its associated constraint
15. Return the modified final slack form
16. Else return “infeasible”

Test solution with \(N = \{1, 2, \ldots, n\}, B = \{n+1, n+2, \ldots, n+m\}, \bar{x}_i = b_i\) for \(i \in B, \bar{x}_i = 0\) otherwise.

\(\ell\) will be the leaving variable so that \(x_\ell\) has the most negative value.

Pivot step with \(x_\ell\) leaving and \(x_0\) entering.

This pivot step does not change the value of any variable.
Example of INITIALIZE-SIMPLEX (1/3)

maximise \[ 2x_1 - x_2 \]
subject to
\[ \begin{align*}
2x_1 - x_2 & \leq 2 \\
x_1 - 5x_2 & \leq -4 \\
\end{align*} \]
\[ x_1, x_2 \geq 0 \]

Basic solution \((0,0,0,2,-4)\) not feasible!
Example of \textit{INITIALIZE-SIMPLEX} (1/3)

maximise $2x_1 - x_2$
subject to

$$
2x_1 - x_2 \leq 2
$$

$$
x_1 - 5x_2 \leq -4
$$

$$
x_1, x_2 \geq 0
$$

Formulating the auxiliary linear program

Basic solution $(0,0,0,2,-4)$ not feasible!
Example of INITIALIZE-SIMPLEX (1/3)

maximise \( 2x_1 - x_2 \)

subject to

\[
\begin{align*}
2x_1 - x_2 & \leq 2 \\
x_1 - 5x_2 & \leq -4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Formulating the auxiliary linear program

maximise \(-x_0\)

subject to

\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_1, x_2, x_0 & \geq 0
\end{align*}
\]
Example of INITIALIZE-SIMPLEX (1/3)

maximise 2\(x_1\) − \(x_2\)
subject to
2\(x_1\) − \(x_2\) ≤ 2
\(x_1\) − 5\(x_2\) ≤ −4
\(x_1\), \(x_2\) ≥ 0

Formulating the auxiliary linear program

maximise − \(x_0\)
subject to
2\(x_1\) − \(x_2\) − \(x_0\) ≤ 2
\(x_1\) − 5\(x_2\) − \(x_0\) ≤ −4
\(x_1\), \(x_2\), \(x_0\) ≥ 0

Converting into slack form
Example of **INITIALIZE-SIMPLEX** (1/3)

maximise \( 2x_1 - x_2 \)
subject to
\[
\begin{align*}
2x_1 - x_2 & \leq 2 \\
x_1 - 5x_2 & \leq -4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

- Formulating the auxiliary linear program
- Converting into slack form

Basic solution \((0,0,0,2,-4)\) not feasible!

---

7. Linear Programming © T. Sauerwald Finding an Initial Solution
Example of INITIALIZE-SIMPLEX (1/3)

maximise $2x_1 - x_2$
subject to
\[
\begin{align*}
2x_1 - x_2 & \leq 2 \\
x_1 - 5x_2 & \leq -4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Formulating the auxiliary linear program

maximise $-x_0$
subject to
\[
\begin{align*}
2x_1 - x_2 - x_0 & \leq 2 \\
x_1 - 5x_2 - x_0 & \leq -4 \\
x_1, x_2, x_0 & \geq 0
\end{align*}
\]

Basic solution $(0, 0, 0, 2, -4)$ not feasible!

Converting into slack form

\[
\begin{align*}
z & = 2 - 2x_1 + x_2 + x_0 \\
x_3 & = 2 - 2x_1 + x_2 + x_0 \\
x_4 & = -4 - x_1 + 5x_2 + x_0
\end{align*}
\]
Example of INITIALIZE-SIMPLEX (2/3)

\[
\begin{align*}
    z &= -x_0 \\
    x_3 &= 2 - 2x_1 + x_2 + x_0 \\
    x_4 &= -4 - x_1 + 5x_2 + x_0
\end{align*}
\]
Example of \textsc{Initialize-Simplex} (2/3)

\[
\begin{align*}
    z &= -x_0 \\
    x_3 &= 2 - 2x_1 + x_2 + x_0 \\
    x_4 &= -4 - x_1 + 5x_2 + x_0
\end{align*}
\]

Pivot with $x_0$ entering and $x_4$ leaving

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Optimal solution has $x_0 = 0$, hence the initial problem was feasible!

---

7. Linear Programming © T. Sauerwald

Finding an Initial Solution
Example of INITIALIZE-SIMPLEX (2/3)

\[ \begin{align*}
    z &= -x_0 \\
    x_3 &= 2 - 2x_1 + x_2 + x_0 \\
    x_4 &= -4 - x_1 + 5x_2 + x_0 \\
\end{align*} \]

Pivot with \( x_0 \) entering and \( x_4 \) leaving

\[ \begin{align*}
    z &= -4 - x_1 + 5x_2 - x_4 \\
    x_0 &= 4 + x_1 - 5x_2 + x_4 \\
    x_3 &= 6 - x_1 - 4x_2 + x_4 \\
\end{align*} \]
Example of **INITIALIZE-SIMPLEX (2/3)**

\[
\begin{align*}
    z &= 2x_1 + x_2 + x_0 \\
    x_3 &= 2 - 2x_1 + x_2 + x_0 \\
    x_4 &= -4 - x_1 + 5x_2 + x_0 \\
\end{align*}
\]

Pivot with \(x_0\) entering and \(x_4\) leaving

\[
\begin{align*}
    z &= -4 - x_1 + 5x_2 - x_4 \\
    x_0 &= 4 + x_1 - 5x_2 + x_4 \\
    x_3 &= 6 - x_1 - 4x_2 + x_4 \\
\end{align*}
\]

Basic solution \((4, 0, 0, 6, 0)\) is feasible!
Example of INITIALIZE-SIMPLEX (2/3)

\[
\begin{align*}
    z &= - x_0 \\
    x_3 &= 2 - 2x_1 + x_2 + x_0 \\
    x_4 &= -4 - x_1 + 5x_2 + x_0
\end{align*}
\]

Pivot with \( x_0 \) entering and \( x_4 \) leaving

\[
\begin{align*}
    z &= -4 - x_1 + 5x_2 - x_4 \\
    x_0 &= 4 + x_1 - 5x_2 + x_4 \\
    x_3 &= 6 - x_1 - 4x_2 + x_4
\end{align*}
\]

Basic solution \((4, 0, 0, 6, 0)\) is feasible!

Pivot with \( x_2 \) entering and \( x_0 \) leaving
Example of INITIALIZE-SIMPLEX (2/3)

\[ z = -x_0 \]
\[ x_3 = 2 - 2x_1 + x_2 + x_0 \]
\[ x_4 = -4 - x_1 + 5x_2 + x_0 \]

Pivot with \( x_0 \) entering and \( x_4 \) leaving

\[ z = -4 - x_1 + 5x_2 - x_4 \]
\[ x_0 = 4 + x_1 - 5x_2 + x_4 \]
\[ x_3 = 6 - x_1 - 4x_2 + x_4 \]

Basic solution \((4, 0, 0, 6, 0)\) is feasible!

Pivot with \( x_2 \) entering and \( x_0 \) leaving

\[ z = 4 - x_0 \]
\[ x_2 = \frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5} \]
\[ x_3 = \frac{14}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \]

Optimal solution has \( x_0 = 0 \), hence the initial problem was feasible!
Example of INITIALIZE-SIMPLEX (2/3)

\[
\begin{align*}
  z &= -x_0 \\
  x_3 &= 2 - 2x_1 + x_2 + x_0 \\
  x_4 &= -4 - x_1 + 5x_2 + x_0
\end{align*}
\]

Pivot with \(x_0\) entering and \(x_4\) leaving

\[
\begin{align*}
  z &= -4 - x_1 + 5x_2 - x_4 \\
  x_0 &= 4 + x_1 - 5x_2 + x_4 \\
  x_3 &= 6 - x_1 - 4x_2 + x_4
\end{align*}
\]

Basic solution \((4, 0, 0, 6, 0)\) is feasible!

Pivot with \(x_2\) entering and \(x_0\) leaving

\[
\begin{align*}
  z &= -x_0 \\
  x_2 &= \frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
  x_3 &= \frac{14}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5}
\end{align*}
\]

Optimal solution has \(x_0 = 0\), hence the initial problem was feasible!
Example of INITIALIZE-SIMPLEX (3/3)

\[ z = -x_0 \]
\[ x_2 = \frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5} \]
\[ x_3 = \frac{14}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \]
Example of INITIALIZE-SIMPLEX (3/3)

\[
\begin{align*}
z &= -x_0 \\
x_2 &= \frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
x_3 &= \frac{14}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5}
\end{align*}
\]

Set \( x_0 = 0 \) and express objective function by non-basic variables.
Example of INITIALIZE-SIMPLEX (3/3)

\[
\begin{align*}
  z &= -x_0 \\
  x_2 &= \frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
  x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{x_4}{5}
\end{align*}
\]

Set \( x_0 = 0 \) and express objective function by non-basic variables

\[
\begin{align*}
  z &= -\frac{4}{5} + \frac{9}{5}x_1 - \frac{x_4}{5} \\
  x_2 &= \frac{4}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
  x_3 &= -\frac{9}{5}x_1 + \frac{x_4}{5}
\end{align*}
\]
Example of **INITIALIZE-SIMPLEX (3/3)**

\[ z = -x_0 \]
\[ x_2 = \frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5} \]
\[ x_3 = \frac{14}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \]

Set \( x_0 = 0 \) and express objective function by non-basic variables

\[ z = -\frac{4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \]
\[ x_2 = \frac{4}{5} + \frac{x_1}{5} + \frac{x_4}{5} \]
\[ x_3 = \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \]

Basic solution \( (0, \frac{4}{5}, \frac{14}{5}, 0) \), which is feasible!
Example of INITIALIZE-SIMPLEX (3/3)

\[
\begin{align*}
  z &= -x_0 + 4 - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
  x_2 &= \frac{4}{5} - \frac{x_0}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \\
  x_3 &= \frac{14}{5} + \frac{4x_0}{5} - \frac{x_1}{5} + \frac{x_4}{5}
\end{align*}
\]

Set \( x_0 = 0 \) and express objective function by non-basic variables

\[
\begin{align*}
  z &= -\frac{4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \\
  x_2 &= \frac{4}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
  x_3 &= \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5}
\end{align*}
\]

Basic solution \((0, \frac{4}{5}, \frac{14}{5}, 0)\), which is feasible!

Lemma 29.12

If a linear program \( L \) has no feasible solution, then INITIALIZE-SIMPLEX returns “infeasible”. Otherwise, it returns a valid slack form for which the basic solution is feasible.
Theorem 29.13 (Fundamental Theorem of Linear Programming)

For any linear program \( L \), given in standard form, either:

1. \( L \) is infeasible \( \Rightarrow \) SIMPLEX returns “infeasible”.
2. \( L \) is unbounded \( \Rightarrow \) SIMPLEX returns “unbounded”.
3. \( L \) has an optimal solution with a finite objective value
   \( \Rightarrow \) SIMPLEX returns an optimal solution with a finite objective value.

Small Technicality: need to equip SIMPLEX with an “anti-cycling strategy” (see extra slides)
Theorem 29.13 (Fundamental Theorem of Linear Programming)

For any linear program \( L \), given in standard form, either:

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2. \( L \) is unbounded \( \Rightarrow \) \texttt{SIMPLEX} returns “unbounded”.
3. \( L \) has an optimal solution with a finite objective value
   \( \Rightarrow \) \texttt{SIMPLEX} returns an optimal solution with a finite objective value.

Small Technicality: need to equip \texttt{SIMPLEX} with an “anti-cycling strategy” (see extra slides)

Proof requires the concept of duality, which is not covered in this course (for details see CLRS3, Chapter 29.4)
Workflow for Solving Linear Programs

Linear Program (in any form)

Standard Form

Slack Form

No Feasible Solution

INITIALIZE-SIMPLEX terminates

Feasible Basic Solution

INITIALIZE-SIMPLEX followed by SIMPLEX

LP unbounded

SIMPLEX terminates

LP bounded

SIMPLEX returns optimum
Linear Programming

Linear Programming is an extremely versatile tool for modeling problems of all kinds. It forms the basis of Integer Programming, to be discussed in later lectures.

In practice, Linear Programming usually terminates in polynomial time, i.e., $O(m + n)$. In theory, however, even with anti-cycling measures, it may need exponential time.

The Simplex Algorithm is a fundamental method for solving Linear Programming problems. A research problem is to find a pivoting rule that makes the Simplex algorithm a polynomial-time algorithm.

Interior-Point Methods are another approach that traverses the interior of the feasible set of solutions, not just the vertices.

Polynomial-Time Algorithms

Finding an Initial Solution

7. Linear Programming © T. Sauerwald

Finding an Initial Solution
Linear Programming

- extremely versatile tool for modelling problems of all kinds
Linear Programming

- extremely versatile tool for modelling problems of all kinds
- basis of Integer Programming, to be discussed in later lectures
Linear Programming

- extremely versatile tool for modelling problems of all kinds
- basis of Integer Programming, to be discussed in later lectures

Simplex Algorithm

- In practice: usually terminates in polynomial time, i.e., $O(m + n)$
Linear Programming

- extremely versatile tool for modelling problems of all kinds
- basis of Integer Programming, to be discussed in later lectures

Simplex Algorithm

- **In practice**: usually terminates in polynomial time, i.e., $O(m + n)$
- **In theory**: even with anti-cycling may need exponential time
Linear Programming and Simplex: Summary and Outlook

Linear Programming
- extremely versatile tool for modelling problems of all kinds
- basis of Integer Programming, to be discussed in later lectures

Simplex Algorithm
- In practice: usually terminates in polynomial time, i.e., $O(m + n)$
- In theory: even with anti-cycling may need exponential time

Research Problem: Is there a pivoting rule which makes SIMPLEX a polynomial-time algorithm?
Linear Programming

- extremely versatile tool for modelling problems of all kinds
- basis of Integer Programming, to be discussed in later lectures

Simplex Algorithm

- In practice: usually terminates in polynomial time, i.e., $O(m + n)$
- In theory: even with anti-cycling may need exponential time

Research Problem: Is there a pivoting rule which makes SIMPLEX a polynomial-time algorithm?

Polynomial-Time Algorithms
Linear Programming

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- basis of Integer Programming, to be discussed in later lectures

Simplex Algorithm

- In practice: usually terminates in polynomial time, i.e., $O(m + n)$
- In theory: even with anti-cycling may need exponential time

Research Problem: Is there a pivoting rule which makes SIMPLEX a polynomial-time algorithm?

Polynomial-Time Algorithms

- Interior-Point Methods: traverses the interior of the feasible set of solutions (not just vertices!)
Linear Programming

- extremely versatile tool for modelling problems of all kinds
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Simplex Algorithm

- In practice: usually terminates in polynomial time, i.e., $O(m + n)$
- In theory: even with anti-cycling may need exponential time

Research Problem: Is there a pivoting rule which makes SIMPLEX a polynomial-time algorithm?

Polynomial-Time Algorithms

- Interior-Point Methods: traverses the interior of the feasible set of solutions (not just vertices!)
1.2 Famous Failures and the Need for Alternatives

For many problems a bit beyond the scope of an undergraduate course, the downside of worst-case analysis rears its ugly head. This section reviews four famous examples in which worst-case analysis gives misleading or useless advice about how to solve a problem. These examples motivate the alternatives to worst-case analysis that are surveyed in Section 1.4 and described in detail in later chapters of the book.

1.2.1 The Simplex Method for Linear Programming

Perhaps the most famous failure of worst-case analysis concerns linear programming, the problem of optimizing a linear function subject to linear constraints (Figure 1.1). Dantzig proposed in the 1940s an algorithm for solving linear programs called the *simplex method*. The simplex method solves linear programs using greedy local

Source: “Beyond the Worst-Case Analysis of Algorithms” by Tim Roughgarden, 2020
Outline

Simplex Algorithm by Example

Details of the Simplex Algorithm

Finding an Initial Solution

Appendix: Cycling and Termination (non-examinable)
Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.
**Degeneracy**: One iteration of SIMPLEX leaves the objective value unchanged.

\[
\begin{align*}
  z &= x_1 + x_2 + x_3 \\
  x_4 &= 8 - x_1 - x_2 \\
  x_5 &= x_2 - x_3
\end{align*}
\]
Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

\[
\begin{align*}
  z &= x_1 + x_2 + x_3 \\
  x_4 &= 8 - x_1 - x_2 \\
  x_5 &= x_2 - x_3
\end{align*}
\]

Pivot with \( x_1 \) entering and \( x_4 \) leaving
Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

\[
\begin{align*}
  z &= x_1 + x_2 + x_3 \\
x_4 &= 8 - x_1 - x_2 \\
x_5 &= x_2 - x_3
\end{align*}
\]

Pivot with \(x_1\) entering and \(x_4\) leaving

\[
\begin{align*}
  z &= 8 + x_3 - x_4 \\
x_1 &= 8 - x_2 - x_4 \\
x_5 &= x_2 - x_3
\end{align*}
\]
Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

\[
\begin{align*}
  z &= x_1 + x_2 + x_3 \\
  x_4 &= 8 - x_1 - x_2 \\
  x_5 &= x_2 - x_3
\end{align*}
\]

Pivot with \( x_1 \) entering and \( x_4 \) leaving

\[
\begin{align*}
  z &= 8 + x_3 - x_4 \\
  x_1 &= 8 - x_2 - x_4 \\
  x_5 &= x_2 - x_3
\end{align*}
\]

Pivot with \( x_3 \) entering and \( x_5 \) leaving
Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

\[
\begin{align*}
z & = x_1 + x_2 + x_3 \\
x_4 & = 8 - x_1 - x_2 \\
x_5 & = x_2 - x_3
\end{align*}
\]

Pivot with \(x_1\) entering and \(x_4\) leaving

\[
\begin{align*}
z & = 8 + x_3 - x_4 \\
x_1 & = 8 - x_2 - x_4 \\
x_5 & = x_2 - x_3
\end{align*}
\]

Pivot with \(x_3\) entering and \(x_5\) leaving

\[
\begin{align*}
z & = 8 + x_2 - x_4 - x_5 \\
x_1 & = 8 - x_2 - x_4 \\
x_3 & = x_2 - x_5
\end{align*}
\]
**Degeneracy:** One iteration of SIMPLEX leaves the objective value unchanged.

\[
z = x_1 + x_2 + x_3
\]
\[
x_4 = 8 - x_1 - x_2
\]
\[
x_5 = x_2 - x_3
\]

**Pivot with** \(x_1\) **entering and** \(x_4\) **leaving**

\[
z = 8 + x_3 - x_4
\]
\[
x_1 = 8 - x_2 - x_4
\]
\[
x_5 = x_2 - x_3
\]

**Cycling:** If additionally slack form at two iterations are identical, SIMPLEX fails to terminate!

\[
z = 8 + x_2 - x_4 - x_5
\]
\[
x_1 = 8 - x_2 - x_4
\]
\[
x_3 = x_2 - x_5
\]
Exercise: Execute one more step of the Simplex Algorithm on the tableau from the previous slide.
Termination and Running Time

**Cycling**: SIMPLEX may fail to terminate.

1. Bland's rule: Choose entering variable with smallest index
2. Random rule: Choose entering variable uniformly at random
3. Perturbation: Perturb the input slightly so that it is impossible to have two solutions with the same objective value

Anti-Cycling Strategies
Assuming INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most \((n + m)\) iterations.

Lemma 29.7: It is theoretically possible, but very rare in practice. Replace each \(b_i\) by \(b_i + \epsilon_i\), where \(\epsilon_i \gg \epsilon_i + 1\) are all small.

Every set \(B\) of basic variables uniquely determines a slack form, and there are at most \((n + m)\) unique slack forms.
Termination and Running Time

Cycling: SIMPLEX may fail to terminate.

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Assuming INITIALIZE -SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most \((n + m)m\) iterations.

Lemma 29.7
It is theoretically possible, but very rare in practice.

Replace each \(b_i\) by \(\hat{b}_i = b_i + \epsilon_i\), where \(\epsilon_i \gg \epsilon_i + 1\) are all small.

Every set \(B\) of basic variables uniquely determines a slack form, and there are at most \((n + m)m\) unique slack forms.
**Termination and Running Time**

**Cycling:** SIMPLEX may fail to terminate.

1. Bland's rule: Choose entering variable with smallest index
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3. Perturbation: Perturb the input slightly so that it is impossible to have two solutions with the same objective value

**Anti-Cycling Strategies**

It is theoretically possible, but very rare in practice.
Cycling: SIMPLEX may fail to terminate.

Anti-Cycling Strategies

1. Bland’s rule: Choose entering variable with smallest index

It is theoretically possible, but very rare in practice.
Termination and Running Time

**Cycling**: SIMPLEX may fail to terminate.

1. **Bland’s rule**: Choose entering variable with smallest index
2. **Random rule**: Choose entering variable uniformly at random
3. **Perturbation**: Perturb the input slightly so that it is impossible to have two solutions with the same objective value

**Anti-Cycling Strategies**

- It is theoretically possible, but very rare in practice.

**Lemma 29.7**

Every set $B$ of basic variables uniquely determines a slack form, and there are at most $(n + m)$ unique slack forms.
Termination and Running Time

Cycling: SIMPLEX may fail to terminate.

1. Bland's rule: Choose entering variable with smallest index
2. Random rule: Choose entering variable uniformly at random
3. Perturbation: Perturb the input slightly so that it is impossible to have two solutions with the same objective value

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It is theoretically possible, but very rare in practice.
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Anti-Cycling Strategies

It is theoretically possible, but very rare in practice.

Replace each $b_i$ by $\hat{b}_i = b_i + \epsilon_i$, where $\epsilon_i \gg \epsilon_{i+1}$ are all small.
Termination and Running Time

Cycling: SIMPLEX may fail to terminate.

Anti-Cycling Strategies

1. Bland’s rule: Choose entering variable with smallest index
2. Random rule: Choose entering variable uniformly at random
3. Perturbation: Perturb the input slightly so that it is impossible to have two solutions with the same objective value

Lemma 29.7

Assuming INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most \( \binom{n+m}{m} \) iterations.
Termination and Running Time

Cycling: SIMPLEX may fail to terminate.

1. Bland’s rule: Choose entering variable with smallest index
2. Random rule: Choose entering variable uniformly at random
3. Perturbation: Perturb the input slightly so that it is impossible to have two solutions with the same objective value

Anti-Cycling Strategies

Replace each $b_i$ by $\hat{b}_i = b_i + \epsilon_i$, where $\epsilon_i \gg \epsilon_{i+1}$ are all small.

Lemma 29.7

Assuming INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most $\binom{n+m}{m}$ iterations.

Every set $B$ of basic variables uniquely determines a slack form, and there are at most $\binom{n+m}{m}$ unique slack forms.

It is theoretically possible, but very rare in practice.
Randomised Algorithms
Lecture 8: Solving a TSP Instance using Linear Programming

Thomas Sauerwald (tms41@cam.ac.uk)
Outline

Introduction

Examples of TSP Instances

Demonstration
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

Formal Definition
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

Formal Definition

- **Given**: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

Formal Definition

- **Given**: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- **Goal**: Find a hamiltonian cycle of $G$ with minimum cost.
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

Formal Definition

- **Given**: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- **Goal**: Find a hamiltonian cycle of $G$ with minimum cost.

Metric TSP: costs satisfy triangle inequality:
\[ c(u, w) \leq c(u, v) + c(v, w) \]

Euclidean TSP: cities are points in the Euclidean space, costs are equal to their (rounded) Euclidean distance.
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

**Formal Definition**

- **Given**: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- **Goal**: Find a hamiltonian cycle of $G$ with minimum cost.

- $3 + 2 + 1 + 3 = 9$
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

Formal Definition

- **Given:** A complete undirected graph \( G = (V, E) \) with nonnegative integer cost \( c(u, v) \) for each edge \( (u, v) \in E \)
- **Goal:** Find a hamiltonian cycle of \( G \) with minimum cost.

\[
2 + 4 + 1 + 1 = 8
\]
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

Formal Definition

- Given: A complete undirected graph \( G = (V, E) \) with nonnegative integer cost \( c(u, v) \) for each edge \((u, v) \in E\)
- Goal: Find a hamiltonian cycle of \( G \) with minimum cost.

Solution space consists of at most \( n! \) possible tours!

---

8. Solving TSP via Linear Programming © T. Sauerwald
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

Formal Definition

- **Given**: A complete undirected graph \( G = (V, E) \) with nonnegative integer cost \( c(u, v) \) for each edge \( (u, v) \in E \)
- **Goal**: Find a hamiltonian cycle of \( G \) with minimum cost.

Solution space consists of at most \( n! \) possible tours!

Actually the right number is \( (n - 1)!/2 \)

Metric TSP: costs satisfy triangle inequality:

\[ c(u, w) \leq c(u, v) + c(v, w) \]

Euclidean TSP: cities are points in the Euclidean space, costs are equal to their (rounded) Euclidean distance

Special Instances

Even this version is NP hard (Ex. 35.2-2)

8. Solving TSP via Linear Programming © T. Sauerwald
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

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- **Given**: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
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Actually the right number is $(n - 1)!/2$

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Even this version is NP hard (Ex. 35.2-2)

8. Solving TSP via Linear Programming © T. Sauerwald
The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*

**Formal Definition**

- **Given:** A complete undirected graph \( G = (V, E) \) with nonnegative integer cost \( c(u, v) \) for each edge \( (u, v) \in E \)
- **Goal:** Find a hamiltonian cycle of \( G \) with minimum cost.

Solution space consists of at most \( n! \) possible tours! Actually the right number is \((n - 1)!/2\)

**Special Instances**

- **Metric TSP:** costs satisfy triangle inequality:
  \[
  \forall u, v, w \in V : \quad c(u, w) \leq c(u, v) + c(v, w).
  \]
The Traveling Salesman Problem (TSP)

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- **Goal:** Find a hamiltonian cycle of $G$ with minimum cost.

Solution space consists of at most $n!$ possible tours!

Actually the right number is $(n - 1)!/2$

Special Instances

- **Metric TSP:** costs satisfy triangle inequality:

  $$\forall u, v, w \in V : c(u, w) \leq c(u, v) + c(v, w).$$

Even this version is NP hard (Ex. 35.2-2)
The Traveling Salesman Problem (TSP)

Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

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- **Given:** A complete undirected graph \( G = (V, E) \) with nonnegative integer cost \( c(u, v) \) for each edge \( (u, v) \in E \)
- **Goal:** Find a hamiltonian cycle of \( G \) with minimum cost.

Solution space consists of at most \( n! \) possible tours!

Actually the right number is \( (n - 1)!/2 \)

Special Instances

- **Metric TSP:** costs satisfy triangle inequality:
  \[
  \forall u, v, w \in V: \quad c(u, w) \leq c(u, v) + c(v, w).
  \]

- **Euclidean TSP:** cities are points in the Euclidean space, costs are equal to their (rounded) Euclidean distance

Even this version is NP hard (Ex. 35.2-2)
Outline

Introduction

Examples of TSP Instances

Demonstration
Traveling Salesman 12

rather simple methods could be found to yield a tour much nearer optimal than 30 percent. However, even a few percent gain would be well worth-while in some cases, so the problem does seem to have practical importance as well as mathematical interest. (p. 65)

Thus Flood realized that the Nearest Neighbor method is not a good estimate of the TSP but it created a decent first solution.

In 1962 a contest brought the TSP national recognition through a contest given by Proctor and Gamble. A flyer of the contest is pictured below.

The traveling salesman problem recently achieved national prominence when a soap company used it as the basis of a promotional contest. Prizes up to $10,000...
532 cities (1987 [Padberg, Rinaldi])
13,509 cities (1999 [Applegate, Bixby, Chavatal, Cook])
SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

G. DANTZIG, R. FULKERSON, AND S. JOHNSON

The Rand Corporation, Santa Monica, California
(Received August 9, 1954)

It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance.

The TRAVELING-SALESMAN PROBLEM might be described as follows: Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the original point of departure. More generally, given an $n$ by $n$ symmetric matrix $D = (d_{ij})$, where $d_{ij}$ represents the 'distance' from $I$ to $J$, arrange the points in a cyclic order in such a way that the sum of the $d_{ij}$ between consecutive points is minimal. Since there are only a finite number of possibilities (at most $\frac{1}{2} (n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of $n$. Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem, little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much; what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so. In particular, it will be shown that a certain arrangement of 49 cities, one in each of the 48 states and Washington, D. C., is best, the $d_{ij}$ used representing road distances as taken from an atlas.
## The 42 (49) Cities

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>Cleveland, Ohio</td>
<td>21. Santa Fe, N. M.</td>
<td>37. Columbia, S. C.</td>
</tr>
<tr>
<td>5</td>
<td>Charleston, W. Va.</td>
<td>22. Denver, Colo.</td>
<td>38. Raleigh, N. C.</td>
</tr>
<tr>
<td>7</td>
<td>Indianapolis, Ind.</td>
<td>24. Omaha, Neb.</td>
<td>40. Washington, D. C.</td>
</tr>
<tr>
<td>9</td>
<td>Milwaukee, Wis.</td>
<td>26. Kansas City, Mo.</td>
<td>42. Portland, Me.</td>
</tr>
<tr>
<td>10</td>
<td>Minneapolis, Minn.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Pierre, S. D.</td>
<td></td>
<td></td>
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<tr>
<td>12</td>
<td>Bismarck, N. D.</td>
<td></td>
<td></td>
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<tr>
<td>13</td>
<td>Helena, Mont.</td>
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<tr>
<td>14</td>
<td>Seattle, Wash.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Portland, Ore.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>Boise, Idaho</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>Salt Lake City, Utah</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. Solving TSP via Linear Programming © T. Sauerwald 
Examples of TSP Instances
Combinatorial Explosion

8. Solving TSP via Linear Programming © T. Sauerwald

Examples of TSP Instances
Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.

http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html
Road Distances

Hence this is an instance of the Metric TSP, but not Euclidean TSP.

TABLE I

Road Distances between Cities in Adjusted Units

The figures in the table are mileages between the two specified numbered cities, less 11, divided by 17, and rounded to the nearest integer.

|   | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 1 | 8 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 2 | 9 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 3 | 10|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 4 | 11|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 5 | 12|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 6 | 13|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 7 | 14|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 8 | 15|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 9 | 16|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 10| 17|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

8. Solving TSP via Linear Programming © T. Sauerwald
Examples of TSP Instances

12
Hence this is an instance of the Metric TSP, but not Euclidean TSP.

TABLE I
ROAD DISTANCES BETWEEN CITIES IN ADJUSTED UNITS

The figures in the table are mileages between the two specified numbered cities, less 11, divided by 17, and rounded to the nearest integer.

8. Solving TSP via Linear Programming © T. Sauerwald

Examples of TSP Instances
Idea: Indicator variable $x(i, j)$, $i > j$, which is one if the tour includes edge $\{i, j\}$ (in either direction)
Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable \( x(i, j), i > j \), which is one if the tour includes edge \( \{i, j\} \) (in either direction)

Minimize

\[
\sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i, j) x(i, j)
\]

Subject to

\[
\sum_{j<i} x(i, j) + \sum_{j>i} x(j, i) = 2 \quad \text{for each } 1 \leq i \leq 42
\]

\[
0 \leq x(i, j) \leq 1 \quad \text{for each } 1 \leq j < i \leq 42
\]

Constraints \( x(i, j) \in \{0, 1\} \) are not allowed in a LP!

Branch & Bound to solve an Integer Program:

As long as solution of LP has fractional \( x(i, j) \in (0, 1) \):

Add \( x(i, j) = 0 \) to the LP, solve it and recurse

Add \( x(i, j) = 1 \) to the LP, solve it and recurse

Return best of these two solutions

If solution of LP integral, return objective value

Bound-Step: If the best known integral solution so far is better than the solution of a LP, no need to explore branch further!
Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable \( x(i, j), i > j \), which is one if the tour includes edge \( \{i, j\} \) (in either direction)

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i, j) x(i, j) \\
\text{subject to} & \quad \sum_{j<i} x(i, j) + \sum_{j>i} x(j, i) = 2 \quad \text{for each } 1 \leq i \leq 42 \\
& \quad 0 \leq x(i, j) \leq 1 \quad \text{for each } 1 \leq j < i \leq 42
\end{align*}
\]

Constraints \( x(i, j) \in \{0, 1\} \) are not allowed in a LP!
Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i, j)$, $i > j$, which is one if the tour includes edge $\{i, j\}$ (in either direction)

minimize $\sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i, j)x(i, j)$
subject to $\sum_{j<i} x(i, j) + \sum_{j>i} x(j, i) = 2$ for each $1 \leq i \leq 42$
$0 \leq x(i, j) \leq 1$ for each $1 \leq j < i \leq 42$

Constraints $x(i, j) \in \{0, 1\}$ are not allowed in a LP!

Branch & Bound to solve an Integer Program:

As long as solution of LP has fractional $x(i, j) \in (0, 1)$:

Add $x(i, j) = 0$ to the LP, solve it and recurse
Add $x(i, j) = 1$ to the LP, solve it and recurse
Return best of these two solutions

If solution of LP integral, return objective value

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Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i, j)$, $i > j$, which is one if the tour includes edge $\{i, j\}$ (in either direction)

$$\text{minimize} \quad \sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i, j)x(i, j)$$

subject to

$$\sum_{j<i} x(i, j) + \sum_{j>i} x(j, i) = 2 \quad \text{for each } 1 \leq i \leq 42$$

$$0 \leq x(i, j) \leq 1 \quad \text{for each } 1 \leq j < i \leq 42$$

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& \quad 0 \leq x(i, j) \leq 1 \quad \text{for each } 1 \leq j < i \leq 42
\end{align*}
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- As long as solution of LP has fractional \( x(i, j) \in (0, 1) \):
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  - Add \( x(i, j) = 1 \) to the LP, solve it and recurse
  - Return best of these two solutions
Modelling TSP as a Linear Program Relaxation

**Idea:** Indicator variable \( x(i, j), i > j \), which is one if the tour includes edge \( \{i, j\} \) (in either direction)

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i, j) x(i, j) \\
\text{subject to} & \quad \sum_{j<i} x(i, j) + \sum_{j>i} x(j, i) = 2 \quad \text{for each } 1 \leq i \leq 42 \\
& \quad 0 \leq x(i, j) \leq 1 \quad \text{for each } 1 \leq j < i \leq 42
\end{align*}
\]

Constraints \( x(i, j) \in \{0, 1\} \) are not allowed in a LP!

**Branch & Bound to solve an Integer Program:**
- As long as solution of LP has fractional \( x(i, j) \in (0, 1) \):
  - Add \( x(i, j) = 0 \) to the LP, solve it and recurse
  - Add \( x(i, j) = 1 \) to the LP, solve it and recurse
  - Return best of these two solutions
- If solution of LP integral, return objective value
Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable \( x(i, j), i > j \), which is one if the tour includes edge \( \{i, j\} \) (in either direction)

\[
\text{minimize} \quad \sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i, j) x(i, j)
\]

subject to

\[
\sum_{j<i} x(i, j) + \sum_{j>i} x(j, i) = 2 \quad \text{for each } 1 \leq i \leq 42
\]

\[
0 \leq x(i, j) \leq 1 \quad \text{for each } 1 \leq j < i \leq 42
\]

Constraints \( x(i, j) \in \{0, 1\} \) are not allowed in a LP!

Branch & Bound to solve an Integer Program:

- As long as solution of LP has fractional \( x(i, j) \in (0, 1) \):
  - Add \( x(i, j) = 0 \) to the LP, solve it and recurse
  - Add \( x(i, j) = 1 \) to the LP, solve it and recurse
  - Return best of these two solutions

- If solution of LP integral, return objective value

Bound-Step: If the best known integral solution so far is better than the solution of a LP, no need to explore branch further!
Outline

Introduction

Examples of TSP Instances

Demonstration
In the following, there are a few different runs of the demo.
Iteration 1:

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations

Disallow subtour $(1, 2, 42, 41)$ by adding this constraint to the LP:

$$x_{(2, 1)} + x_{(41, 1)} + x_{(42, 1)} + x_{(41, 2)} + x_{(42, 2)} + x_{(42, 41)} \leq 3$$

Equivalent to:

$$S = \{1, 2, 41, 42\}, \sum_{i \in S, j \in V \setminus S} x_{(\max(i, j), \min(i, j))} \geq 2$$
Iteration 1: Eliminate Subtour 1, 2, 41, 42

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations
Iteration 1: Eliminate Subtour 1, 2, 41, 42

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations

Disallow subtour (1, 2, 42, 41) by adding this constraint to the LP:

$$x(2, 1) + x(41, 1) + x(42, 1) + x(41, 2) + x(42, 2) + x(42, 41) \leq 3$$
Iteration 1: Eliminate Subtour 1, 2, 41, 42

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations

Disallow subtour $(1, 2, 42, 41)$ by adding this constraint to the LP:

$$x(2, 1) + x(41, 1) + x(42, 1) + x(41, 2) + x(42, 2) + x(42, 41) \leq 3$$

Equivalent to:

$S = \{1, 2, 41, 42\}$,

$$\sum_{i \in S, j \in V \setminus S} x(\max(i, j), \min(i, j)) \geq 2$$
Iteration 2:

Objective value: $-676.000000$, $861$ variables, $946$ constraints, $1802$ iterations
Iteration 2: Eliminate Subtour 3 – 9

Objective value: $-676.000000$, 861 variables, 946 constraints, 1802 iterations
Iteration 3:

Objective value: $-681.000000$, 861 variables, 947 constraints, 1984 iterations
**Iteration 3: Eliminate Subtour 24, 25, 26, 27**

Objective value: $-681.000000$, 861 variables, 947 constraints, 1984 iterations
Iteration 4:

Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations.
Iteration 4: Eliminate Cut 11 – 23

Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations
Iteration 4: Eliminate Cut 11 – 23

Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations

Tour has to include at least two edges between $S = \{11, 12, \ldots, 23\}$ and $V \setminus S$:

$$\sum_{i \in S, j \in V \setminus S} x(\max(i, j), \min(i, j)) \geq 2.$$
Iteration 5:

Objective value: $-686.000000$, 861 variables, 949 constraints, 2446 iterations
Iteration 5: Eliminate Subtour 13 – 23

Objective value: \(-686.000000\), 861 variables, 949 constraints, 2446 iterations
Iteration 6:

Objective value: $-694.500000$, 861 variables, 950 constraints, 1690 iterations
Iteration 6: Eliminate Cut 13 – 17

Objective value: $-694.500000$, 861 variables, 950 constraints, 1690 iterations
Iteration 7:

Objective value: $-697.000000$, 861 variables, 951 constraints, 2212 iterations
Iteration 7: Branch 1a $x_{18,15} = 0$

Objective value: $-697.000000$, 861 variables, 951 constraints, 2212 iterations
Iteration 8:

Objective value: \(-698.000000\), 861 variables, 952 constraints, 1878 iterations
Iteration 8: Branch 2a $x_{17,13} = 0$

Objective value: $-698.000000$, 861 variables, 952 constraints, 1878 iterations
Iteration 9:

Objective value: $-699.000000$, 861 variables, 953 constraints, 2281 iterations
Iteration 9: Branch 2b $x_{17,13} = 1$

Objective value: $-699.000000$, 861 variables, 953 constraints, 2281 iterations
Iteration 10:

Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations
Iteration 10:

Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations

Branch & Bound procedure would stop here, since value of the best LP solution for $x_{18,15} = 0$ is worse than a previously found tour.
Iteration 10: Branch 1b $x_{18,15} = 1$

Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations

Branch & Bound procedure would stop here, since value of the best LP solution for $x_{18,15} = 0$ is worse than a previously found tour.
Iteration 11:

Objective value: $-701.000000$, 861 variables, 953 constraints, 2506 iterations
Iteration 11: Branch & Bound terminates

Objective value: $-701.000000$, 861 variables, 953 constraints, 2506 iterations
Branch & Bound Overview

1: LP solution 641

Eliminate Subtour 1, 2, 41, 42
Eliminate Subtour 3, 9
Eliminate Subtour 24, 25, 26, 27
Eliminate Cut 11, 23
Eliminate Subtour 10, 11, 12
Eliminate Cut 13, 17, 18, 15 = 0
x 17, 13 = 0
x 17, 13 = 1
x 18, 15 = 1

Cut branch, since LP solution worse than current best possible tour.

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration
Branch & Bound Overview

1: LP solution 641

Eliminate Subtour 1, 2, 41, 42
Branch & Bound Overview

1: LP solution 641
Eliminate Subtour 1, 2, 41, 42
2: LP solution 676

3: LP solution 681
4: LP solution 682.5
5: LP solution 686
6: LP solution 694.5
7: LP solution 697
8: LP solution 698
9: Valid tour 699
10: LP solution 700
11: Valid tour 701

Eliminate Subtour 1, 2, 41, 42

Eliminate Subtour 3, 9
Eliminate Subtour 24, 25, 26
Eliminate Cut 11, 23
Eliminate Subtour 10, 11, 12
Eliminate Cut 13, 17, 18, 15
$x_{17}, x_{13} = 0$
$x_{17}, x_{13} = 1$
$x_{18}, x_{15} = 1$

Cut branch, since LP solution worse than current best possible tour.
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

Cut branch, since LP solution worse than current best possible tour.
Branch & Bound Overview

1: LP solution 641
   \[\text{Eliminate Subtour 1, 2, 41, 42}\]
2: LP solution 676
   \[\text{Eliminate Subtour 3 – 9}\]
3: LP solution 681
   \[\text{Eliminate Subtour 10, 11, 12}\]
4: LP solution 682.5
5: LP solution 686
6: LP solution 694.5
7: LP solution 697
8: LP solution 698
9: Valid tour 699

Eliminate Cut 11:
\[x_{17}, x_{18}, x_{15} = 0\]
\[x_{17}, x_{13} = 1\]
\[x_{18}, x_{15} = 1\]

Cut branch, since LP solution worse than current best possible tour.
8. Solving TSP via Linear Programming © T. Sauerwald

Branch & Bound Overview

1: LP solution 641
Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
Eliminate Subtour 3 – 9

3: LP solution 681
Eliminate Subtour 24, 25, 26, 27
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23
Branch & Bound Overview

1: LP solution 641
   - Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   - Eliminate Subtour 3 – 9

3: LP solution 681
   - Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   - Eliminate Cut 11 – 23

5: LP solution 686

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
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   Eliminate Subtour 10, 11, 12
6: LP solution 694.5

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration
8. Solving TSP via Linear Programming © T. Sauerwald

Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
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   Eliminate Subtour 24, 25, 26, 27
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   Eliminate Cut 11 – 23
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   Eliminate Cut 13 – 17
7: LP solution 697
Eliminate Subtour 1, 2, 41, 42

Eliminate Subtour 3 – 9

Eliminate Subtour 24, 25, 26, 27

Eliminate Cut 11 – 23

Eliminate Subtour 10, 11, 12

Eliminate Cut 13 – 17

\[ x_{18,15} = 0 \]
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 11 – 23
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   Eliminate Cut 13 – 17
7: LP solution 697

$x_{18,15} = 0$

8: LP solution 698
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   $x_{18,15} = 0$
   $x_{17,13} = 0$

Cut branch, since LP solution worse than current best possible tour.
Eliminate Subtour 1, 2, 41, 42
Eliminate Subtour 3 – 9
Eliminate Subtour 24, 25, 26, 27
Eliminate Cut 11 – 23
Eliminate Subtour 10, 11, 12
Eliminate Cut 13 – 17
$x_{18,15} = 0$
$x_{17,13} = 0$
9: Valid tour 699
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 11 – 23
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   Eliminate Cut 13 – 17
7: LP solution 697
   \[ x_{18,15} = 0 \]
   \[ x_{17,13} = 0 \]
8: LP solution 698
9: Valid tour 699

8. Solving TSP via Linear Programming © T. Sauerwald
Branch & Bound Overview

1: LP solution 641
   - Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   - Eliminate Subtour 3 – 9
3: LP solution 681
   - Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   - Eliminate Cut 11 – 23
5: LP solution 686
   - Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   - Eliminate Cut 13 – 17
7: LP solution 697
   - $x_{18,15} = 0$
8: LP solution 698
   - $x_{17,13} = 0$
   - $x_{17,13} = 1$
9: Valid tour 699

Cut branch, since LP solution worse than current best possible tour.
Branch & Bound Overview

1: LP solution 641
   - Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   - Eliminate Subtour 3 – 9

3: LP solution 681
   - Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   - Eliminate Cut 11 – 23

5: LP solution 686
   - Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   - Eliminate Cut 13 – 17

7: LP solution 697

8: LP solution 698
   - \( x_{18,15} = 0 \)
   - \( x_{17,13} = 0 \)
   - \( x_{17,13} = 1 \)

9: Valid tour 699

10: LP solution 700
   - Cut branch, since LP solution worse than current best possible tour.
Branch & Bound Overview

1: LP solution 641
   - Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   - Eliminate Subtour 3 – 9

3: LP solution 681
   - Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   - Eliminate Cut 11 – 23

5: LP solution 686

6: LP solution 694.5
   - Eliminate Subtour 10, 11, 12

7: LP solution 697
   - Eliminate Cut 13 – 17

8: LP solution 698
   - $x_{18,15} = 0$
   - $x_{17,13} = 0$
   - $x_{17,13} = 1$

9: Valid tour 699
10: LP solution 700

Cut branch, since LP solution worse than current best possible tour.

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration
1: LP solution 641
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   $x_{18,15} = 0$
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Cut branch, since LP solution worse than current best possible tour.
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   - Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   - Eliminate Cut 13 – 17
7: LP solution 697
   - \[ x_{18,15} = 0 \]
   - \[ x_{18,15} = 1 \]
   - \[ x_{17,13} = 0 \]
   - \[ x_{17,13} = 1 \]
8: LP solution 698
9: Valid tour 699
10: LP solution 700
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 11 – 23

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 694.5
   Eliminate Cut 13 – 17

7: LP solution 697
   \[ x_{18,15} = 0 \]
   \[ x_{17,13} = 0 \]

8: LP solution 698
   \[ x_{18,15} = 1 \]
   \[ x_{17,13} = 0 \]

9: Valid tour 699

10: LP solution 700
   \[ x_{17,13} = 1 \]

11: Valid tour 701

8. Solving TSP via Linear Programming © T. Sauerwald

Demonstration
Branch & Bound Overview

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 11 – 23
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 694.5
   Eliminate Cut 13 – 17
7: LP solution 697
   $x_{18,15} = 0$, $x_{17,13} = 0$
8: LP solution 698
   $x_{18,15} = 1$
   9: Valid tour 699
10: LP solution 700
   $x_{17,13} = 1$
11: Valid tour 701

Cut branch, since LP solution worse than current best possible tour.

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration 27
Iteration 7: Objective 697

What about choosing a different branching variable?
Iteration 7: Objective 697

What about choosing a different branching variable?

8. Solving TSP via Linear Programming © T. Sauerwald
Eliminate Subtour 1, 2, 41, 42

Eliminate Subtour 3 – 9

Eliminate Subtour 24, 25, 26, 27

Eliminate Cut 13 – 17

Eliminate Subtour 10, 11, 12

Eliminate Subtour 13 – 23

Eliminate Subtour 11 – 23

Eliminate Subtour 1, 2, 41, 42

Eliminate Subtour 3 – 9

Eliminate Subtour 24, 25, 26, 27

Eliminate Cut 13 – 17

Eliminate Subtour 10, 11, 12

Eliminate Subtour 13 – 23

Eliminate Subtour 11 – 23

8: LP solution 697
Solving Progress (Alternative Branch 1)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 13 – 17
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 686
   Eliminate Subtour 13 – 23
7: LP solution 688
   Eliminate Subtour 11 – 23
8: LP solution 697
   \[ x_{15,18} = 1 \]
   \[ x_{15,18} = 0 \]
9: ???
10: ???

8. Solving TSP via Linear Programming © T. Sauerwald
Alternative Branch 1: $x_{18,15}$, Objective 697
Alternative Branch 1: $x_{18,15}$, Objective 697
Alternative Branch 1a: $x_{18,15} = 1$, Objective 701 (Valid Tour)
Alternative Branch 1b: $x_{18,15} = 0$, Objective 698
Solving Progress (Alternative Branch 1)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42
2: LP solution 676
   Eliminate Subtour 3 – 9
3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27
4: LP solution 682.5
   Eliminate Cut 13 – 17
5: LP solution 686
   Eliminate Subtour 10, 11, 12
6: LP solution 686
   Eliminate Subtour 13 – 23
7: LP solution 688
   Eliminate Subtour 11 – 23
8: LP solution 697
   \[ x_{18,15} = 1 \]
9: valid tour 701
10: LP solution 698
   \[ x_{18,15} = 0 \]

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration 33
Solving Progress (Alternative Branch 2)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 13 – 17

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 686
   Eliminate Subtour 13 – 23

7: LP solution 688
   Eliminate Subtour 11 – 23

8: LP solution 697
Solving Progress (Alternative Branch 2)

1: LP solution 641
Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
Eliminate Subtour 3 – 9

3: LP solution 681
Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
Eliminate Cut 13 – 17

5: LP solution 686
Eliminate Subtour 10, 11, 12

6: LP solution 686
Eliminate Subtour 13 – 23

7: LP solution 688
Eliminate Subtour 11 – 23

8: LP solution 697

\[ x_{27,22} = 1 \]

\[ x_{27,22} = 0 \]

9: ???

10: ???
Alternative Branch 2: $x_{27,22}$, Objective 697
Alternative Branch 2: $x_{27,22}$, Objective 697
Alternative Branch 2a: $x_{27,22} = 1$, Objective 708 (Valid tour)
Alternative Branch 2b: $x_{27,22} = 0$, Objective $697.75$
Solving Progress (Alternative Branch 2)

1: LP solution 641
   - Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   - Eliminate Subtour 3 – 9

3: LP solution 681
   - Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   - Eliminate Cut 13 – 17

5: LP solution 686
   - Eliminate Subtour 10, 11, 12

6: LP solution 686
   - Eliminate Subtour 13 – 23

7: LP solution 688
   - Eliminate Subtour 11 – 23

8: LP solution 697
   - $x_{27,22} = 1$
   - $x_{27,22} = 0$

9: valid tour 708
10: LP solution 697.75
Solving Progress (Alternative Branch 3)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 13 – 17

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 686
   Eliminate Subtour 13 – 23

7: LP solution 688
   Eliminate Subtour 11 – 23

8: LP solution 697

8. Solving TSP via Linear Programming © T. Sauerwald
Solving Progress (Alternative Branch 3)

1: LP solution 641
   Eliminate Subtour 1, 2, 41, 42

2: LP solution 676
   Eliminate Subtour 3 – 9

3: LP solution 681
   Eliminate Subtour 24, 25, 26, 27

4: LP solution 682.5
   Eliminate Cut 13 – 17

5: LP solution 686
   Eliminate Subtour 10, 11, 12

6: LP solution 686
   Eliminate Subtour 13 – 23

7: LP solution 688
   Eliminate Subtour 11 – 23

8: LP solution 697
   \[ x_{27,24} = 1 \]
   \[ x_{27,24} = 0 \]

9: ???
10: ???
Alternative Branch 3: $x_{27,24}$, Objective 697

8. Solving TSP via Linear Programming © T. Sauerwald
Demonstration
40
Alternative Branch 3: $x_{27,24}$, Objective 697
Alternative Branch 3a: \( x_{27,24} = 1 \), Objective 697.75
Alternative Branch 3b: $x_{27,24} = 0$, Objective 698
Eliminate Subtour 1, 2, 41, 42

Eliminate Subtour 3 – 9

Eliminate Subtour 24, 25, 26, 27

Eliminate Subtour 13 – 17

Eliminate Subtour 10, 11, 12

Eliminate Subtour 13 – 23

Eliminate Subtour 11 – 23

\[ x_{27,24} = 1 \]

\[ x_{27,24} = 0 \]

9: LP solution 697.75

10: LP solution 698

Not only do we have to explore (and branch further in) both subtrees, but also the optimal tour is in the subtree with larger LP solution!
Solving Progress (Alternative Branch 3)

Not only do we have to explore (and branch further in) both subtrees, but also the optimal tour is in the subtree with larger LP solution!

8. Solving TSP via Linear Programming © T. Sauerwald Demonstration
Conclusion (1/2)

- How can one generate these constraints automatically?
Conclusion (1/2)

- How can one generate these constraints automatically?
  - Subtour Elimination: Finding Connected Components
  - Small Cuts: Finding the Minimum Cut in Weighted Graphs
Conclusion (1/2)

- How can one generate these constraints automatically?
  - Subtour Elimination: Finding Connected Components
  - Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don’t we add all possible Subtour Elimination constraints to the LP?
How can one generate these constraints automatically?

- **Subtour Elimination**: Finding Connected Components
- **Small Cuts**: Finding the Minimum Cut in Weighted Graphs

Why don’t we add all possible Subtour Elimination constraints to the LP?

There are exponentially many of them!
Conclusion (1/2)

- How can one generate these constraints automatically?
  **Subtour Elimination**: Finding Connected Components
  **Small Cuts**: Finding the Minimum Cut in Weighted Graphs

- Why don’t we add all possible Subtour Elimination constraints to the LP?
  There are exponentially many of them!

- Should the search tree be explored by BFS or DFS?
How can one generate these constraints automatically?
Subtour Elimination: Finding Connected Components
Small Cuts: Finding the Minimum Cut in Weighted Graphs

Why don’t we add all possible Subtour Elimination constraints to the LP?
There are exponentially many of them!

Should the search tree be explored by BFS or DFS?
BFS may be more attractive, even though it might need more memory.
Conclusion (1/2)

- How can one generate these constraints automatically?
  - Subtour Elimination: Finding Connected Components
  - Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don’t we add all possible Subtour Elimination constraints to the LP?
  There are exponentially many of them!

- Should the search tree be explored by BFS or DFS?
  BFS may be more attractive, even though it might need more memory.

CONCLUDING REMARK

It is clear that we have left unanswered practically any question one might pose of a theoretical nature concerning the traveling-salesman problem; however, we hope that the feasibility of attacking problems involving a moderate number of points has been successfully demonstrated, and that perhaps some of the ideas can be used in problems of similar nature.
Conclusion (2/2)

- Eliminate Subtour 1, 2, 41, 42
- Eliminate Subtour 3 – 9
- Eliminate Subtour 10, 11, 12
- Eliminate Subtour 11 – 23
- Eliminate Subtour 13 – 23
- Eliminate Cut 13 – 17
- Eliminate Subtour 24, 25, 26, 27

We are indebted to I. Glicksberg of Rand for pointing out relations of this kind to us.
Conclusion (2/2)

- Eliminate Subtour 1, 2, 41, 42
- Eliminate Subtour 3 – 9
- **Eliminate Subtour 10, 11, 12**
- Eliminate Subtour 11 – 23
- Eliminate Subtour 13 – 23
- Eliminate Cut 13 – 17
- Eliminate Subtour 24, 25, 26, 27

**THE 49-CITY PROBLEM***

The optimal tour $\tilde{x}$ is shown in Fig. 16. The proof that it is optimal is given in Fig. 17. To make the correspondence between the latter and its programming problem clear, we will write down in addition to 42 relations in non-negative variables (2), a set of 25 relations which suffice to prove that $D(x)$ is a minimum for $\tilde{x}$. We distinguish the following subsets of the 42 cities:

$S_1 = \{1, 2, 41, 42\}$  
$S_2 = \{3, 4, \ldots, 9\}$  
$S_3 = \{1, 2, \ldots, 9, 29, 30, \ldots, 42\}$  
$S_4 = \{11, 12, \ldots, 23\}$  
$S_5 = \{13, 14, \ldots, 23\}$  
$S_6 = \{13, 14, 15, 16, 17\}$  
$S_7 = \{24, 25, 26, 27\}$.  

---
*8. Solving TSP via Linear Programming © T. Sauerwald*
IBM ILOG CPLEX Optimization Studio (often informally referred to simply as CPLEX) is an optimization software package. In 2004, the work on CPLEX earned the first INFORMS Impact Prize.

The CPLEX Optimizer was named for the simplex method as implemented in the C programming language, although today it also supports other types of mathematical optimization and offers interfaces other than just C. It was originally developed by Robert E. Bixby and was offered commercially starting in 1988 by CPLEX Optimization Inc., which was acquired by ILOG in 1997; ILOG was subsequently acquired by IBM in January 2009. CPLEX continues to be actively developed under IBM.

The IBM ILOG CPLEX Optimizer solves integer programming problems, very large linear programming problems using either primal or dual variants of the simplex method or the barrier interior
Welcome to IBM(R) ILOG(R) CPLEX(R) Interactive Optimizer 12.6.1.0
with Simplex, Mixed Integer & Barrier Optimizers
5725-A06 5725-A29 5724-Y48 5724-Y49 5724-Y54 5724-Y55 5655-Y21
Copyright IBM Corp. 1988, 2014. All Rights Reserved.

Type 'help' for a list of available commands.
Type 'help' followed by a command name for more information on commands.

CPLEX> read tsp.lp
Problem 'tsp.lp' read.
Read time = 0.00 sec. (0.06 ticks)
CPLEX> primopt
Tried aggregator 1 time.
LP Presolve eliminated 1 rows and 1 columns.
Reduced LP has 49 rows, 860 columns, and 2483 nonzeros.
Presolve time = 0.00 sec. (0.36 ticks)

Iteration log...
Iteration:   1  Infeasibility = 33.999999
Iteration:  26  Objective    = 1510.000000
Iteration:  90  Objective    = 923.000000
Iteration: 155  Objective    = 711.000000

Primal simplex - Optimal:  Objective = 6.9900000000e+02
Solution time = 0.00 sec.  Iterations = 168 (25)
Deterministic time = 1.16 ticks  (288.86 ticks/sec)

CPLEX>
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<th>Solution Value</th>
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</thead>
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<tr>
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</tbody>
</table>

All other variables in the range 1-861 are 0.
Randomised Algorithms
Lecture 9: Approximation Algorithms: MAX-3-CNF and Vertex-Cover

Thomas Sauerwald (tms41@cam.ac.uk)
Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover
A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size $n$, the expected cost (value) $E[C]$ of the returned solution and optimal cost $C^*$ satisfy:

$$\max \left( \frac{E[C]}{C^*}, \frac{C^*}{E[C]} \right) \leq \rho(n).$$
Approximation Ratio for Randomised Approximation Algorithms

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size $n$, the expected cost (value) $E[C]$ of the returned solution and optimal cost $C^*$ satisfy:

$$\max \left( \frac{E[C]}{C^*}, \frac{C^*}{E[C]} \right) \leq \rho(n).$$

- **Maximisation problem:** $\frac{C^*}{E[C]} \geq 1$
- **Minimisation problem:** $\frac{E[C]}{C^*} \geq 1$
Approximation Ratio for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size $n$, the expected cost (value) $E[C]$ of the returned solution and optimal cost $C^*$ satisfy:

$$\max \left( \frac{E[C]}{C^*}, \frac{C^*}{E[C]} \right) \leq \rho(n).$$

Randomised Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1 + \epsilon)$-approximation algorithm.
A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size $n$, the expected cost (value) $\mathbb{E}[C]$ of the returned solution and optimal cost $C^*$ satisfy:

$$\max \left( \frac{\mathbb{E}[C]}{C^*}, \frac{C^*}{\mathbb{E}[C]} \right) \leq \rho(n).$$

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1 + \epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in $n$. For example, $O(n^2/\epsilon)$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and $n$. For example, $O((1/\epsilon)^2 \cdot n^3)$. 

Randomised Approximation Schemes

not covered here (non-examinable)
Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover
MAX-3-CNF Satisfiability

- **Given**: 3-CNF formula, e.g.: $(x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots$

Idea: What about assigning each variable uniformly and independently at random?
MAX-3-CNF Satisfiability

- **Given**: 3-CNF formula, e.g.: \((x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots\)
- **Goal**: Find an assignment of the variables that satisfies as many clauses as possible.
MAX-3-CNF Satisfiability

- **Given:** 3-CNF formula, e.g.: \((x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots\)
- **Goal:** Find an assignment of the variables that satisfies as many clauses as possible.

**Relaxation of the satisfiability problem.** Want to compute how “close” the formula to being satisfiable is.
MAX-3-CNF Satisfiability

- **Given**: 3-CNF formula, e.g.: \((x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor x_5) \land \cdots\)
- **Goal**: Find an assignment of the variables that satisfies as many clauses as possible.

Assume that no literal (including its negation) appears more than once in the same clause.

Relaxation of the satisfiability problem. Want to compute how “close” the formula to being satisfiable is.

Idea: What about assigning each variable uniformly and independently at random?
MAX-3-CNF Satisfiability

- **Given:** 3-CNF formula, e.g.: $(x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots$
- **Goal:** Find an assignment of the variables that satisfies as many clauses as possible.

Relaxation of the satisfiability problem. Want to compute how “close” the formula to being satisfiable is.

Example:

$(x_1 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_3} \lor x_5) \land (x_2 \lor \overline{x_4} \lor x_5) \land (\overline{x_1} \lor x_2 \lor \overline{x_3})$
MAX-3-CNF Satisfiability

- **Given:** 3-CNF formula, e.g.: \((x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots\)
- **Goal:** Find an assignment of the variables that satisfies as many clauses as possible.

Relaxation of the satisfiability problem. Want to compute how “close” the formula to being satisfiable is.

Example:

\[(x_1 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_3} \lor x_5) \land (x_2 \lor x_4 \lor x_5) \land (\overline{x_1} \lor x_2 \lor \overline{x_3})\]

\[x_1 = 1, \ x_2 = 0, \ x_3 = 1, \ x_4 = 0 \text{ and } x_5 = 1 \text{ satisfies 3 (out of 4 clauses)}\]
MAX-3-CNF Satisfiability

- **Given**: 3-CNF formula, e.g.: \((x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots\)

- **Goal**: Find an assignment of the variables that satisfies as many clauses as possible.

Relaxation of the satisfiability problem. Want to compute how “close” the formula to being satisfiable is.

Example:

\[(x_1 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_3} \lor x_5) \land (x_2 \lor x_4 \lor x_5) \land (\overline{x_1} \lor x_2 \lor \overline{x_3})\]

\(x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0\) and \(x_5 = 1\) satisfies 3 (out of 4 clauses)

Idea: What about assigning each variable uniformly and independently at random?
Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised \( \frac{8}{7} \)-approximation algorithm.
Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

Proof:
Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised \( 8/7 \)-approximation algorithm.

Proof:
- For every clause \( i = 1, 2, \ldots, m \), define a random variable:
  \[ Y_i = 1 \{ \text{clause } i \text{ is satisfied} \} \]
Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

Proof:
- For every clause $i = 1, 2, \ldots, m$, define a random variable:
  \[ Y_i = 1 \{ \text{clause } i \text{ is satisfied} \} \]
- Since each literal (including its negation) appears at most once in clause $i$, \[ E[Y_i] = P[Y_i = 1] \cdot 1 = \frac{7}{8} \cdot m \]
Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised \( 8/7 \)-approximation algorithm.

Proof:

- For every clause \( i = 1, 2, \ldots, m \), define a random variable:
  \[
  Y_i = \mathbf{1}\{\text{clause } i \text{ is satisfied}\}
  \]
- Since each literal (including its negation) appears at most once in clause \( i \),
  \[
  \mathbb{P}[\text{clause } i \text{ is not satisfied}] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}
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Analysis

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  \]
  \[
  \Rightarrow P[ \text{clause } i \text{ is satisfied}] = 1 - \frac{1}{8} = \frac{7}{8}
  \]
Analysis

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  \[
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  \]
  \[
  \Rightarrow \quad P [ \text{clause } i \text{ is satisfied} ] = 1 - \frac{1}{8} = \frac{7}{8}
  \]
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  \Rightarrow \quad E [ Y_i ] = P [ Y_i = 1 ] \cdot 1 = \frac{7}{8}.
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  \]
  \[
  \Rightarrow E[ Y_i] = P[ Y_i = 1] \cdot 1 = \frac{7}{8}.
  \]

- Let \( Y := \sum_{i=1}^{m} Y_i \) be the number of satisfied clauses. Then,
Theorem 35.6

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised 8/7-approximation algorithm.

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  \Rightarrow \quad \Pr[\text{clause } i \text{ is satisfied}] = 1 - \frac{1}{8} = \frac{7}{8}
  \]

  \[
  \Rightarrow \quad \mathbb{E}[Y_i] = \Pr[Y_i = 1] \cdot 1 = \frac{7}{8}.
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- Let \( Y := \sum_{i=1}^{m} Y_i \) be the number of satisfied clauses. Then,
  \[
  \mathbb{E}[Y]
  \]
Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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  \[
  \Rightarrow \quad \mathbb{P}[\text{clause } i \text{ is satisfied}] = 1 - \frac{1}{8} = \frac{7}{8}
  \]
  \[
  \Rightarrow \quad \mathbb{E}[Y_i] = \mathbb{P}[Y_i = 1] \cdot 1 = \frac{7}{8}.
  \]

- Let $Y := \sum_{i=1}^{m} Y_i$ be the number of satisfied clauses. Then,
  
  \[
  \mathbb{E}[Y] = \mathbb{E}\left[ \sum_{i=1}^{m} Y_i \right]
  \]
Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

**Theorem 35.6**

**Proof:**

- For every clause $i = 1, 2, \ldots, m$, define a random variable:
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  \]
  \[
  \Rightarrow \quad P [ \text{clause } i \text{ is satisfied}] = 1 - \frac{1}{8} = \frac{7}{8}
  \]
  \[
  \Rightarrow \quad E [ Y_i] = P [ Y_i = 1] \cdot 1 = \frac{7}{8}.
  \]

- Let $Y := \sum_{i=1}^{m} Y_i$ be the number of satisfied clauses. Then,
  \[
  E [ Y] = E \left[ \sum_{i=1}^{m} Y_i \right]
  \]

**Linearity of Expectations**
Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with \(n\) variables \(x_1, x_2, \ldots, x_n\) and \(m\) clauses, the randomised algorithm that sets each variable independently at random is a randomised 8/7-approximation algorithm.

Proof:

- For every clause \(i = 1, 2, \ldots, m\), define a random variable:
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  \]

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  \]
  \[
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  \]
  \[
  \Rightarrow \mathbf{E}[Y_i] = P[Y_i = 1] \cdot 1 = \frac{7}{8}.
  \]

- Let \(Y := \sum_{i=1}^{m} Y_i\) be the number of satisfied clauses. Then,
  \[
  \mathbf{E}[Y] = \mathbf{E}\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} \mathbf{E}[Y_i]
  \]
  (Linearity of Expectations)
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Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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  \[ \Rightarrow \quad \Pr[\text{clause } i \text{ is satisfied}] = 1 - \frac{1}{8} = \frac{7}{8} \]
  \[ \Rightarrow \quad \mathbb{E}[Y_i] = \Pr[Y_i = 1] \cdot 1 = \frac{7}{8}. \]

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Analysis

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised \( 8/7 \)-approximation algorithm.

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  \]
- Let \( Y := \sum_{i=1}^{m} Y_i \) be the number of satisfied clauses. Then,
  \[
  E[ Y ] = E \left[ \sum_{i=1}^{m} Y_i \right] = \sum_{i=1}^{m} E[ Y_i ] = \sum_{i=1}^{m} \frac{7}{8} = \frac{7}{8} \cdot m.
  \]

\**Linearity of Expectations**
Analysis

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Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised \( \frac{8}{7} \)-approximation algorithm.

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  \]

Linearity of Expectations

maximum number of satisfiable clauses is \( m \)
Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised \( 8/7 \)-approximation algorithm.

**Proof:**

- For every clause \( i = 1, 2, \ldots, m \), define a random variable:
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- Let \( Y := \sum_{i=1}^{m} Y_i \) be the number of satisfied clauses. Then,
  \[
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  \]
  \( \square \)

**Linearity of Expectations**

maximum number of satisfiable clauses is \( m \)
Interesting Implications

Theorem 35.6

Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

Theorem 35.6

For any instance of MAX-3-CNF, there exists an assignment which satisfies at least $\frac{7}{8}$ of all clauses.

Corollary

Any instance of MAX-3-CNF with at most 7 clauses is satisfiable.

Corollary

There is $\omega \in \Omega$ such that $Y(\omega) \geq E[Y]$. Probabilistic Method: powerful tool to show existence of a non-obvious property. Follows from the previous Corollary.
Interesting Implications

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For any instance of MAX-3-CNF, there exists an assignment which satisfies at least $\frac{7}{8}$ of all clauses.

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For any instance of MAX-3-CNF, there exists an assignment which satisfies at least $\frac{7}{8}$ of all clauses.

There is $\omega \in \Omega$ such that $Y(\omega) \geq \mathbb{E}[Y]$.

**Probabilistic Method:** powerful tool to show existence of a non-obvious property.
Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

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Any instance of MAX-3-CNF with at most 7 clauses is satisfiable.
Interesting Implications

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Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised \( \frac{8}{7} \)-approximation algorithm.

Corollary

For any instance of MAX-3-CNF, there exists an assignment which satisfies at least \( \frac{7}{8} \) of all clauses.

There is \( \omega \in \Omega \) such that \( Y(\omega) \geq E[Y] \)

Probabilistic Method: powerful tool to show existence of a non-obvious property.

Corollary

Any instance of MAX-3-CNF with at most 7 clauses is satisfiable.

Follows from the previous Corollary.
Expected Approximation Ratio

Theorem 35.6

Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.
Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised \( \frac{8}{7} \)-approximation algorithm.

One could prove that the probability to satisfy \( \left( \frac{7}{8} \right) \cdot m \) clauses is at least \( \frac{1}{8m} \).
Theorem 35.6

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised \( 8/7 \)-approximation algorithm.

One could prove that the probability to satisfy \((7/8) \cdot m\) clauses is at least \(1/(8m)\).

\[
E[Y] = \frac{1}{2} \cdot E[Y \mid x_1 = 1] + \frac{1}{2} \cdot E[Y \mid x_1 = 0].
\]

\( Y \) is defined as in the previous proof.
Expected Approximation Ratio

Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

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$$E[Y] = \frac{1}{2} \cdot E[Y \mid x_1 = 1] + \frac{1}{2} \cdot E[Y \mid x_1 = 0].$$

$Y$ is defined as in the previous proof.

One of the two conditional expectations is at least $E[Y]$.
Expected Approximation Ratio

**Theorem 35.6**

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised \( \frac{8}{7} \)-approximation algorithm.

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\[
E[Y] = \frac{1}{2} \cdot E[Y | x_1 = 1] + \frac{1}{2} \cdot E[Y | x_1 = 0].
\]

\( Y \) is defined as in the previous proof.

One of the two conditional expectations is at least \( E[Y] \).

**Algorithm:** Assign \( x_1 \) so that the conditional expectation is maximised and recurse.
Expected Approximation Ratio

Theorem 35.6
Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised \( 8/7 \)-approximation algorithm.

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\[
E[Y] = \frac{1}{2} \cdot E[Y | x_1 = 1] + \frac{1}{2} \cdot E[Y | x_1 = 0].
\]

Y is defined as in the previous proof.

One of the two conditional expectations is at least \( E[Y] \)

Algorithm:

Greedy-3-CNF(\( \phi, n, m \))
1: \textbf{for} \( j = 1, 2, \ldots, n \)
2: \hspace{1em} Compute \( E[Y | x_1 = v_1 \ldots, x_{j-1} = v_{j-1}, x_j = 1] \)
3: \hspace{1em} Compute \( E[Y | x_1 = v_1 \ldots, x_{j-1} = v_{j-1}, x_j = 0] \)
4: \hspace{1em} Let \( x_j = v_j \) so that the conditional expectation is maximised
5: \textbf{return} the assignment \( v_1, v_2, \ldots, v_n \)
**Expected Approximation Ratio**

**Theorem 35.6**

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised \( 8/7 \)-approximation algorithm.

One could prove that the probability to satisfy \((7/8) \cdot m\) clauses is at least \(1/(8m)\).

\[
E[Y] = \frac{1}{2} \cdot E[Y \mid x_1 = 1] + \frac{1}{2} \cdot E[Y \mid x_1 = 0].
\]

\(Y\) is defined as in the previous proof.

One of the two conditional expectations is at least \(E[Y]\).

**Algorithm:**

1. **for** \( j = 1, 2, \ldots, n \)
2. Compute \( E[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1] \)
3. Compute \( E[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 0] \)
4. Let \( x_j = v_j \) so that the conditional expectation is maximised
5. **return** the assignment \( v_1, v_2, \ldots, v_n \)

Skip Analysis


MAX-3-CNF
Run of **GREEDY-3-CNF** $(\varphi, n, m)$

$$(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor \overline{x_3}) \land (x_1 \lor x_2 \lor x_4) \land (\overline{x_1} \lor x_2 \lor \overline{x_3}) \land (x_1 \lor \overline{x_2} \lor x_4) \land (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_4})$$
Run of GREEDY-3-CNF($\varphi, n, m$)

\[(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor \overline{x_3}) \land (x_1 \lor x_2 \lor \overline{x_3}) \land (\overline{x_1} \lor x_2 \lor x_4) \land (x_1 \lor x_2 \lor x_4) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \land (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor x_4) \land (x_2 \lor \overline{x_3} \lor \overline{x_4})\]
Run of \textsc{Greedy-3-CNF}(\(\varphi, n, m\))

\((x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor \overline{x_4}) \land (x_1 \lor x_2 \lor \overline{x_4}) \land (x_1 \lor \overline{x_3} \lor x_4) \land (x_1 \lor x_2 \lor \overline{x_3}) \land (\overline{x_1} \lor x_2 \lor \overline{x_3}) \land (\overline{x_1} \lor \overline{x_2} \lor x_3) \land (x_1 \lor x_3 \lor x_4) \land (x_2 \lor \overline{x_3} \lor \overline{x_4})\)
Run of GREEDY-3-CNF($\varphi, n, m$)

\[(x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \lor \overline{x_3}) \land (x_1 \lor x_2 \lor x_4) \land (\overline{x_1} \lor \overline{x_3} \lor x_4) \land (x_1 \lor x_2 \lor \overline{x_4}) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \land (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor x_4) \land (x_2 \lor x_3 \lor \overline{x_4})\]
Run of \textbf{GREEDY-3-CNF}(\(\varphi, n, m\))

\[
(\overline{x_1} \lor \overline{x_2} \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_4) \land (x_1 \lor x_2 \lor \overline{x_4}) \land (\overline{x_1} \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor \overline{x_3}) \land (x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor x_4) \land (x_2 \lor x_3 \lor \overline{x_4})
\]
Run of GREEDY-3-CNF(ϕ, n, m)

\[ 1 \land 1 \land 1 \land (\overline{x_3} \lor x_4) \land 1 \land (\overline{x_2} \lor \overline{x_3}) \land (x_2 \lor x_3) \land (\overline{x_2} \lor x_3) \land 1 \land (x_2 \lor \overline{x_3} \lor \overline{x_4}) \]

Go to Analysis
Run of GREEDY-3-CNF($\varphi, n, m$)

\[ 1 \land 1 \land 1 \land (\overline{x_3} \lor x_4) \land 1 \land (\overline{x_2} \lor \overline{x_3}) \land (x_2 \lor x_3) \land (\overline{x_2} \lor x_3) \land 1 \land (x_2 \lor \overline{x_3} \lor \overline{x_4}) \]

\[ x_1 = 0 \quad \text{8.625} \]
\[ x_1 = 1 \quad \text{8.75} \]

\[ x_2 = 0 \quad \text{8.75} \]
\[ x_2 = 1 \quad \text{8.875} \]

\[ x_3 = 0 \quad \text{9} \]
\[ x_3 = 1 \quad \text{8.75} \]
Run of **GREEDY-3-CNF**($\varphi, n, m$)

$$1 \land 1 \land 1 \land (\bar{x}_3 \lor x_4) \land 1 \land (\bar{x}_2 \lor \bar{x}_3) \land (x_2 \lor x_3) \land (\bar{x}_2 \lor x_3) \land 1 \land (x_2 \lor \bar{x}_3 \lor \bar{x}_4)$$

![Decision tree for GREEDY-3-CNF](image)
Run of GREEDY-3-CNF($\varphi, n, m$)

$$1 \land 1 \land 1 \land (\overline{x_3} \lor x_4) \land 1 \land (\overline{x_2} \lor \overline{x_3}) \land (x_2 \lor x_3) \land (\overline{x_2} \lor x_3) \land 1 \land (x_2 \lor x_3 \lor \overline{x_4})$$

Go to Analysis

Run of GREEDY-3-CNF($\varphi, n, m$)

$1 \land 1 \land 1 \land (\bar{x}_3 \lor x_4) \land 1 \land 1 \land (x_3) \land 1 \land 1 \land (\bar{x}_3 \lor \bar{x}_4)$
Run of **GREEDY-3-CNF** $(\varphi, n, m)$

\[ 1 \land 1 \land 1 \land (\overline{x}_3 \lor x_4) \land 1 \land 1 \land (x_3) \land 1 \land 1 \land (\overline{x}_3 \lor \overline{x}_4) \]

\[ x_1 \]

\[ 8.75 \]

\[ x_2 \]

\[ 8.625 \]

\[ 8.75 \]

\[ x_3 \]

\[ 9 \]

\[ 8.875 \]

\[ 9 \]

\[ 8.75 \]

\[ 8.75 \]

\[ 9 \]

\[ 9 \]

\[ 8.75 \]

\[ 9 \]

\[ 9 \]

\[ 8.75 \]
Run of GREEDY-3-CNF($\varphi, n, m$)

\[ 1 \land 1 \land 1 \land (\overline{x_3} \lor x_4) \land 1 \land 1 \land (x_3) \land 1 \land 1 \land (\overline{x_3} \lor \overline{x_4}) \]

\[ x_1 = 0 \]
\[ x_1 = 1 \]
\[ x_2 = 0 \]
\[ x_2 = 1 \]
\[ x_3 = 0 \]
\[ x_3 = 1 \]
\[ x_4 = 0 \]
\[ x_4 = 1 \]

Go to Analysis
Run of **GREEDY-3-CNF**($\varphi, n, m$)

\[
1 \land 1 \land 1 \land (\overline{x_3} \lor x_4) \land 1 \land 1 \land (x_3) \land 1 \land 1 \land (\overline{x_3} \lor \overline{x_4})
\]
Run of **GREEDY-3-CNF** ($\varphi, n, m$)

$$1 \land 1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1$$

 Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.
Run of **GREEDY-3-CNF**($\varphi, n, m$)

$$1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1$$
Run of GREEDY-3-CNF($\varphi, n, m$)

$1 \land 1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1$

$\varphi, n, m$

$1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1$

$\varphi, n, m$

$1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1$

$\varphi, n, m$
Run of GREEDY-3-CNF($\varphi, n, m$)

$1 \land 1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1$

$\begin{align*}
\text{x}_1 &= 0 \\
\text{x}_2 &= 0 \\
\text{x}_3 &= 0
\end{align*}$

$\begin{align*}
\text{x}_1 &= 1 \\
\text{x}_2 &= 0 \\
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\text{x}_1 &= 1 \\
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$\begin{align*}
\text{x}_1 &= 1 \\
\text{x}_2 &= 1 \\
\text{x}_3 &= 1
\end{align*}$
Run of GREEDY-3-CNF(\(\varphi, n, m\))

\[1 \land 1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1\]
Run of **GREEDY-3-CNF** $(\varphi, n, m)$

\[ 1 \land 1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1 \land 1 \]

\[ x_1 = 0 \quad 8.75 \quad x_1 = 1 \]

\[ x_2 = 0 \quad 8.625 \quad x_2 = 1 \]

\[ x_3 = 0 \quad 8.25 \quad x_3 = 1 \]

\[ x_4 = 0 \quad 8 \quad x_4 = 1 \]

\[ \text{Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.} \]
Run of GREEDY-3-CNF($\phi, n, m$)

\[1 \land 1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1\]

Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.
Run of **GREEDY-3-CNF**\( (\varphi, n, m) \)

\[
1 \land 1 \land 1 \land 1 \land 1 \land 1 \land 0 \land 1 \land 1 \land 1
\]
Analysis of **GREEDY-3-CNF** (\(\phi, n, m\))

**Theorem**

**GREEDY-3-CNF** (\(\phi, n, m\)) is a polynomial-time 8/7-approximation.
Analysis of $\text{GREEDY-3-CNF}(\phi, n, m)$

Theorem

GREEDY-3-CNF($\phi, n, m$) is a polynomial-time $8/7$-approximation.

This algorithm is deterministic.
Analysis of **GREEDY-3-CNF**$(\phi, n, m)$

**Theorem**

**GREEDY-3-CNF**$(\phi, n, m)$ is a polynomial-time $8/7$-approximation.

**Proof:**

This algorithm is deterministic.
Analysis of \textbf{GREEDY-3-CNF}(\(\phi, n, m\))

\begin{itemize}
  \item \textbf{Step 1:} polynomial-time algorithm
\end{itemize}
Analysis of \texttt{GREEDY-3-CNF}(\phi, n, m)

\textbf{Theorem}

\texttt{GREEDY-3-CNF}(\phi, n, m) is a polynomial-time 8/7-approximation.

\textbf{Proof:}

- **Step 1:** polynomial-time algorithm
  - In iteration \( j = 1, 2, \ldots, n \), \( Y = Y(\phi) \) averages over \( 2^{n-j+1} \) assignments

\textbf{This algorithm is deterministic.}
Analysis of **GREEDY-3-CNF**($\phi, n, m$)

**Theorem**

**GREEDY-3-CNF**($\phi, n, m$) is a polynomial-time $8/7$-approximation.

**Proof:**

- **Step 1:** polynomial-time algorithm
  - In iteration $j = 1, 2, \ldots, n$, $Y = Y(\phi)$ averages over $2^{n-j+1}$ assignments
  - A smarter way is to use linearity of (conditional) expectations:

\[
E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right]
\]

This algorithm is deterministic.
**Theorem**

**GREEDY-3-CNF**\((\phi, n, m)\) is a polynomial-time \(8/7\)-approximation.

**Proof:**

- **Step 1:** polynomial-time algorithm
  - In iteration \(j = 1, 2, \ldots, n\), \(Y = Y(\phi)\) averages over \(2^{n-j+1}\) assignments
  - A smarter way is to use linearity of (conditional) expectations:

  \[
  E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right] = \sum_{i=1}^{m} E \left[ Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right]
  \]
Analysis of GREEDY-3-CNF($\phi, n, m$)

**Theorem**

GREEDY-3-CNF($\phi, n, m$) is a polynomial-time 8/7-approximation.

**Proof:**

- **Step 1:** polynomial-time algorithm
  - In iteration $j = 1, 2, \ldots, n$, $Y = Y(\phi)$ averages over $2^{n-j+1}$ assignments
  - A smarter way is to use linearity of (conditional) expectations:
    \[
    E \left[ \frac{Y}{x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1} \right] = \sum_{i=1}^{m} E \left[ \frac{Y_i}{x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1} \right]
    \]
    computable in $O(1)$
Analysis of \textbf{GREEDY-3-CNF}(\phi, n, m)

This algorithm is deterministic.

\textbf{Theorem}

\textbf{GREEDY-3-CNF}(\phi, n, m) is a polynomial-time 8/7-approximation.

\textbf{Proof:}

\begin{itemize}
    \item \textbf{Step 1:} polynomial-time algorithm \checkmark
        \begin{itemize}
            \item In iteration \(j = 1, 2, \ldots, n\), \(Y = Y(\phi)\) averages over \(2^{n-j+1}\) assignments
            \item A smarter way is to use linearity of (conditional) expectations:
        \end{itemize}
    \end{itemize}

\[
\mathbb{E}\left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right] = \sum_{i=1}^{m} \mathbb{E}\left[ Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right]
\]

\text{computable in } O(1)
Analysis of \textsc{Greedy-3-Cnf}(\(\phi, n, m\))

Theorem

\textsc{Greedy-3-Cnf}(\(\phi, n, m\)) is a polynomial-time 8/7-approximation.

Proof:

- **Step 1:** polynomial-time algorithm ✓
  - In iteration \(j = 1, 2, \ldots, n\), \(Y = Y(\phi)\) averages over \(2^{n-j+1}\) assignments
  - A smarter way is to use linearity of (conditional) expectations:

\[
E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right] = \sum_{i=1}^{m} E \left[ Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right]
\]

- **Step 2:** satisfies at least \(7/8 \cdot m\) clauses

This algorithm is deterministic.
Analysis of \textsc{Greedy-3-Cnf}(\phi, n, m)

\textbf{Theorem}

\textsc{Greedy-3-Cnf}(\phi, n, m) is a polynomial-time 8/7-approximation.

\textbf{Proof:}

- **Step 1**: polynomial-time algorithm \( \checkmark \)
  - In iteration \( j = 1, 2, \ldots, n \), \( Y = Y(\phi) \) averages over \( 2^{n-j+1} \) assignments
  - A smarter way is to use linearity of (conditional) expectations:
    \[
    E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right] = \sum_{i=1}^{m} E \left[ Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right]
    \]

- **Step 2**: satisfies at least \( 7/8 \cdot m \) clauses
  - Due to the greedy choice in each iteration \( j = 1, 2, \ldots, n \),
Analysis of \( \text{GREEDY-3-CNF}(\phi, n, m) \)

This algorithm is deterministic.

\( \text{GREEDY-3-CNF}(\phi, n, m) \) is a polynomial-time 8/7-approximation.

Proof:
- **Step 1:** polynomial-time algorithm
  - In iteration \( j = 1, 2, \ldots, n \), \( Y = Y(\phi) \) averages over \( 2^{n-j+1} \) assignments
  - A smarter way is to use linearity of (conditional) expectations:
    \[
    E \left[ Y \middle| x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right] = \sum_{i=1}^{m} E \left[ Y_i \middle| x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right]
    \]

- **Step 2:** satisfies at least \( 7/8 \cdot m \) clauses
  - Due to the greedy choice in each iteration \( j = 1, 2, \ldots, n \),
    \[
    E \left[ Y \middle| x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v_j \right] \geq E \left[ Y \middle| x_1 = v_1, \ldots, x_{j-1} = v_{j-1} \right]
    \]
Analysis of \textsc{Greedy-3-CNF}(\(\phi, n, m\))

This algorithm is deterministic.

\textsc{Greedy-3-CNF}(\(\phi, n, m\)) is a polynomial-time 8/7-approximation.

**Theorem**

This algorithm is deterministic.

\textsc{Greedy-3-CNF}(\(\phi, n, m\)) is a polynomial-time 8/7-approximation.

**Proof:**

- **Step 1:** polynomial-time algorithm √
  - In iteration \(j = 1, 2, \ldots, n\), \(Y = Y(\phi)\) averages over \(2^{n-j+1}\) assignments.
  - A smarter way is to use linearity of (conditional) expectations:

  \[
  \mathbb{E}[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1] = \sum_{i=1}^{m} \mathbb{E}[Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1]
  \]

- **Step 2:** satisfies at least \(7/8 \cdot m\) clauses
  - Due to the greedy choice in each iteration \(j = 1, 2, \ldots, n\),

  \[
  \mathbb{E}[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v_j] \geq \mathbb{E}[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}]
  \]
  \[
  \geq \mathbb{E}[Y \mid x_1 = v_1, \ldots, x_{j-2} = v_{j-2}]
  \]
Analysis of GREEDY-3-CNF($\phi, n, m$)

**Theorem**

GREEDY-3-CNF($\phi, n, m$) is a polynomial-time $8/7$-approximation.

**Proof:**

- **Step 1:** polynomial-time algorithm ✓
  - In iteration $j = 1, 2, \ldots, n$, $Y = Y(\phi)$ averages over $2^{n-j+1}$ assignments
  - A smarter way is to use linearity of (conditional) expectations:

  \[
  \mathbb{E}[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1] = \sum_{i=1}^{m} \mathbb{E}[Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1]
  \]

- **Step 2:** satisfies at least $7/8 \cdot m$ clauses
  - Due to the greedy choice in each iteration $j = 1, 2, \ldots, n$,
    \[
    \mathbb{E}[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v_j] \geq \mathbb{E}[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}]
    \]
    \[
    \geq \mathbb{E}[Y \mid x_1 = v_1, \ldots, x_{j-2} = v_{j-2}]
    \]
    \[
    \vdots
    \]
    \[
    \geq \mathbb{E}[Y]
    \]
Analysis of **GREEDY-3-CNF**($\phi, n, m$)

**Theorem**

GREEDY-3-CNF($\phi, n, m$) is a polynomial-time $\frac{8}{7}$-approximation.

**Proof:**

- **Step 1:** polynomial-time algorithm ✓
  - In iteration $j = 1, 2, \ldots, n$, $Y = Y(\phi)$ averages over $2^{n-j+1}$ assignments
  - A smarter way is to use linearity of (conditional) expectations:
    \[
    \mathbb{E}[Y | x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1] = \sum_{i=1}^{m} \mathbb{E}[Y_i | x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1]
    \]

- **Step 2:** satisfies at least $\frac{7}{8} \cdot m$ clauses
  - Due to the greedy choice in each iteration $j = 1, 2, \ldots, n$,
    \[
    \mathbb{E}[Y | x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v_j] \geq \mathbb{E}[Y | x_1 = v_1, \ldots, x_{j-1} = v_{j-1}]
    \geq \mathbb{E}[Y | x_1 = v_1, \ldots, x_{j-2} = v_{j-2}]
    \geq \cdots
    \geq \mathbb{E}[Y] = \frac{7}{8} \cdot m.
    \]
Analysis of GREEDY-3-CNF($\phi, n, m$)

This algorithm is deterministic.

**Theorem**

GREEDY-3-CNF($\phi, n, m$) is a polynomial-time $8/7$-approximation.

**Proof:**

- **Step 1:** polynomial-time algorithm ✓
  - In iteration $j = 1, 2, \ldots, n$, $Y = Y(\phi)$ averages over $2^{n-j+1}$ assignments
  - A smarter way is to use linearity of (conditional) expectations:

  \[ E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right] = \sum_{i=1}^{m} E \left[ Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right] \]

- **Step 2:** satisfies at least $7/8 \cdot m$ clauses ✓
  - Due to the greedy choice in each iteration $j = 1, 2, \ldots, n$,

  \[ E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v_j \right] \geq E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1} \right] \]

  \[ \geq E \left[ Y \mid x_1 = v_1, \ldots, x_{j-2} = v_{j-2} \right] \]

  \[ \vdots \]

  \[ \geq E \left[ Y \right] = \frac{7}{8} \cdot m. \]
Analysis of \textsc{Greedy-3-CNF}(\(\phi, n, m\))

\textbf{Theorem}

\textsc{Greedy-3-CNF}(\(\phi, n, m\)) is a polynomial-time \(8/7\)-approximation.

\textbf{Proof:}

\begin{itemize}
  \item \textbf{Step 1:} polynomial-time algorithm \(\checkmark\)
    \begin{itemize}
      \item In iteration \(j = 1, 2, \ldots, n\), \(Y = Y(\phi)\) averages over \(2^{n-j+1}\) assignments
      \item A smarter way is to use linearity of (conditional) expectations:
    \end{itemize}

    \[
    \mathbb{E}\left[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1\right] = \sum_{i=1}^{m} \mathbb{E}\left[Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1\right]
    \]

  \item \textbf{Step 2:} satisfies at least \(7/8 \cdot m\) clauses \(\checkmark\)
    \begin{itemize}
      \item Due to the greedy choice in each iteration \(j = 1, 2, \ldots, n\),
      \[
      \mathbb{E}\left[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v_j\right] \geq \mathbb{E}\left[Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}\right]
      \geq \mathbb{E}\left[Y \mid x_1 = v_1, \ldots, x_{j-2} = v_{j-2}\right]
      \]
      \[
      \vdots
      \]
      \[
      \geq \mathbb{E}[Y] = \frac{7}{8} \cdot m.
      \]
    \end{itemize}
\end{itemize}
**Analysis of GREEDY-3-CNF(φ, n, m)**

**Theorem**

GREEDY-3-CNF(φ, n, m) is a polynomial-time 8/7-approximation.

**Proof:**

- **Step 1:** polynomial-time algorithm ✓
  - In iteration $j = 1, 2, \ldots, n$, $Y = Y(\phi)$ averages over $2^{n-j+1}$ assignments
  - A smarter way is to use linearity of (conditional) expectations:

  $$E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right] = \sum_{i=1}^{m} E \left[ Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \right]$$

- **Step 2:** satisfies at least $7/8 \cdot m$ clauses ✓
  - Due to the greedy choice in each iteration $j = 1, 2, \ldots, n$,

  $$E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v \right] \geq E \left[ Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1} \right]$$
  $$\geq E \left[ Y \mid x_1 = v_1, \ldots, x_{j-2} = v_{j-2} \right]$$
  $$\vdots$$
  $$\geq E \left[ Y \right] = \frac{7}{8} \cdot m.$$

This algorithm is deterministic.
Theorem 35.6

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised \( 8/7 \)-approximation algorithm.
Theorem 35.6

Given an instance of MAX-3-CNF with \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses, the randomised algorithm that sets each variable independently at random is a randomised \( 8/7 \)-approximation algorithm.

Theorem

\[ \text{GREEDY-3-CNF}(\phi, n, m) \] is a polynomial-time \( 8/7 \)-approximation.
MAX-3-CNF: Concluding Remarks

--- Theorem 35.6 ---

Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

--- Theorem ---

$\text{GREEDY-3-CNF}(\phi, n, m)$ is a polynomial-time $8/7$-approximation.

--- Theorem (Hastad'97) ---

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-CNF unless P=NP.
Given an instance of MAX-3-CNF with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

**Theorem 35.6**

**Theorem**

$\text{GREEDY-3-CNF}(\phi, n, m)$ is a polynomial-time $8/7$-approximation.

**Theorem (Hastad’97)**

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-CNF unless P=NP.

Essentially there is nothing smarter than just guessing!
So you said you have been studying the field of algorithms for MAX-3-SAT?

Yes, my research has finally concluded...

...the best approach is to randomly guess a solution.

Source of Image: Stefan Szeider, TU Vienna
So you said you have been studying the field of algorithms for MAX-3-SAT?

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So you said you have been studying the field of algorithms for MAX-3-SAT?

Source of Image: Stefan Szeider, TU Vienna
Yes, my research has finally concluded...

So you said you have been studying the field of algorithms for MAX-3-SAT?

...the best approach is to **randomly guess** a solution.

Source of Image: Stefan Szeider, TU Vienna
Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover
The Weighted Vertex-Cover Problem

Given: Undirected, vertex-weighted graph $G = (V, E)$

Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $\{u, v\} \in E(G)$, then $u \in V'$ or $v \in V'$.

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $\{u, v\} \in E(G)$, then $u \in V'$ or $v \in V'$.

Applications:
Every edge forms a task, and every vertex represents a person/machine which can execute that task
Weight of a vertex could be salary of a person
Perform all tasks with the minimal amount of resources
The **Weighted Vertex-Cover Problem**

---

**Vertex Cover Problem**

- **Given:** Undirected, vertex-weighted graph $G = (V, E)$
- **Goal:** Find a minimum-weight subset $V' \subseteq V$ such that if $\{u, v\} \in E(G)$, then $u \in V'$ or $v \in V'$.

**Question:** How can we deal with graphs that have negative weights?
The **Weighted Vertex-Cover Problem**

- **Given:** Undirected, vertex-weighted graph \( G = (V, E) \)
- **Goal:** Find a minimum-weight subset \( V' \subseteq V \) such that if \( \{u, v\} \in E(G) \), then \( u \in V' \) or \( v \in V' \).
The **Weighted Vertex-Cover Problem**

**Vertex Cover Problem**

- **Given:** Undirected, vertex-weighted graph $G = (V, E)$
- **Goal:** Find a minimum-weight subset $V' \subseteq V$ such that if $\{u, v\} \in E(G)$, then $u \in V'$ or $v \in V'$.
The **Weighted Vertex-Cover Problem**

- **Given:** Undirected, vertex-weighted graph $G = (V, E)$
- **Goal:** Find a minimum-weight subset $V' \subseteq V$ such that if $\{u, v\} \in E(G)$, then $u \in V'$ or $v \in V'$.

This is (still) an NP-hard problem.
The Weighted Vertex-Cover Problem

Given: Undirected, vertex-weighted graph \( G = (V, E) \)
Goal: Find a minimum-weight subset \( V' \subseteq V \) such that if \( \{u, v\} \in E(G) \), then \( u \in V' \) or \( v \in V' \).

This is (still) an NP-hard problem.

Applications:
**The Weighted Vertex-Cover Problem**

- **Given:** Undirected, vertex-weighted graph $G = (V, E)$
- **Goal:** Find a minimum-weight subset $V' \subseteq V$ such that if $\{u, v\} \in E(G)$, then $u \in V'$ or $v \in V'$.

This is (still) an NP-hard problem.

**Applications:**
- Every edge forms a task, and every vertex represents a person/machine which can execute that task
The Weighted Vertex-Cover Problem

**Vertex Cover Problem**

- **Given:** Undirected, vertex-weighted graph $G = (V, E)$
- **Goal:** Find a minimum-weight subset $V' \subseteq V$ such that if $\{u, v\} \in E(G)$, then $u \in V'$ or $v \in V'$.

This is (still) an NP-hard problem.

**Applications:**

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- **Weight** of a vertex could be salary of a person
The Weighted Vertex-Cover Problem

Vertex Cover Problem

- **Given**: Undirected, vertex-weighted graph \( G = (V, E) \)
- **Goal**: Find a minimum-weight subset \( V' \subseteq V \) such that if \( \{u, v\} \in E(G) \), then \( u \in V' \) or \( v \in V' \).

This is (still) an NP-hard problem.

Applications:

- Every edge forms a **task**, and every vertex represents a person/machine which can execute that task
- **Weight** of a vertex could be salary of a person
- Perform all tasks with the **minimal amount of resources**
A Greedy Approach working for Unweighted Vertex Cover

**APPROX-VERTEX-COVER** \((G)\)

1. \(C = \emptyset\)
2. \(E' = G \cdot E\)
3. **while** \(E' \neq \emptyset\)
   4. let \((u, v)\) be an arbitrary edge of \(E'\)
   5. \(C = C \cup \{u, v\}\)
   6. remove from \(E'\) every edge incident on either \(u\) or \(v\)
4. **return** \(C\)
A Greedy Approach working for Unweighted Vertex Cover

**APPROX-VERTEX-COVER** (*G*)

1. \( C = \emptyset \)
2. \( E' = G.E \)
3. **while** \( E' \neq \emptyset \)
   4. let \((u, v)\) be an arbitrary edge of \( E' \)
   5. \( C = C \cup \{u, v\} \)
   6. remove from \( E' \) every edge incident on either \( u \) or \( v \)
4. **return** \( C \)

This algorithm is a 2-approximation for **unweighted graphs**!
**Figure 35.1** The operation of $\text{APPROX-VERTEX-COVER}$. (a) The input graph $G$, which has 7 vertices and 8 edges. (b) The edge $bc$ is the first edge chosen by $\text{APPROX-VERTEX-COVER}$. Vertices $b$ and $c$, shown lightly shaded, are added to the set $C$ containing the vertex cover being created. Edges $ab$, $ce$, and $cd$, shown as dashed, are removed since they are covered by some vertex in $C$. (c) Edge $ef$ is chosen; vertices $e$ and $f$ are added to $C$. (d) Edge $dg$ is chosen; vertices $d$ and $g$ are added to $C$. (e) The set $C$, which is the vertex cover produced by $\text{APPROX-VERTEX-COVER}$, contains the vertices $b, c, d, e, f, g$. (f) The optimal vertex cover for this problem contains only three vertices: $b, d$, and $e$.

**Algorithm**

$\text{APPROX-VERTEX-COVER}(G)$

1. $C = \emptyset$
2. $E' = G.E$
3. while $E' \neq \emptyset$
   4. let $(u, v)$ be an arbitrary edge of $E'$
   5. $C = C \cup \{u, v\}$
   6. remove from $E'$ every edge incident on either $u$ or $v$
4. return $C$
A Greedy Approach working for Unweighted Vertex Cover

**Algorithm: APPROX-VERTEX-COVER**

1. \( C = \emptyset \)
2. \( E' = G \cdot E \)
3. \( \textbf{while } E' \neq \emptyset \)
   4. let \((u, v)\) be an arbitrary edge of \( E' \)
   5. \( C = C \cup \{u, v\} \)
   6. remove from \( E' \) every edge incident on either \( u \) or \( v \)
4. return \( C \)

This algorithm is a 2-approximation for unweighted graphs.

**Computed solution has weight 101**

**Optimal solution has weight 4**
A Greedy Approach working for Unweighted Vertex Cover

**APPROX-VERTEX-COVER**\((G)\)

1. \(C = \emptyset\)
2. \(E' = G \cdot E\)
3. while \(E' \neq \emptyset\)
   4. let \((u, v)\) be an arbitrary edge of \(E'\)
   5. \(C = C \cup \{u, v\}\)
   6. remove from \(E'\) every edge incident on either \(u\) or \(v\)
4. return \(C\)

This algorithm is a 2-approximation for unweighted graphs!
Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

0-1 Integer Program

\[
\begin{align*}
\text{minimize} & \quad \sum_{v \in V} w(v) x(v) \\
\text{subject to} & \quad x(u) + x(v) \geq 1 \quad \text{for each} \quad (u, v) \in E \\
x(v) & \in \{0, 1\} \quad \text{for each} \quad v \in V
\end{align*}
\]

Linear Program

\[
\begin{align*}
\text{optimum is a lower bound on the optimal weight of a minimum weight-cover.}
\end{align*}
\]

Rounding Rule:

\[x(v) \geq \frac{1}{2} \text{ then round up, otherwise round down.}\]
Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

0-1 Integer Program

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\begin{align*}
\text{minimize} & \quad \sum_{v \in V} w(v)x(v) \\
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Invoking an (Integer) Linear Program

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\[
\begin{align*}
& \text{minimize } \sum_{v \in V} w(v) x(v) \\
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optimum is a lower bound on the optimal weight of a minimum weight-cover.

Linear Program

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\begin{align*}
& \text{minimize } \sum_{v \in V} w(v) x(v) \\
& \text{subject to } \\
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& \quad x(v) \in [0, 1] \quad \text{for each } v \in V
\end{align*}
\]
Invoking an (Integer) Linear Program

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0-1 Integer Program

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\begin{align*}
\text{minimize} & \quad \sum_{v \in V} w(v)x(v) \\
\text{subject to} & \quad x(u) + x(v) \geq 1 \quad \text{for each } (u, v) \in E \\
& \quad x(v) \in \{0, 1\} \quad \text{for each } v \in V
\end{align*}
\]

Linear Program

\[
\begin{align*}
\text{minimize} & \quad \sum_{v \in V} w(v)x(v) \\
\text{subject to} & \quad x(u) + x(v) \geq 1 \quad \text{for each } (u, v) \in E \\
& \quad x(v) \in [0, 1] \quad \text{for each } v \in V
\end{align*}
\]

Rounding Rule: if \(x(v) \geq 1/2\) then round up, otherwise round down.
The Algorithm

`APPROX-MIN-WEIGHT-VC(G, w)`
1  \( C = \emptyset \)
2  compute \( \tilde{x} \), an optimal solution to the linear program
3  \textbf{for} each \( v \in V \)
4    \textbf{if} \( \tilde{x}(v) \geq 1/2 \)
5      \( C = C \cup \{v\} \)
6  \textbf{return} \( C \)
The Algorithm

**Theorem 35.7**

**APPROX-MIN-WEIGHT-VC** is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

**APPROX-MIN-WEIGHT-VC**($G, w$)

1. $C = \emptyset$
2. compute $\bar{x}$, an optimal solution to the linear program
3. for each $v \in V$
   4. if $\bar{x}(v) \geq 1/2$
   5. $C = C \cup \{v\}$
6. return $C$
The Algorithm

**APPROX-MIN-WEIGHT-VC** \((G, w)\)
1. \(C = \emptyset\)
2. compute \(\tilde{x}\), an optimal solution to the linear program
3. for each \(v \in V\)
   4. if \(\tilde{x}(v) \geq 1/2\)
   5. \(C = C \cup \{v\}\)
4. return \(C\)

**Theorem 35.7**

**APPROX-MIN-WEIGHT-VC** is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

is polynomial-time because we can solve the linear program in polynomial time
Example of \textsc{Approx-Min-Weight-VC}

\[ \bar{x}(a) = \bar{x}(b) = \bar{x}(e) = \frac{1}{2}, \quad \bar{x}(d) = 1, \quad \bar{x}(c) = 0 \]

fractional solution of LP with weight $= 5.5$
Example of **APPROX-MIN-WEIGHT-VC**

\[
\bar{x}(a) = \bar{x}(b) = \bar{x}(e) = \frac{1}{2}, \quad \bar{x}(d) = 1, \quad \bar{x}(c) = 0
\]

\[
x(a) = x(b) = x(e) = 1, \quad x(d) = 1, \quad x(c) = 0
\]

Rounding

Fractional solution of LP with weight = 5.5

Rounded solution of LP with weight = 10
Example of **APPROX-MIN-WEIGHT-VC**

\[
\begin{align*}
\bar{x}(a) &= \bar{x}(b) = \bar{x}(e) = \frac{1}{2}, \quad \bar{x}(d) = 1, \quad \bar{x}(c) = 0 \\
\end{align*}
\]

\[
\begin{align*}
x(a) &= x(b) = x(e) = 1, \quad x(d) = 1, \quad x(c) = 0 \\
\end{align*}
\]

Fractional solution of LP with weight = 5.5

Rounded solution of LP with weight = 10

Optimal solution with weight = 6
Proof (Approximation Ratio is 2 and Correctness):

Let $C^*$ be an optimal solution to the minimum-weight vertex cover problem.

Let $z^*$ be the value of an optimal solution to the linear program, so

$$z^* \leq w(C^*)$$

Step 1:
The computed set $C$ covers all vertices:

Consider any edge $(u, v) \in E$ which imposes the constraint $x(u) + x(v) \geq 1$.

$\Rightarrow$ at least one of $x(u)$ and $x(v)$ is at least $1/2$.

$\Rightarrow C$ covers edge $(u, v)$.

Step 2:
The computed set $C$ satisfies $w(C) \leq 2z^*$:

$$w(C^*) \geq z^* = \sum_{v \in V} w(v) x(v) \geq \sum_{v \in V} : x(v) \geq 1/2 w(v) = \frac{1}{2} w(C).$$
Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

Let $C^*$ be an optimal solution to the minimum-weight vertex cover problem. Let $z^*$ be the value of an optimal solution to the linear program, so $z^* \leq w(C^*)$.

Step 1: The computed set $C$ covers all vertices. Consider any edge $(u, v) \in E$ which imposes the constraint $x(u) + x(v) \geq 1 \Rightarrow$ at least one of $x(u)$ and $x(v)$ is at least $1/2$ $\Rightarrow C$ covers edge $(u, v)$.

Step 2: The computed set $C$ satisfies $w(C) \leq 2z^*$.

$w(C^*) \geq z^* = \sum_{v \in V} w(v) x(v) \geq \sum_{v \in V} x(v) \geq 1/2 w(v) \cdot 1/2 = 1/4 w(C)$. 

Rounding
Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

- Let $C^*$ be an optimal solution to the minimum-weight vertex cover problem.
Approximation Ratio

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Approximation Ratio

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- Let $z^*$ be the value of an optimal solution to the linear program, so
  \[ z^* \leq w(C^*) \]

- **Step 1:** The computed set $C$ covers all vertices:

![Diagram of a graph with vertices and edges labeled with weights, showing the rounding process from vertices to edges.](image)
Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):
- Let $C^*$ be an optimal solution to the minimum-weight vertex cover problem
- Let $z^*$ be the value of an optimal solution to the linear program, so
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  - Consider any edge $(u, v) \in E$ which imposes the constraint $x(u) + x(v) \geq 1$
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**Step 1:** The computed set $C$ covers all vertices:

- Consider any edge $(u, v) \in E$ which imposes the constraint $x(u) + x(v) \geq 1$
- $\Rightarrow$ at least one of $x(u)$ and $x(v)$ is at least $1/2$
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- Let $C^*$ be an optimal solution to the minimum-weight vertex cover problem.
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  - $\Rightarrow$ at least one of $x(u)$ and $x(v)$ is at least $1/2$ $\Rightarrow$ $C$ covers edge $(u, v)$

- **Step 2:** The computed set $C$ satisfies $w(C) \leq 2z^*$:

![Rounding diagram]


Weighted Vertex Cover
Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

- Let $C^*$ be an optimal solution to the minimum-weight vertex cover problem
- Let $z^*$ be the value of an optimal solution to the linear program, so

$$z^* \leq w(C^*)$$

- **Step 1:** The computed set $C$ covers all vertices:
  - Consider any edge $(u, v) \in E$ which imposes the constraint $x(u) + x(v) \geq 1$
  - $\Rightarrow$ at least one of $x(u)$ and $x(v)$ is at least $1/2$ $\Rightarrow$ $C$ covers edge $(u, v)$
- **Step 2:** The computed set $C$ satisfies $w(C) \leq 2z^*$:
Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):
- Let $C^*$ be an optimal solution to the minimum-weight vertex cover problem
- Let $z^*$ be the value of an optimal solution to the linear program, so
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  - Consider any edge $(u, v) \in E$ which imposes the constraint $x(u) + x(v) \geq 1$
    \[ \Rightarrow \] at least one of $x(u)$ and $x(v)$ is at least $1/2 \Rightarrow C$ covers edge $(u, v)$

- **Step 2:** The computed set $C$ satisfies $w(C) \leq 2z^*$:
  \[ w(C^*) \geq z^* \]
Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

- Let \( C^* \) be an optimal solution to the minimum-weight vertex cover problem.
- Let \( z^* \) be the value of an optimal solution to the linear program, so

\[ z^* \leq w(C^*) \]

- **Step 1:** The computed set \( C \) covers all vertices:
  - Consider any edge \((u, v) \in E\) which imposes the constraint \( x(u) + x(v) \geq 1 \)
  \[ \Rightarrow \text{at least one of } \bar{x}(u) \text{ and } \bar{x}(v) \text{ is at least } 1/2 \Rightarrow C \text{ covers edge } (u, v) \]

- **Step 2:** The computed set \( C \) satisfies \( w(C) \leq 2z^* \):

\[ w(C^*) \geq z^* = \sum_{v \in V} w(v)\bar{x}(v) \]
Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

- Let \( C^* \) be an optimal solution to the minimum-weight vertex cover problem.
- Let \( z^* \) be the value of an optimal solution to the linear program, so
  \[
  z^* \leq w(C^*)
  \]

**Step 1:** The computed set \( C \) covers all vertices:
- Consider any edge \((u, v) \in E\) which imposes the constraint \( x(u) + x(v) \geq 1 \) \(\Rightarrow\) at least one of \( x(u) \) and \( x(v) \) is at least \( 1/2 \) \(\Rightarrow\) \( C \) covers edge \((u, v)\).

**Step 2:** The computed set \( C \) satisfies \( w(C) \leq 2z^* \):

\[
\begin{align*}
  w(C^*) &\geq z^* = \sum_{v \in V} w(v) \bar{x}(v) \\
  &\geq \sum_{v \in V: \bar{x}(v) \geq 1/2} w(v) \cdot \frac{1}{2}
\end{align*}
\]
Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

- Let $C^*$ be an optimal solution to the minimum-weight vertex cover problem
- Let $z^*$ be the value of an optimal solution to the linear program, so

$$z^* \leq w(C^*)$$

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  \]
Outline

Weighted Set Cover

MAX-CNF
The **Weighted Set-Cover Problem**

### Set Cover Problem

- **Given:** set $X$ and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- **Goal:** Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

\[ \text{s.t. } X = \bigcup_{S \in \mathcal{C}} S. \]
The **Weighted Set-Cover Problem**

- **Given**: set $X$ and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \rightarrow \mathbb{R}^+$
- **Goal**: Find a minimum-cost subset $C \subseteq \mathcal{F}$ such that $X = \bigcup_{S \in C} S$.

**Set Cover Problem**

- Sum over the costs of all sets in $C$
The Weighted Set-Cover Problem

Given: set $X$ and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \rightarrow \mathbb{R}^+$

Goal: Find a minimum-cost subset $C \subseteq \mathcal{F}$ s.t. $X = \bigcup_{S \in C} S$.

Remarks:
- generalisation of the weighted Vertex-Cover problem
- models resource allocation problems

Set Cover Problem

Sum over the costs of all sets in $C$

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The Weighted Set-Cover Problem

Given: set $X$ and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \rightarrow \mathbb{R}^+$

Goal: Find a minimum-cost subset $C \subseteq \mathcal{F}$ s.t. $X = \bigcup_{S \in C} S$

Set Cover Problem

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Goal: Find a minimum-cost subset $C \subseteq \mathcal{F}$ such that $X = \bigcup_{S \in C} S$.

Set Cover Problem

Question: How can we reduce the Vertex-Cover problem to the Set-Cover problem?

Remarks:

- generalisation of the weighted Vertex-Cover problem
- models resource allocation problems
Setting up an Integer Program

**Question:** Try to formulate the integer program and linear program of the weighted SET-COVER problem (solution on next slide!)

\[
\begin{align*}
\text{minimize} & \quad \sum_{S \in F} c(S) y(S) \\
\text{subject to} & \quad \sum_{S \in F: x \in S} y(S) \geq 1 \text{ for each } x \in X \\
& \quad y(S) \in \{0, 1\} \text{ for each } S \in F
\end{align*}
\]
Setting up an Integer Program

0-1 Integer Program

minimize \[ \sum_{S \in \mathcal{F}} c(S)y(S) \]

subject to \[ \sum_{S \in \mathcal{F} : x \in S} y(S) \geq 1 \quad \text{for each } x \in X \]
\[ y(S) \in \{0, 1\} \quad \text{for each } S \in \mathcal{F} \]
Setting up an Integer Program

\[ \text{minimize} \quad \sum_{S \in \mathcal{F}} c(S)y(S) \]

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Linear Program

\[ \text{minimize} \quad \sum_{S \in \mathcal{F}} c(S)y(S) \]

subject to \[ \sum_{S \in \mathcal{F}: x \in S} y(S) \geq 1 \quad \text{for each } x \in X \]
\[ y(S) \in [0, 1] \quad \text{for each } S \in \mathcal{F} \]
The strategy employed for Vertex-Cover would take all 6 sets! Even worse: If all $y$’s were below $\frac{1}{2}$, we would not even return a valid cover!

$\begin{array}{ccccccc}
  S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
  c : & 2 & 3 & 3 & 5 & 1 & 2 \\
\end{array}$
Back to the Example

Cost equals 8.5.

The strategy employed for Vertex-Cover would take all 6 sets!

Even worse: If all \( y \)'s were below \( 1/2 \), we would not even return a valid cover!
Back to the Example

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Back to the Example

The strategy employed for Vertex-Cover would take all 6 sets!

Cost equals 8.5

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Weighted Set Cover
Back to the Example

The strategy employed for Vertex-Cover would take all 6 sets!

Even worse: If all $\bar{y}$'s were below $1/2$, we would not even return a valid cover!

Cost equals 8.5
Randomised Rounding

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Idea: Interpret the $y$-values as probabilities for picking the respective set.

The expected cost satisfies

$$E[c(C)] = \sum_{S \in F} c(S) \cdot y(S)$$

The probability that an element $x \in X$ is covered satisfies

$$P[x \in \bigcup_{S \in C} S] \geq 1 - \frac{1}{e}$$

Lemma 10. Approximation Algorithms © T. Sauerwald

Weighted Set Cover
Randomised Rounding

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Idea: Interpret the \(\bar{y}\)-values as probabilities for picking the respective set.
Randomised Rounding

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Idea: Interpret the $\bar{y}$-values as probabilities for picking the respective set.

Randomised Rounding

- Let $C \subseteq F$ be a random set with each set $S$ being included independently with probability $\bar{y}(S)$.
- More precisely, if $\bar{y}$ denotes the optimal solution of the LP, then we compute an integral solution $y$ by:

$$y(S) = \begin{cases} 
1 & \text{with probability } \bar{y}(S) \\
0 & \text{otherwise.} 
\end{cases}$$

for all $S \in F$. 

Therefore, $E[y(S)] = \bar{y}(S)$.

The probability that an element $x \in X$ is covered satisfies

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Randomised Rounding

The expected cost satisfies

$$\mathbb{E}[c(C)] = \sum_{S \in \mathcal{F}} c(S) \cdot \bar{y}(S).$$

The probability that an element $x \in X$ is covered satisfies

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Randomised Rounding

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Idea: Interpret the $\bar{y}$-values as probabilities for picking the respective set.

Lemma

- The expected cost satisfies

$$E \left[ c(C) \right] = \sum_{S \in \mathcal{F}} c(S) \cdot \bar{y}(S)$$
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Lemma 10. Approximation Algorithms © T. Sauerwald

- The expected cost satisfies
  \[
  E[ c(C) ] = \sum_{S \in \mathcal{F}} c(S) \cdot \bar{y}(S)
  \]
- The probability that an element $x \in X$ is covered satisfies
  \[
  P \left[ x \in \bigcup_{S \in C} S \right] \geq 1 - \frac{1}{e}.
  \]
Proof of Lemma

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set $S$ being included independently with probability $\overline{y}(S)$.

- The expected cost satisfies $\mathbb{E}[c(\mathcal{C})] = \sum_{S \in \mathcal{F}} c(S) \cdot \overline{y}(S)$.
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Proof of Lemma

Let \( C \subseteq \mathcal{F} \) be a random subset with each set \( S \) being included independently with probability \( \overline{y}(S) \).

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Proof:

- **Step 1**: The expected cost of the random set \( C \)
Proof of Lemma

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\[
\mathbb{E}[c(C)]
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\mathbb{E}[c(C)] = \mathbb{E} \left[ \sum_{S \in C} c(S) \right]
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Let $C \subseteq F$ be a random subset with each set $S$ being included independently with probability $\overline{y}(S)$.

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Let $C \subseteq \mathcal{F}$ be a random subset with each set $S$ being included independently with probability $\overline{y}(S)$.

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$$E[\ c(C)\ ] = E\left[\ \sum_{S \in C} c(S)\right] = E\left[\ \sum_{S \in \mathcal{F}} 1_{S \in C} \cdot c(S)\right] = \sum_{S \in \mathcal{F}} P[\ S \in C\ ] \cdot c(S) = \sum_{S \in \mathcal{F}} \overline{y}(S) \cdot c(S).$$
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  = \sum_{S \in \mathcal{F}} \mathbb{P}[S \in C] \cdot c(S) = \sum_{S \in \mathcal{F}} \bar{y}(S) \cdot c(S).
  \]

- **Step 2**: The probability for an element to be (not) covered

  $\mathbb{P}[x \notin \bigcup_{S \in C} S] = \prod_{S \in \mathcal{F}: x \in S} \mathbb{P}[S \notin C] = \prod_{S \in \mathcal{F}: x \in S} (1 - \bar{y}(S)) \\
  \leq \prod_{S \in \mathcal{F}: x \in S} e^{-\bar{y}(S)} = e^{-\sum_{S \in \mathcal{F}: x \in S} \bar{y}(S)} \\
  = e^{-1}$

\[\text{for any } x \in \mathbb{R}\]

$\bar{y}$ solves the LP!
Proof of Lemma

Let $C \subseteq F$ be a random subset with each set $S$ being included independently with probability $\overline{y}(S)$.

- The expected cost satisfies $E[c(C)] = \sum_{S \in F} c(S) \cdot \overline{y}(S)$.
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Proof:

- **Step 1:** The expected cost of the random set $C$

  $\mathbb{E}[c(C)] = \mathbb{E}\left[ \sum_{S \in C} c(S) \right] = \mathbb{E}\left[ \sum_{S \in F} 1_{S \in C} \cdot c(S) \right] = \sum_{S \in F} \mathbb{P}[S \in C] \cdot c(S) = \sum_{S \in F} \overline{y}(S) \cdot c(S)$.

- **Step 2:** The probability for an element to be (not) covered

  $\mathbb{P}[x \not\in \bigcup_{S \in C} S] = \prod_{S \in F : x \in S} \mathbb{P}[S \not\in C]$.
Proof of Lemma

**Lemma**

Let $C \subseteq \mathcal{F}$ be a random subset with each set $S$ being included independently with probability $\overline{y}(S)$.

- The expected cost satisfies $E[c(C)] = \sum_{S \in \mathcal{F}} c(S) \cdot \overline{y}(S)$.
- The probability that $x$ is covered satisfies $P[x \in \bigcup_{S \in C} S] \geq 1 - \frac{1}{e}$.

**Proof:**

- **Step 1:** The expected cost of the random set $C$

  $$E[c(C)] = E \left[ \sum_{S \in C} c(S) \right] = E \left[ \sum_{S \in \mathcal{F}} 1_{S \in C} \cdot c(S) \right] = \sum_{S \in \mathcal{F}} P[S \in C] \cdot c(S) = \sum_{S \in \mathcal{F}} \overline{y}(S) \cdot c(S).$$

- **Step 2:** The probability for an element to be (not) covered

  $$P[x \notin \bigcup_{S \in C} S] = \prod_{S \in \mathcal{F}: x \notin S} P[S \notin C] = \prod_{S \in \mathcal{F}: x \notin S} (1 - \overline{y}(S))$$
Proof of Lemma

Lemma

Let $C \subseteq F$ be a random subset with each set $S$ being included independently with probability $\overline{y}(S)$.

- The expected cost satisfies $\mathbb{E}[c(C)] = \sum_{S \in F} c(S) \cdot \overline{y}(S)$.
- The probability that $x$ is covered satisfies $\mathbb{P}[x \in \bigcup_{S \in C} S] \geq 1 - \frac{1}{e}$.

Proof:

- **Step 1**: The expected cost of the random set $C$

  $$\mathbb{E}[c(C)] = \mathbb{E} \left[ \sum_{S \in C} c(S) \right] = \mathbb{E} \left[ \sum_{S \in F} 1_{S \in C} \cdot c(S) \right] = \sum_{S \in F} \mathbb{P}[S \in C] \cdot c(S) = \sum_{S \in F} \overline{y}(S) \cdot c(S).$$

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  $$\mathbb{P}[x \not\in \bigcup_{S \in C} S] = \prod_{S \in F: x \in S} \mathbb{P}[S \not\in C] = \prod_{S \in F: x \in S} (1 - \overline{y}(S))$$

  $$1 + x \leq e^x \text{ for any } x \in \mathbb{R}$$
Proof of Lemma

Let \( C \subseteq \mathcal{F} \) be a random subset with each set \( S \) being included independently with probability \( y(S) \).

- The expected cost satisfies \( \mathbb{E}[c(C)] = \sum_{S \in \mathcal{F}} c(S) \cdot y(S) \).
- The probability that \( x \) is covered satisfies \( \mathbb{P}(x \in \bigcup_{S \in C} S) \geq 1 - \frac{1}{e} \).

Proof:

- **Step 1:** The expected cost of the random set \( C \)
  \[
  \mathbb{E}[c(C)] = \mathbb{E}\left[ \sum_{S \in C} c(S) \right] = \mathbb{E}\left[ \sum_{S \in \mathcal{F}} 1_{S \in C} \cdot c(S) \right] \\
  = \sum_{S \in \mathcal{F}} \mathbb{P}(S \in C) \cdot c(S) = \sum_{S \in \mathcal{F}} y(S) \cdot c(S).
  \]

- **Step 2:** The probability for an element to be (not) covered
  \[
  \mathbb{P}(x \notin \bigcup_{S \in C} S) = \prod_{S \in \mathcal{F}: x \in S} \mathbb{P}(S \notin C) = \prod_{S \in \mathcal{F}: x \in S} (1 - y(S)) \\
  \leq \prod_{S \in \mathcal{F}: x \in S} e^{-y(S)}
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  \[1 + x \leq e^x \text{ for any } x \in \mathbb{R} \]
Proof of Lemma

Let $C \subseteq \mathcal{F}$ be a random subset with each set $S$ being included independently with probability $\overline{y}(S)$.

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Proof of Lemma

Let \( C \subseteq F \) be a random subset with each set \( S \) being included independently with probability \( \bar{y}(S) \).

- The expected cost satisfies \( \mathbb{E}[c(C)] = \sum_{S \in F} c(S) \cdot \bar{y}(S) \).
- The probability that \( x \) is covered satisfies \( \mathbb{P}[x \in \bigcup_{S \in C} S] \geq 1 - \frac{1}{e} \).

Proof:

- **Step 1**: The expected cost of the random set \( C \)

\[
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\mathbb{P}[x \notin \bigcup_{S \in C} S] = \prod_{S \in F: x \in S} \mathbb{P}[S \notin C] = \prod_{S \in F: x \in S} (1 - \bar{y}(S))
\]

\[1 + x \leq e^x \text{ for any } x \in \mathbb{R}
\]

\[
\leq \prod_{S \in F: x \in S} e^{-\bar{y}(S)} \quad \bar{y} \text{ solves the LP!}
\]

\[
= e^{-\sum_{S \in F: x \in S} \bar{y}(S)}
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Proof of Lemma

Let \( C \subseteq F \) be a random subset with each set \( S \) being included independently with probability \( y(S) \).

- The expected cost satisfies \( \mathbb{E} [ c(C) ] = \sum_{S \in F} c(S) \cdot y(S) \).
- The probability that \( x \) is covered satisfies \( \mathbb{P} [ x \in \bigcup_{S \in C} S ] \geq 1 - \frac{1}{e} \).

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\mathbb{P} [ x \notin \bigcup_{S \in C} S ] = \prod_{S \in F : x \in S} \mathbb{P} [ S \notin C ] = \prod_{S \in F : x \in S} (1 - y(S)) \\
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= e^{-\sum_{S \in F : x \in S} y(S)} \leq e^{-1}
\]

\( y \) solves the LP!
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1 + \( x \) \leq e\( x \) for any \( x \in \mathbb{R} \)

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Let \( C \subseteq \mathcal{F} \) be a random subset with each set \( S \) being included independently with probability \( y(S) \).

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\]

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\mathbb{P}[x \notin \bigcup_{S \in C} S] = \prod_{S \in \mathcal{F} : x \in S} \mathbb{P}[S \notin C] = \prod_{S \in \mathcal{F} : x \in S} (1 - y(S)) \leq \prod_{S \in \mathcal{F} : x \in S} e^{-y(S)} = e^{-\sum_{S \in \mathcal{F} : x \in S} y(S)} \leq e^{-1} \quad \Box
\]

1 + \( x \) ≤ \( e^x \) for any \( x \in \mathbb{R} \)

\( \bar{y} \) solves the LP!
**The Final Step**

**Lemma**

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set $S$ being included independently with probability $y(S)$.

- The expected cost satisfies $\mathbb{E}[c(\mathcal{C})] = \sum_{S \in \mathcal{F}} c(S) \cdot y(S)$.
- The probability that $x$ is covered satisfies $\mathbb{P}[x \in \bigcup_{S \in \mathcal{C}} S] \geq 1 - \frac{1}{e}$.
Let \( C \subseteq F \) be a random subset with each set \( S \) being included independently with probability \( y(S) \).

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Idea: Amplify this probability by taking the union of $\Omega(\log n)$ random sets $C$. 
The Final Step

Let $C \subseteq \mathcal{F}$ be a random subset with each set $S$ being included independently with probability $y(S)$.

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**Weighted Set Cover-LP**

1: compute $\bar{y}$, an optimal solution to the linear program
2: $C = \emptyset$
3: repeat $2 \ln n$ times
4: for each $S \in \mathcal{F}$
5: let $C = C \cup \{S\}$ with probability $\bar{y}(S)$
6: return $C$
The Final Step

Lemma

Let $C \subseteq \mathcal{F}$ be a random subset with each set $S$ being included independently with probability $y(S)$.

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**Weighted Set Cover-LP ($X, \mathcal{F}, c$)**

1: compute $\overline{y}$, an optimal solution to the linear program
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Clearly runs in polynomial-time!
**Analysis of WEIGHTED SET COVER-LP**

**Theorem**

- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
- The expected approximation ratio is $2 \ln(n)$.
Analysis of **WEIGHTED SET COVER-LP**

---

**Theorem**

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### Theorem

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### Proof:

- **Step 1:** The probability that $C$ is a cover

---

**Exercise Question (9/10)** gives a different perspective on the amplification procedure through non-linear randomised rounding.
Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
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Proof:

**Step 1:** The probability that $C$ is a cover

- By previous Lemma, an element $x \in X$ is covered in one of the $2 \ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that
Analysis of Weighted Set Cover-LP

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Proof:

- **Step 1**: The probability that $C$ is a cover
  - By previous Lemma, an element $x \in X$ is covered in one of the $2 \ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that
    \[
    P\left[x \notin \bigcup_{S \in C} S\right] \leq \left(\frac{1}{e}\right)^{2 \ln n}
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Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
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$$P \left[ x \notin \bigcup_{S \in C} S \right] \leq \left( \frac{1}{e} \right)^{2 \ln n} = \frac{1}{n^2}.$$
Analysis of **Weighted Set Cover**-LP

---

**Theorem**

- With probability at least \(1 - \frac{1}{n}\), the returned set \(C\) is a valid cover of \(X\).
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    \]
  - This implies for the event that all elements are covered:
Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
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Proof:

- **Step 1**: The probability that $C$ is a cover
  - By previous Lemma, an element $x \in X$ is covered in one of the $2 \ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that
    
    $$\Pr[x \notin \bigcup_{S \in C} S] \leq \left(\frac{1}{e}\right)^{2 \ln n} = \frac{1}{n^2}.$$ 
  - This implies for the event that all elements are covered:
    
    $$\Pr[X = \bigcup_{S \in C} S] =$$
Analysis of Weighted Set Cover - LP

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    \[
    \Pr[X = \bigcup_{S \in C} S] = 1 - \Pr[\bigcup_{x \in X} \{x \notin \bigcup_{S \in C} S\}]
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Analysis of WEIGHTED SET COVER -LP

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Analysis of **WEIGHTED SET COVER**-LP

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- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
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    \]

- **Step 2:** The expected approximation ratio
  - By previous lemma, the expected cost of one iteration is
    \[
    \sum_{S \in F} c(S) \cdot y(S).
    \]
  - Linearity \(\Rightarrow\)
    \[
    E[c(C)] \leq 2 \ln(n) \cdot \sum_{S \in F} c(S) \cdot y(S) \leq 2 \ln(n) \cdot c(C^*)
    \]
Analysis of **WEIGHTED SET COVER**-LP

---

**Theorem**

- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
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- **Step 1**: The probability that $C$ is a cover
  - By previous Lemma, an element $x \in X$ is covered in one of the $2 \ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that
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  - This implies for the event that all elements are covered:
    \[
    P \left[ X = \bigcup_{S \in C} S \right] = 1 - P \left[ \bigcup_{x \in X} \{ x \notin \bigcup_{S \in C} S \} \right].
    \]

- By the union bound inequality,
  \[
  P(A \cup B) \leq P(A) + P(B),
  \]

  \[
  \geq 1 - \sum_{x \in X} P \left[ x \notin \bigcup_{S \in C} S \right] \geq 1 - n \cdot \frac{1}{n^2}.
  \]
Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
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Proof:

- **Step 1:** The probability that $C$ is a cover
  - By previous Lemma, an element $x \in X$ is covered in one of the $2 \ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that
    $$
    P[x \notin \bigcup_{S \in C} S] \leq \left(\frac{1}{e}\right)^{2\ln n} = \frac{1}{n^2}.
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    P[X = \bigcup_{S \in C} S] = 1 - P\left[\bigcup_{x \in X} \{x \notin \bigcup_{S \in C} S\}\right] \geq 1 - \sum_{x \in X} P[x \notin \bigcup_{S \in C} S] \geq 1 - n \cdot \frac{1}{n^2} = 1 - \frac{1}{n}.
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Analysis of **Weighted Set Cover-LP**

**Theorem**

- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
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- By previous Lemma, an element $x \in X$ is covered in one of the $2 \ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that

$$P \left[ x \not\in \bigcup_{S \in C} S \right] \leq \left( \frac{1}{e} \right)^{2 \ln n} = \frac{1}{n^2}.$$  

- This implies for the event that all elements are covered:

$$P \left[ X = \bigcup_{S \in C} S \right] = 1 - P \left[ \bigcup_{x \in X} \left\{ x \not\in \bigcup_{S \in C} S \right\} \right] \geq 1 - \sum_{x \in X} P \left[ x \not\in \bigcup_{S \in C} S \right] \geq 1 - n \cdot \frac{1}{n^2} = 1 - \frac{1}{n}.$$
**Theorem**

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  \]

- **Step 2:** The expected approximation ratio

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Weighted Set Cover
Analysis of WEIGHTED SET COVER-LP

Theorem

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  - By previous lemma, the expected cost of one iteration is $\sum_{S \in \mathcal{F}} c(S) \cdot \bar{y}(S)$.

- **Step 2**: The expected approximation ratio
  - By previous lemma, the expected cost of one iteration is $\sum_{S \in \mathcal{F}} c(S) \cdot \bar{y}(S)$. 
Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
- The expected approximation ratio is $2\ln(n)$.

Proof:

**Step 1:** The probability that $C$ is a cover

- By previous Lemma, an element $x \in X$ is covered in one of the $2\ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that

\[
P[ x \notin \bigcup_{S \in C} S ] \leq \left( \frac{1}{e} \right)^{2\ln n} = \frac{1}{n^2}.
\]

- This implies for the event that all elements are covered:

\[
P[ X = \bigcup_{S \in C} S ] = 1 - \sum_{x \in X} P[ x \notin \bigcup_{S \in C} S ] \geq 1 - n \cdot \frac{1}{n^2} = 1 - \frac{1}{n}.
\]

**Step 2:** The expected approximation ratio

- By previous lemma, the expected cost of one iteration is $\sum_{S \in \mathcal{F}} c(S) \cdot \bar{y}(S)$.
- Linearity $\Rightarrow$ $E[ c(C) ] \leq 2\ln(n) \cdot \sum_{S \in \mathcal{F}} c(S) \cdot \bar{y}(S)$.
Analysis of WEIGHTED SET COVER-LP

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Analysis of \textsc{Weighted Set Cover-LP}

\begin{itemize}
  \item With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
  \item The expected approximation ratio is $2 \ln(n)$.
\end{itemize}

\textbf{Proof:}

\textbf{Step 1: The probability that $C$ is a cover $\checkmark$}

\begin{itemize}
  \item By previous Lemma, an element $x \in X$ is covered in one of the $2 \ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that
  \[ P \left[ x \notin \bigcup_{S \in C} S \right] \leq \left( \frac{1}{e} \right)^{2 \ln n} = \frac{1}{n^2}. \]
  \item This implies for the event that all elements are covered:
  \[ P \left[ X = \bigcup_{S \in C} S \right] = 1 - P \left[ \bigcup_{x \in X} \{ x \notin \bigcup_{S \in C} S \} \right] \geq 1 - n \cdot \frac{1}{n^2} = 1 - \frac{1}{n}. \]
\end{itemize}

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  \item By previous lemma, the expected cost of one iteration is $\sum_{S \in \mathcal{F}} c(S) \cdot \bar{y}(S)$.
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Analysis of WEIGHTED SET COVER -LP

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  - By previous Lemma, an element $x \in X$ is covered in one of the $2 \ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that
    
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- With probability at least $1 - \frac{1}{n}$, the returned set $C$ is a valid cover of $X$.
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By Markov’s inequality, $\Pr[c(C) \leq 4 \ln(n) \cdot c(C^*]) \geq 1/2$. 
Analysis of **Weighted Set Cover**-LP

---

**Theorem**

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Hence with probability at least \(1 - \frac{1}{n} - \frac{1}{2} > \frac{1}{3}\), solution is **valid** and within a factor of \(4 \ln(n)\) of the optimum.
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Hence with probability at least $1 - \frac{1}{n} - \frac{1}{2} > \frac{1}{3}$, solution is valid and within a factor of $4 \ln(n)$ of the optimum.

Probability could be further increased by repeating...
Analysis of WEIGHTED SET COVER-LP

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Hence with probability at least $1 - \frac{1}{n} - \frac{1}{2} > \frac{1}{3}$, solution is valid and within a factor of $4 \ln(n)$ of the optimum. Probability could be further increased by repeating.

Typical Approach for Designing Approximation Algorithms based on LPs
Theorem

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Typical Approach for Designing Approximation Algorithms based on LPs

[Exercise Question (9/10).10] gives a different perspective on the amplification procedure through non-linear randomised rounding.
Outline

Weighted Set Cover

MAX-CNF
Recall:

**MAX-3-CNF Satisfiability**

- **Given:** 3-CNφ formula, e.g.: \((x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots\)
- **Goal:** Find an assignment of the variables that satisfies as many clauses as possible.

**MAX-CNφ Satisfiability (MAX-SAT)**
**MAX-CNF**

Recall:

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- **Given**: CNF formula, e.g.: \((x_1 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor x_4 \lor \overline{x_5}) \land \cdots\)
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Why study this generalised problem?
Recall:

- **MAX-3-CNF Satisfiability**
  - **Given:** 3-CNF formula, e.g.: \((x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots\)
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  - **Goal:** Find an assignment of the variables that satisfies as many clauses as possible.

Why study this generalised problem?
- Allowing arbitrary clause lengths makes the problem more interesting (we will see that simply guessing is not the best!)
- A nice concluding example where we can practice previously learned approaches.
Approach 1: Guessing the Assignment

Assign each variable true or false uniformly and independently at random.

Recall: This was the successful approach to solve MAX-3-CNF!

For any clause \( i \) which has length \( \ell \),

\[
P\left[ \text{clause } i \text{ is satisfied} \right] = 1 - 2^{-\ell} = \alpha^\ell.
\]

In particular, the guessing algorithm is a randomised 2-approximation.

Analysis

Proof:

First statement as in the proof of Theorem 35.6. For clause \( i \) not to be satisfied, all \( \ell \) occurring variables must be set to a specific value. As before, let \( Y = \sum_{i=1}^{m} Y_i \) be the number of satisfied clauses. Then,

\[
E[Y] = E[\sum_{i=1}^{m} Y_i] = \sum_{i=1}^{m} E[Y_i] \geq \sum_{i=1}^{m} \frac{1}{2} = \frac{1}{2} m.
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Approach 1: Guessing the Assignment

Assign each variable true or false uniformly and independently at random.

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For any clause $i$ which has length $\ell$,

$$P\left[\text{clause } i \text{ is satisfied}\right] = 1 - 2^{-\ell} := \alpha_{\ell}.$$ 

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E[ Y ] = E \left[ \sum_{i=1}^{m} Y_i \right] = \sum_{i=1}^{m} E[ Y_i ] \geq \sum_{i=1}^{m} \frac{1}{2} = \frac{1}{2} \cdot m.
\]
Approach 2: Guessing with a “Hunch” (Randomised Rounding)

First solve a linear program and use fractional values for a biased coin flip.

\[
\begin{align*}
\max \sum_{i=1}^{m} z_i \\
\text{s.t.} \quad \sum_{j \in C^+ i} y_j + \sum_{j \in C^- i} (1 - y_j) \geq z_i \\
\quad \forall i = 1, 2, \ldots, m \\
\quad z_i \in \{0, 1\} \\
\quad y_j \in \{0, 1\} 
\end{align*}
\]
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The same as randomised rounding!
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First solve a linear program and use fractional values for a biased coin flip.

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0-1 Integer Program

maximize \( \sum_{i=1}^{m} z_i \)

subject to \( \sum_{j \in C_i^+} y_j + \sum_{j \in C_i^-} (1 - y_j) \geq z_i \) for each \( i = 1, 2, \ldots, m \)

\( z_i \in \{0, 1\} \) for each \( i = 1, 2, \ldots, m \)

\( y_j \in \{0, 1\} \) for each \( j = 1, 2, \ldots, n \)
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\( C_i^+ \) is the index set of the unnegated variables of clause \( i \).
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\( C_i^+ \) is the index set of the unnegated variables of clause \( i \).

- In the corresponding LP each \( \in \{0, 1\} \) is replaced by \( \in [0, 1] \)
- Let \((\bar{y}, \bar{z})\) be the optimal solution of the LP
- Obtain an integer solution \( y \) through randomised rounding of \( \bar{y} \)
Lemma

For any clause $i$ of length $\ell$,

$$P \left[ \text{clause } i \text{ is satisfied} \right] \geq \left( 1 - \left( 1 - \frac{1}{\ell} \right)^\ell \right) \cdot Z_i.$$
Lemma

For any clause $i$ of length $\ell$,

$$P[\text{clause } i \text{ is satisfied}] \geq \left(1 - \left(1 - \frac{1}{\ell}\right)^\ell\right) \cdot \bar{z}_i.$$ 

Proof of Lemma (1/2):
Analysis of Randomised Rounding

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For any clause $i$ of length $\ell$,

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Proof of Lemma (1/2):

- Assume w.l.o.g. all literals in clause $i$ appear non-negated (otherwise replace every occurrence of $x_j$ by $\overline{x_j}$ in the whole formula)
Analysis of Randomised Rounding

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For any clause $i$ of length $\ell$,

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- Assume w.l.o.g. all literals in clause $i$ appear non-negated (otherwise replace every occurrence of $x_j$ by $\overline{x}_j$ in the whole formula)
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For any clause $i$ of length $\ell$,

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$$\Rightarrow \Pr[\text{clause } i \text{ is satisfied}] = 1 - \prod_{j=1}^\ell \Pr[y_j \text{ is false}].$$
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For any clause $i$ of length $\ell$,

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$$

Arithmetic vs. geometric mean:

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\frac{a_1 + \cdots + a_k}{k} \geq \sqrt[k]{a_1 \times \cdots \times a_k}.
$$
Analysis of Randomised Rounding

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For any clause $i$ of length $\ell$,

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**Arithmetic vs. geometric mean:**

$$\frac{a_1 + \ldots + a_k}{k} \geq \sqrt[k]{a_1 \times \ldots \times a_k} \geq 1 - \left(\frac{\sum_{j=1}^{\ell}(1 - \overline{y}_j)}{\ell}\right)^\ell$$
Analysis of Randomised Rounding

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\[
= 1 - \left(1 - \frac{\sum_{j=1}^{\ell} \overline{y}_j}{\ell}\right)^\ell
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Analysis of Randomised Rounding

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For any clause $i$ of length $\ell$,

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Proof of Lemma (2/2):

- So far we have shown:

$$P \left[ \text{clause } i \text{ is satisfied} \right] \geq 1 - \left( 1 - \frac{\overline{Z}_i}{\ell} \right)^\ell$$
Analysis of Randomised Rounding

Lemma

For any clause $i$ of length $\ell$,

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Proof of Lemma (2/2):

- So far we have shown:

$$P[\text{clause } i \text{ is satisfied}] \geq 1 - \left(1 - \frac{\overline{Z}_i}{\ell}\right)^\ell$$

- For any $\ell \geq 1$, define $g(z) := 1 - (1 - \frac{z}{\ell})^\ell$. 
Analysis of Randomised Rounding

**Lemma**

For any clause $i$ of length $\ell$,

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P\left[\text{clause } i \text{ is satisfied}\right] \geq \left(1 - \left(1 - \frac{1}{\ell}\right)^\ell\right) \cdot \overline{z}_i.
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For any clause $i$ of length $\ell$,

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\[ g(z) = \begin{cases} 
1 - (1 - \frac{1}{3})^3 & \text{for } z = 0 \\
1 & \text{for } z = 1 
\end{cases} \]
Analysis of Randomised Rounding

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- Therefore, $P \left[ \text{clause } i \text{ is satisfied} \right] \geq \beta_{\ell} \cdot \overline{z}_i$. 

![Graph](image-url)
Analysis of Randomised Rounding

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Analysis of Randomised Rounding

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**Theorem**

Randomised Rounding yields a $1/(1 - 1/e) \approx 1.5820$ randomised approximation algorithm for MAX-CNF.
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Proof of Theorem:
Analysis of Randomised Rounding

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- For any clause $i = 1, 2, \ldots, m$, let $\ell_i$ be the corresponding length.
Analysis of Randomised Rounding

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For any clause $i$ of length $\ell$,

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Proof of Theorem:

- For any clause $i = 1, 2, \ldots, m$, let $\ell_i$ be the corresponding length.
- Then the expected number of satisfied clauses is:

$$\mathbb{E}[Y] = \sum_{i=1}^{m} \mathbb{E}[Y_i] \geq$$
Analysis of Randomised Rounding

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By Lemma
Analysis of Randomised Rounding

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$$

By Lemma \hspace{1cm} Since $(1 - 1/x)^x \leq 1/e$
Analysis of Randomised Rounding

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By Lemma

Since $(1 - 1/x)^x \leq 1/e$

LP solution at least as good as optimum
Approach 3: Hybrid Algorithm

Summary

- Approach 1 (Guessing) achieves better guarantee on longer clauses
- Approach 2 (Rounding) achieves better guarantee on shorter clauses

Idea:

Consider a hybrid algorithm which interpolates between the two approaches

\[ \text{HYBRID-MAX-CNF}(\phi, n, m) \]

1: Let \( b \in \{0, 1\} \) be the flip of a fair coin
2: If \( b = 0 \) then perform random guessing
3: If \( b = 1 \) then perform randomised rounding
4: return the computed solution

Algorithm sets each variable \( x_i \) to TRUE with prob. \( \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot y_i \).

Note, however, that variables are not independently assigned!
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$$\text{HYBRID-MAX-CNF}(\varphi, n, m)$$
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2. **If** \( b = 0 \) **then** perform random guessing
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Algorithm sets each variable \( x_i \) to TRUE with prob. \( \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \overline{y}_i \). Note, however, that variables are not independently assigned!
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Analysis of Hybrid Algorithm

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HYBRID-MAX-CNF(\(\varphi, n, m\)) is a randomised 4/3-approx. algorithm.

Proof:
Analysis of Hybrid Algorithm

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- It suffices to prove that clause \( i \) is satisfied with probability at least \( 3/4 \cdot \overline{z}_i \).
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\[
\text{Algorithm 1 satisfies it with probability } 1 - 2^{-\ell} \geq \alpha \cdot z_i.
\]

\[
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\]

HYBRID-MAX-CNF(φ, n, m) satisfies it with probability $1/2 \cdot \alpha \ell \cdot z_i + 1/2 \cdot \beta \ell \cdot z_i$.

Note $\alpha \ell + \beta \ell^2 = 3/4$ for $\ell \in \{1, 2\}$, and for $\ell \geq 3$, $\alpha \ell + \beta \ell^2 \geq 3/4$ (see figure).
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Analysis of Hybrid Algorithm

**Theorem**

\( \text{HYBRID-MAX-CNF}(\varphi, n, m) \) is a randomised 4/3-approx. algorithm.

**Proof:**

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- Note \( \frac{\alpha_\ell + \beta_\ell}{2} = 3/4 \) for \( \ell \in \{1, 2\} \), and for \( \ell \geq 3 \), \( \frac{\alpha_\ell + \beta_\ell}{2} \geq 3/4 \) (see figure)
- \( \Rightarrow \) \( \text{HYBRID-MAX-CNF}(\varphi, n, m) \) satisfies it with prob. at least \( 3/4 \cdot Z_i \)
Since $\alpha_2 = \beta_2 = 3/4$, we cannot achieve a better approximation ratio than 4/3 by combining Algorithm 1 & 2 in a different way.

The 4/3-approximation algorithm can be easily derandomised:

- Idea: use the conditional expectation trick for both Algorithm 1 & 2 and output the better solution.

The 4/3-approximation algorithm applies unchanged to a weighted version of MAX-CNF, where each clause has a non-negative weight.

Even MAX-2-CNF (every clause has length 2) is NP-hard!
Randomised Algorithms
Lecture 11: Spectral Graph Theory

Thomas Sauerwald (tms41@cam.ac.uk)
Outline

Introduction to (Spectral) Graph Theory and Clustering

Matrices, Spectrum and Structure

A Simplified Clustering Problem
Origin of Graph Theory

Leonhard Euler (1707-1783)

Seven Bridges at Königsberg 1737

Is there a tour which crosses each bridge exactly once?


Seven Bridges at Königsberg 1737
Origin of Graph Theory

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Is there a tour which crosses each bridge \textbf{exactly once}?


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Graphs Nowadays: Clustering

Goal:
Use spectrum of graphs (unstructured data) to extract clustering (communities) or other structural information.
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Graph Clustering (applications)

Applications of Graph Clustering

- Community detection
- Group webpages according to their topics
- Find proteins performing the same function within a cell
- Image segmentation
- Identify bottlenecks in a network
- ...
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- ...

Unsupervised learning method
(there is no ground truth (usually), and we cannot learn from mistakes!)
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- Different formalisations for different applications
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    - \( k \)-means, \( k \)-medians, \( k \)-centres, etc.
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  - **Graph Clustering**: partition vertices in a graph
    - modularity, conductance, min-cut, etc.
Graph Clustering (applications)

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Graphs and Matrices

Graphs

```
1---2
<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
</table>
4---3
```

Matrices

```
\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{pmatrix}
```
Graphs and Matrices

Graphs

1 2
3 4

Connectivity
Bipartiteness
Number of triangles
Graph Clustering
Graph isomorphism
Maximum Flow
Shortest Paths

Matrices

\[
\begin{pmatrix}
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1 & 0 & 1 & 0 \\
\end{pmatrix}
\]

Eigenvalues
Eigenvectors
Inverse
Determinant
Matrix-powers
...
Graphs and Matrices

Graphs

- Connectivity
- Bipartiteness
- Number of triangles
- Graph Clustering
- Graph isomorphism
- Maximum Flow
- Shortest Paths
- ...

Matrices

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\begin{pmatrix}
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- Eigenvalues
- Eigenvectors
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- ...

...
Outline

Introduction to (Spectral) Graph Theory and Clustering

Matrices, Spectrum and Structure

A Simplified Clustering Problem
Adjacency Matrix

Let $G = (V, E)$ be an undirected graph. The adjacency matrix of $G$ is the $n$ by $n$ matrix $A$ defined as

$$A_{u,v} = \begin{cases} 1 & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise.} \end{cases}$$
Adjacency Matrix

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\end{cases}$$

Properties of $A$:

- The sum of elements in each row/column $i$ equals the degree of the corresponding vertex $i$, $\deg(i)$
- Since $G$ is undirected, $A$ is symmetric
Eigenvalues and Graph Spectrum of $A$

**Eigenvalues and Eigenvectors**

Let $M \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$ is an eigenvalue of $M$ if and only if there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$Mx = \lambda x.$$  

We call $x$ an eigenvector of $M$ corresponding to the eigenvalue $\lambda$. 

**Graph Spectrum**

An undirected graph $G$ is $d$-regular if every degree is $d$, i.e., every vertex has exactly $d$ connections.

Remark: For symmetric matrices we have algebraic multiplicity $=$ geometric multiplicity (otherwise $\geq$).
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Let $A$ be the adjacency matrix of a $d$-regular graph $G$ with $n$ vertices.
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Graph Spectrum

Let \( A \) be the adjacency matrix of a \( d \)-regular graph \( G \) with \( n \) vertices. Then, \( A \) has \( n \) real eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \) and \( n \) corresponding orthonormal eigenvectors \( f_1, \ldots, f_n \).
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11. Spectral Graph Theory © T. Sauerwald 

Matrices, Spectrum and Structure
Eigenvalues and Graph Spectrum of A

Eigenvalues and Eigenvectors

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Remark: For symmetric matrices we have algebraic multiplicity \( = \) geometric multiplicity (otherwise \( \geq \)).
Example 1

**Question:** What are the Eigenvalues and Eigenvectors?

**Solution:**

The three eigenvalues are $\lambda_1 = \lambda_2 = -1, \lambda_3 = 2$.

The three eigenvectors are (for example):

$f_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$,
$f_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ -1 \\ 2 \end{pmatrix}$,
$f_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
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**Question:** What are the Eigenvalues and Eigenvectors?

**Bonus:** Can you find a short-cut to \( \text{det}(A - \lambda \cdot I) \)?

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A = \begin{pmatrix}
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\]
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Bonus: Can you find a short-cut to $\text{det}(A - \lambda \cdot I)$?

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- The three eigenvectors are (for example):

  $f_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$,  $f_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$,  $f_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Laplacian Matrix

Let $G = (V, E)$ be a $d$-regular undirected graph. The (normalised) Laplacian matrix of $G$ is the $n$ by $n$ matrix $L$ defined as

$$L = I - \frac{1}{d}A,$$

where $I$ is the $n \times n$ identity matrix.
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Properties of $L$:
- The sum of elements in each row/column equals zero
- $L$ is symmetric
Correspondence between Adjacency and Laplacian Matrix

\[ A \text{ and } L \text{ have the same set of eigenvectors.} \]

**Exercise:** Prove this correspondence. Hint: Use that \( L = I - \frac{1}{d} A. \)

*Exercise 11/12.1*
Eigenvalues and Graph Spectrum of $L$

Let $M \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$ is an eigenvalue of $M$ if and only if there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that $Mx = \lambda x$.

We call $x$ an eigenvector of $M$ corresponding to the eigenvalue $\lambda$.

Graph Spectrum

Let $L$ be the Laplacian matrix of a $d$-regular graph $G$ with $n$ vertices. Then, $L$ has $n$ real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and $n$ corresponding orthonormal eigenvectors $f_1, \ldots, f_n$. These eigenvalues associated with their multiplicities constitute the spectrum of $G$. 
Useful Facts of Graph Spectrum

**Lemma**

Let $L$ be the Laplacian matrix of an undirected, regular graph $G = (V, E)$ with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$.

1. $\lambda_1 = 0$ with eigenvector $\mathbf{1}$
2. the multiplicity of the eigenvalue 0 is equal to the number of connected components in $G$
Useful Facts of Graph Spectrum

Lemma

Let \( L \) be the Laplacian matrix of an undirected, regular graph \( G = (V, E) \) with eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \).

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2. the multiplicity of the eigenvalue 0 is equal to the number of connected components in \( G \)
3. \( \lambda_n \leq 2 \)
4. \( \lambda_n = 2 \) iff there exists a bipartite connected component.
Useful Facts of Graph Spectrum

Let $L$ be the Laplacian matrix of an undirected, regular graph $G = (V, E)$ with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$.

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2. the multiplicity of the eigenvalue 0 is equal to the number of connected components in $G$
3. $\lambda_n \leq 2$
4. $\lambda_n = 2$ iff there exists a bipartite connected component.

Lemma

The proof of these properties is based on a powerful characterisation of eigenvalues/vectors!
A Min-Max Characterisation of Eigenvalues and Eigenvectors

Let \( M \) be an \( n \) by \( n \) symmetric matrix with eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \). Then,

\[
\lambda_k = \min_{x^{(1)}, \ldots, x^{(k)} \in \mathbb{R}^n \setminus \{0\}, \ x^{(i)} \perp x^{(j)}, \ i \in \{1, \ldots, k\}} \max_{x^{(i)}} \frac{x^{(i)^T M x^{(i)}}}{x^{(i)^T x^{(i)}}}.
\]

The eigenvectors corresponding to \( \lambda_1, \ldots, \lambda_k \) minimise such expression.

Courant-Fischer Min-Max Formula
A Min-Max Characterisation of Eigenvalues and Eigenvectors

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The eigenvectors corresponding to $\lambda_1, \ldots, \lambda_k$ minimise such expression.

$$\lambda_1 = \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^\top \mathbf{M} x}{x^\top x}$$

minimised by an eigenvector $f_1$ for $\lambda_1$. 

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$$\lambda_2 = \min_{x \in \mathbb{R}^n \setminus \{0\}, x \perp f_1} \frac{x^T M x}{x^T x}$$

minimised by $f_2$
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Quadratic Forms of the Laplacian

Let \( L \) be the Laplacian matrix of a \( d \)-regular graph \( G = (V, E) \) with \( n \) vertices. For any \( x \in \mathbb{R}^n \),

\[
x^T L x = \sum_{\{u,v\} \in E} \frac{(x_u - x_v)^2}{d}.
\]
Quadratic Forms of the Laplacian

Lemma

Let $L$ be the Laplacian matrix of a $d$-regular graph $G = (V, E)$ with $n$ vertices. For any $x \in \mathbb{R}^n$,

$$x^T L x = \sum_{\{u,v\} \in E} \frac{(x_u - x_v)^2}{d}.$$ 

Proof:

$$x^T L x = x^T \left( I - \frac{1}{d} A \right) x = x^T x - \frac{1}{d} x^T A x$$

$$= \sum_{u \in V} x_u^2 - \frac{2}{d} \sum_{\{u,v\} \in E} x_u x_v$$

$$= \frac{1}{d} \sum_{\{u,v\} \in E} (x_u^2 + x_v^2 - 2x_u x_v)$$

$$= \sum_{\{u,v\} \in E} \frac{(x_u - x_v)^2}{d}.$$
Question: How can we visualize a complicated object like an unknown graph with many vertices in low-dimensional space?
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Embedding onto Line

Coordinates given by \( x \)

The coordinates in the vector \( x \) indicate how similar/dissimilar vertices are. Edges between dissimilar vertices are penalised quadratically.
Visualising a Graph

**Question:** How can we visualize a complicated object like an unknown graph with many vertices in low-dimensional space?

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Coordinates given by $x$

$$\lambda_2 = \frac{1}{d} \cdot \min_{x \in \mathbb{R}^n \setminus \{0\}, x \perp f_1} \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{\|x\|_2^2}$$
**Visualising a Graph**

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Introduction to (Spectral) Graph Theory and Clustering

Matrices, Spectrum and Structure

A Simplified Clustering Problem
Partition the graph into **connected components** so that any pair of vertices in the same component is connected, but vertices in different components are not.
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A Simplified Clustering Problem

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A Simplified Clustering Problem

Partition the graph into **connected components** so that any pair of vertices in the same component is connected, but vertices in different components are not.

We could obviously solve this easily using DFS/BFS, but let's see how we can tackle this using the **spectrum of L**!
Example 2

Question: What are the Eigenvectors with Eigenvalue 0 of \( L \)?

Solution:

Two smallest eigenvalues are \( \lambda_1 = \lambda_2 = 0 \).

The corresponding two eigenvectors are:

\[
\mathbf{f}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

Thus we can easily solve the simplified clustering problem by computing the eigenvectors with eigenvalue 0.

Next Lecture: A fine-grained approach works even if the clusters are sparsely connected!
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$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

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$$L = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

Solution:

Two smallest eigenvalues are $\lambda_1 = \lambda_2 = 0$. The corresponding two eigenvectors are:

$$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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\end{pmatrix}$

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$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, \hspace{1cm} $f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

(or $f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, \hspace{1cm} $f_2 = \begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$)
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$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,  $f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Thus we can easily solve the simplified clustering problem by computing the eigenvectors with eigenvalue 0.
Question: What are the Eigenvectors with Eigenvalue 0 of $L$?

Solution:
- Two smallest eigenvalues are $\lambda_1 = \lambda_2 = 0$.
- The corresponding two eigenvectors are:

$$ f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} $$

Thus we can easily solve the simplified clustering problem by computing the eigenvectors with eigenvalue 0.

Next Lecture: A fine-grained approach works even if the clusters are sparsely connected!
Let us generalise and formalise the previous example!
Proof of Lemma, 2nd statement (non-examinable)

Let us generalise and formalise the previous example!

Proof (multiplicity of 0 equals the no. of connected components):

1. \( \Rightarrow \) \( \chi_{C_i}(u) = 1 \) for all \( u \in V \) such that \( \chi_{C_i} \)'s are orthogonal.

2. \( \Leftarrow \) \( \chi_{T_i} \)'s are orthogonal if \( \chi_{C_i} \)'s are orthogonal.

there exist \( f_1, \ldots, f_k \) orthonormal such that

\[
\sum_{\{u, v\} \in E} (f_i(u) - f_i(v))^2 = 0 \Rightarrow f_1, \ldots, f_k \text{ constant on connected components.}
\]

as \( f_1, \ldots, f_k \) are pairwise orthogonal.

G must have \( k \) different connected components.
Proof of Lemma, 2nd statement (non-examinable)

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1. (“$\implies$” $cc(G) \leq \text{mult}(0)$). We will show:
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Randomised Algorithms
Lecture 12: Spectral Graph Clustering

Thomas Sauerwald (tms41@cam.ac.uk)
Outline

Conductance, Cheeger’s Inequality and Spectral Clustering

Illustrations of Spectral Clustering and Extension to Non-Regular Graphs

Appendix: Relating Spectrum to Mixing Times (non-examinable)
Partition the graph into pieces (clusters) so that vertices in the same piece have, on average, more connections among each other than with vertices in other clusters.
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Partition the graph into \textbf{pieces (clusters)} so that vertices in the same piece have, on average, more connections among each other than with vertices in other clusters.

Let us for simplicity focus on the case of two clusters!
Conductance

Let $G = (V, E)$ be a $d$-regular and undirected graph and $\emptyset \neq S \subsetneq V$. The **conductance** (edge expansion) of $S$ is

$$\phi(S) := \frac{e(S, S^c)}{d \cdot |S|}$$
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Moreover, the conductance (edge expansion) of the graph $G$ is

$$\phi(G) := \min_{S \subseteq V : 1 \leq |S| \leq n/2} \phi(S)$$
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- $\phi(S) = \frac{5}{9}$
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- $\phi(S) = \frac{5}{9}$
- $\phi(G) \in [0, 1]$ and $\phi(G) = 0$ iff $G$ is disconnected
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- \( \phi(S) = \frac{5}{9} \)
- \( \phi(G) \in [0, 1] \) and \( \phi(G) = 0 \) iff \( G \) is disconnected
- If \( G \) is a complete graph, then
  \[
e(S, V \setminus S) = |S| \cdot (n - |S|) \]
  and
  \[
  \phi(G) \approx 1/2.
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Moreover, the conductance (edge expansion) of the graph $G$ is

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Conductance is NP-hard to compute!

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- If $G$ is a complete graph, then $e(S, V \setminus S) = |S| \cdot (n - |S|)$ and $\phi(G) \approx 1/2$. 

$\phi(G) = 0 \iff G$ is disconnected \iff $\lambda_2(G) = 0$.

What is the relationship between $\phi(G)$ and $\lambda_2(G)$ for connected graphs?
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What is the relationship between \( \phi(G) \) and \( \lambda_2(G) \) for connected graphs?
\( \lambda_2 \) versus Conductance (2/2)

1D Grid (Path)

\[
\begin{align*}
\lambda_2 & \sim n^{-2} \\
\phi & \sim n^{-1}
\end{align*}
\]

2D Grid

\[
\begin{align*}
\lambda_2 & \sim n^{-1} \\
\phi & \sim n^{-1/2}
\end{align*}
\]

3D Grid

\[
\begin{align*}
\lambda_2 & \sim n^{-2/3} \\
\phi & \sim n^{-1/3}
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$\lambda_2$ versus Conductance (2/2)

1D Grid (Path)

$\lambda_2 \sim n^{-2}$

$\phi \sim n^{-1}$

2D Grid

$\lambda_2 \sim n^{-1}$

$\phi \sim n^{-1/2}$

3D Grid

$\lambda_2 \sim n^{-2/3}$

$\phi \sim n^{-1/3}$

Hypercube

$\lambda_2 \sim (\log n)^{-1}$

$\phi \sim (\log n)^{-1}$

Random Graph (Expanders)

$\lambda_2 = \Theta(1)$

$\phi = \Theta(1)$

Binary Tree

$\lambda_2 \sim n^{-1}$

$\phi \sim n^{-1}$
Relating $\lambda_2$ and Conductance

Cheeger’s inequality

Let $G$ be a $d$-regular undirected graph and $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of its Laplacian matrix. Then,

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}.$$
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Spectral Clustering:

1. Compute the eigenvector $x$ corresponding to $\lambda_2$.
2. Order the vertices so that $x_1 \leq x_2 \leq \cdots \leq x_n$ (embed $V$ on $\mathbb{R}$).
3. Try all $n-1$ sweep cuts of the form $(\{1, 2, \ldots, k\}, \{k+1, \ldots, n\})$ and return the one with smallest conductance.

It returns cluster $S \subseteq V$ such that $\phi(S) \leq \sqrt{2\lambda_2}$.

No constant factor worst-case guarantee, but usually works well in practice (see examples later!).
Relating $\lambda_2$ and Conductance

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- It returns cluster $S \subseteq V$ such that $\phi(S) \leq \sqrt{2\lambda_2} \leq 2\sqrt{\phi(G)}$
- no constant factor worst-case guarantee, but usually works well in practice (see examples later!)
- very fast: can be implemented in $O(|E| \log |E|)$ time
Proof of Cheeger’s Inequality (non-examinable)

Proof (of the easy direction):

By the Courant-Fischer Formula,

\[ \lambda_2 = \min_{x \in \mathbb{R}^n \atop x \neq 0, x \perp 1} \frac{x^T L x}{x^T x} \]

Optimisation Problem:
Embed vertices on a line such that sum of squared distances is minimised

Let \( S \subseteq V \) be the subset for which \( \phi(G) \) is minimised. Define \( y \in \mathbb{R}^n \) by:

\[ y_u = \begin{cases} 1 \mid S \mid & \text{if } u \in S, \\ -1 \mid V \setminus S \mid & \text{if } u \in V \setminus S. \end{cases} \]

Since \( y \perp 1 \), it follows that

\[ \lambda_2 \leq 1 \cdot \frac{\sum_{u \sim v} (y_u - y_v)^2}{\sum u y_u^2} = 1 \cdot \frac{|E(S, V \setminus S)| \cdot (\frac{1}{|S|} + \frac{1}{|V \setminus S|})^2}{|S| + |V \setminus S|} \leq 1 \cdot 2 \cdot \phi(G). \]
Proof of Cheeger’s Inequality (non-examinable)

Proof (of the easy direction):

- By the Courant-Fischer Formula,

\[
\lambda_2 = \min_{x \in \mathbb{R}^n} \frac{\sum_{u \sim v} (x_u - x_v)^2}{\sum_{u} x_u^2} \leq \frac{1}{d} \cdot \left| E(S, V \setminus S) \right| \cdot \left( \frac{1}{|S|} + \frac{1}{|V \setminus S|} \right) \leq 2 \cdot \phi(G).
\]

\(d\)
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\]

**Optimisation Problem:** Embed vertices on a line such that sum of squared distances is minimised.
Proof of Cheeger’s Inequality (non-examinable)

Proof (of the easy direction):

- By the Courant-Fischer Formula,

\[
\lambda_2 = \min_{x \in \mathbb{R}^n, x \neq 0, x \perp 1} \frac{x^T L x}{x^T x} = \frac{1}{d} \cdot \min_{x \in \mathbb{R}^n, x \neq 0, x \perp 1} \frac{\sum_{u \sim v} (x_u - x_v)^2}{\sum_u x_u^2}.
\]

- Let \( S \subseteq V \) be the subset for which \( \phi(G) \) is minimised. Define \( y \in \mathbb{R}^n \) by:

\[
y_u = \begin{cases} 
\frac{1}{|S|} & \text{if } u \in S, \\
-\frac{1}{|V\setminus S|} & \text{if } u \in V \setminus S.
\end{cases}
\]
Proof of Cheeger’s Inequality (non-examinable)

Proof (of the easy direction):

- By the Courant-Fischer Formula,

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\lambda_2 = \min_{\substack{x \in \mathbb{R}^n \\
 x \neq 0, x \perp 1}} \frac{x^T L x}{x^T x} = \frac{1}{d} \cdot \min_{\substack{x \in \mathbb{R}^n \\
 x \neq 0, x \perp 1}} \frac{\sum_{u \sim v} (x_u - x_v)^2}{\sum_u x_u^2}.
\]

- Let \( S \subseteq V \) be the subset for which \( \phi(G) \) is minimised. Define \( y \in \mathbb{R}^n \) by:

\[
y_u = \begin{cases} 
\frac{1}{|S|} & \text{if } u \in S, \\
-\frac{1}{|V \setminus S|} & \text{if } u \in V \setminus S.
\end{cases}
\]

- Since \( y \perp 1 \), it follows that

\[
\lambda_2 \leq \frac{1}{d} \cdot \frac{\sum_{u \sim v} (y_u - y_v)^2}{\sum_u y_u^2} = \frac{1}{d} \cdot \frac{|E(S, V \setminus S)| \cdot \left( \frac{1}{|S|} + \frac{1}{|V \setminus S|} \right)^2}{\frac{1}{|S|} + \frac{1}{|V \setminus S|}}
\]

\[
= \frac{1}{d} \cdot |E(S, V \setminus S)| \cdot \left( \frac{1}{|S|} + \frac{1}{|V \setminus S|} \right)
\]

\[
\leq \frac{1}{d} \cdot 2 \cdot \frac{|E(S, V \setminus S)|}{|S|} = 2 \cdot \phi(G). \quad \square
\]
Outline

Conductance, Cheeger’s Inequality and Spectral Clustering

Illustrations of Spectral Clustering and Extension to Non-Regular Graphs

Appendix: Relating Spectrum to Mixing Times (non-examinable)
Illustration on a small Example

\[ A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \]

\[ L = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25 \]

\[ v = \left( -0.425, +0.263, -0.263, -0.425, +0.425, +0.263, -0.263, +0.425 \right)^T \]
Illustration on a small Example

\[ A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix} \]

\[ L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix} \]
Illustration on a small Example

\[
A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \\
\end{pmatrix}
\]

\[
\lambda_2 = 1 - \sqrt{5}/3 \approx 0.25
\]

\[
v = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T
\]
Illustration on a small Example

\[ \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \]

\[ \mathbf{L} = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \]

\[ \mathbf{\lambda}_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25 \]

\[ \mathbf{v} = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]
Illustration on a small Example

\[ A = \begin{pmatrix}
  0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
  1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
  1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
  0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
  1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 
\end{pmatrix}, \quad L = \begin{pmatrix}
  1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
  0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
 -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
 -\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
 0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
 -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 
\end{pmatrix}\]

\[ \lambda_2 = 1 - \sqrt{5}/3 \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]
Illustration on a small Example

\[ A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix} \quad \quad L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix} \]

\[ \lambda_2 = 1 - \frac{\sqrt{5}}{3} \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]
Illustration on a small Example

\[ A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & \frac{-1}{3} & \frac{-1}{3} & 0 & 0 & \frac{-1}{3} & 0 \\ 0 & 1 & 0 & 0 & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & 0 \\ \frac{-1}{3} & 0 & 1 & \frac{-1}{3} & 0 & 0 & \frac{-1}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{-1}{3} & 1 & 0 & 0 & \frac{-1}{3} & 0 \\ 0 & \frac{-1}{3} & 0 & 0 & 1 & \frac{-1}{3} & 0 & \frac{-1}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & \frac{-1}{3} & 1 & 0 & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & 0 & \frac{-1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{3} & 0 & \frac{-1}{3} & \frac{-1}{3} & 0 & 1 \end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]
Illustration on a small Example

\[
A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix}
\]

\[
\lambda_2 = 1 - \sqrt{5}/3 \approx 0.25
\]

\[
v = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T
\]
Illustration on a small Example

\[ A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \]

\[ L = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{5}/3 \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]
Illustration on a small Example

\[
A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix}
\]

\[\lambda_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25\]

\[v = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T\]
Illustration on a small Example

\[ A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}, \quad
L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{5}/3 \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]
Illustration on a small Example

\[ A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix} \quad L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25 \]

\[ v = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]
Illustration on a small Example

\[
A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix}
\]

\[\lambda_2 = 1 - \frac{\sqrt{5}}{3} \approx 0.25\]
\[v = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T\]
Illustration on a small Example

\[ A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}, \quad
L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix}\]

\[ \lambda_2 = 1 - \sqrt{5}/3 \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]
Illustration on a small Example

\[
A = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix}
\]

\[
\lambda_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25
\]

\[
v = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T
\]

Sweep: 2
Conductance: 0.666
Illustration on a small Example

\[ A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix} \quad L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]

Sweep: 3

Conductance: 0.333
Illustration on a small Example

\[ A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \]

\[ L = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \]

\[ \lambda_2 = 1 - \frac{\sqrt{5}}{3} \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]

Sweep: 4
Conductance: 0.166
Illustration on a small Example

\[ A = \begin{pmatrix}
  0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
  1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
  1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
  0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
  1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix} \quad L = \begin{pmatrix}
  1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
  0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
  -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
  -\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
  0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
  0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
  -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
  0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]

Sweep: 5
Conductance: 0.333
Illustration on a small Example

\[ A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \]

\[ L = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{\frac{5}{3}} \approx 0.25 \]

\[ \nu = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]

Sweep: 6
Conductance: 0.666
Illustration on a small Example

\[ A = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 
\end{pmatrix} \quad L = \begin{pmatrix}
1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 
\end{pmatrix} \]

\[ \lambda_2 = 1 - \sqrt{5}/3 \approx 0.25 \]

\[ v = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T \]

Sweep: 7
Conductance: 1
Physical Interpretation of the Minimisation Problem

- For each edge \( \{u, v\} \in E(G) \), add spring between pins at \( x_u \) and \( x_v \)
Physical Interpretation of the Minimisation Problem

- For each edge \( \{u, v\} \in E(G) \), add spring between pins at \( x_u \) and \( x_v \).
Physical Interpretation of the Minimisation Problem

- For each edge \( \{u, v\} \in E(G) \), add spring between pins at \( x_u \) and \( x_v \)
- The potential energy at each spring is \( (x_u - x_v)^2 \)
Physical Interpretation of the Minimisation Problem

- For each edge \( \{u, v\} \in E(G) \), add spring between pins at \( x_u \) and \( x_v \)
- The potential energy at each spring is \( (x_u - x_v)^2 \)
- Courant-Fisher characterisation:

\[
\lambda_2 = \min_{\substack{x \in \mathbb{R}^n \setminus \{0\} \\
 x \perp 1}} \frac{x^T L x}{x^T x}
\]
Physical Interpretation of the Minimisation Problem

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\]

- In our example, we found out that \( \lambda_2 \approx 0.25 \)
Physical Interpretation of the Minimisation Problem

- For each edge \( \{u, v\} \in E(G) \), add spring between pins at \( x_u \) and \( x_v \)
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- Courant-Fisher characterisation:
  \[
  \lambda_2 = \min_{x \in \mathbb{R}^n \setminus \{0\}, x \perp 1} \frac{x^T L x}{x^T x}
  \]

- In our example, we found out that \( \lambda_2 \approx 0.25 \)
- The eigenvector \( x \) on the last slide is normalised (i.e., \( \|x\|_2^2 = 1 \)). Hence,
Physical Interpretation of the Minimisation Problem

- For each edge \( \{u, v\} \in E(G) \), add spring between pins at \( x_u \) and \( x_v \)
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- Courant-Fisher characterisation:

\[
\lambda_2 = \min_{\substack{x \in \mathbb{R}^n \setminus \{0\} \\
\text{x \perp 1}}} \frac{x^T L x}{x^T x} = \frac{1}{d} \cdot \min_{\substack{x \in \mathbb{R}^n \\
\|x\|_2^2 = 1, x \perp 1}} (x_u - x_v)^2
\]

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Physical Interpretation of the Minimisation Problem

- For each edge \( \{u, v\} \in E(G) \), add spring between pins at \( x_u \) and \( x_v \)
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\lambda_2 = \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T L x}{x^T x} = \frac{1}{d} \cdot \min_{\|x\|_2^2 = 1, x \perp 1} (x_u - x_v)^2
\]

- In our example, we found out that \( \lambda_2 \approx 0.25 \)
- The eigenvector \( x \) on the last slide is normalised (i.e., \( \|x\|_2^2 = 1 \)). Hence,

\[
\lambda_2 = \frac{1}{3} \cdot \left( (x_1 - x_3)^2 + (x_1 - x_4)^2 + (x_1 - x_7)^2 + \cdots + (x_6 - x_8)^2 \right)
\]
For each edge \( \{u, v\} \in E(G) \), add spring between pins at \( x_u \) and \( x_v \).

The potential energy at each spring is \( (x_u - x_v)^2 \).

**Courant-Fisher characterisation:**

\[
\lambda_2 = \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T L x}{x^T x} = \frac{1}{d} \cdot \min_{x \in \mathbb{R}^n} \frac{1}{\|x\|_2^2} (x_u - x_v)^2
\]

In our example, we found out that \( \lambda_2 \approx 0.25 \).

The eigenvector \( x \) on the last slide is normalised (i.e., \( \|x\|_2^2 = 1 \)). Hence,

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\]
Let us now look at an example of a non-regular graph!
The (normalised) Laplacian matrix of $G = (V, E, w)$ is the $n$ by $n$ matrix

$$L = I - D^{-1/2} AD^{-1/2}$$

where $D$ is a diagonal $n \times n$ matrix such that $D_{uu} = \text{deg}(u) = \sum_{v: \{u, v\} \in E} w(u, v)$, and $A$ is the weighted adjacency matrix of $G$. 

12. Clustering © T. Sauerwald Illustrations of Spectral Clustering and Extension to Non-Regular Graphs
The (normalised) Laplacian matrix of $G = (V, E, w)$ is the $n$ by $n$ matrix

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$$L = \begin{pmatrix}
1 & -16/25 & 0 & -9/20 \\
-16/25 & 1 & -9/20 & 0 \\
0 & -9/20 & 1 & -7/16 \\
-9/20 & 0 & -7/16 & 1
\end{pmatrix}$$
The (normalised) Laplacian matrix of \( G = (V, E, w) \) is the \( n \) by \( n \) matrix

\[
L = I - D^{-1/2}AD^{-1/2}
\]

where \( D \) is a diagonal \( n \times n \) matrix such that \( D_{uu} = \text{deg}(u) = \sum_{v: \{u,v\} \in E} w(u,v) \), and \( A \) is the weighted adjacency matrix of \( G \).

- \( L_{uv} = -\frac{w(u,v)}{\sqrt{d_u d_v}} \) for \( u \neq v \)
- \( L \) is symmetric
- If \( G \) is \( d \)-regular, \( L = I - \frac{1}{d} \cdot A \).
Conductance (General Version)

Let $G = (V, E, w)$ and $\emptyset \subsetneq S \subsetneq V$. The conductance (edge expansion) of $S$ is

$$\phi(S) := \frac{w(S, S^c)}{\min \{\text{vol}(S), \text{vol}(S^c)\}},$$

where $w(S, S^c) := \sum_{u \in S, v \in S^c} w(u, v)$ and $\text{vol}(S) := \sum_{u \in S} d(u)$. Moreover, the conductance (edge expansion) of $G$ is

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Spectral Clustering (General Version):

1. Compute the eigenvector $x$ corresponding to $\lambda_2$ and $y = D^{-1/2}x$. 
Conductance and Spectral Clustering (General Version)

Let $G = (V, E, w)$ and $\emptyset \subsetneq S \subsetneq V$. The conductance (edge expansion) of $S$ is

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Spectral Clustering (General Version):

1. Compute the eigenvector $x$ corresponding to $\lambda_2$ and $y = D^{-1/2}x$.
2. Order the vertices so that $y_1 \leq y_2 \leq \cdots \leq y_n$ (embed $V$ on $\mathbb{R}$).
Let $G = (V, E, w)$ and $\emptyset \subset S \subset V$. The conductance (edge expansion) of $S$ is

$$\phi(S) := \frac{w(S, S^c)}{\min \{\text{vol}(S), \text{vol}(S^c)\}},$$

where $w(S, S^c) := \sum_{u \in S, v \in S^c} w(u, v)$ and $\text{vol}(S) := \sum_{u \in S} d(u)$. Moreover, the conductance (edge expansion) of $G$ is

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### Spectral Clustering (General Version):
1. Compute the eigenvector $x$ corresponding to $\lambda_2$ and $y = D^{-1/2}x$.
2. Order the vertices so that $y_1 \leq y_2 \leq \cdots \leq y_n$ (embed $V$ on $\mathbb{R}$).
3. Try all $n - 1$ sweep cuts of the form ($\{1, 2, \ldots, k\}, \{k + 1, \ldots, n\}$) and return the one with smallest conductance.
Stochastic Block Model and 1D-Embedding

Stochastic Block Model

\[ G = (V, E) \text{ with clusters } S_1, S_2 \subseteq V, 0 \leq q < p \leq 1 \]

\[ P\left[\{u, v\} \in E\right] = \begin{cases} 
  p & \text{if } u, v \in S_i, \\
  q & \text{if } u \in S_i, v \in S_j, i \neq j. 
\end{cases} \]
Stochastic Block Model and 1D-Embedding

**Stochastic Block Model**

\[ G = (V, E) \text{ with clusters } S_1, S_2 \subseteq V, \; 0 \leq q < p \leq 1 \]

\[
P[\{u, v\} \in E] = \begin{cases} 
  p & \text{if } u, v \in S_i, \\
  q & \text{if } u \in S_i, v \in S_j, i \neq j.
\end{cases}
\]

**Here:**
- \(|S_1| = 80,\)
- \(|S_2| = 120\)
- \(p = 0.08\)
- \(q = 0.01\)
Stochastic Block Model

\[ G = (V, E) \text{ with clusters } S_1, S_2 \subseteq V, \ 0 \leq q < p \leq 1 \]

\[ P \left[ \{u, v\} \in E \right] = \begin{cases} 
  p & \text{if } u, v \in S_i, \\
  q & \text{if } u \in S_i, v \in S_j, i \neq j. 
\end{cases} \]

Here:
- \( |S_1| = 80 \), \( |S_2| = 120 \)
- \( p = 0.08 \)
- \( q = 0.01 \)

Number of Vertices: 200
Number of Edges: 919
Eigenvalue 1 : -1.196843147956568e-16
Eigenvalue 2 : 0.1543784937248489
Eigenvalue 3 : 0.37049909753568877
Eigenvalue 4 : 0.39770640242147404
Eigenvalue 5 : 0.4316114413430584
Eigenvalue 6 : 0.44379221120189777
Eigenvalue 7 : 0.4564011652684181
Eigenvalue 8 : 0.4632911204500282
Eigenvalue 9 : 0.474638606357877
Eigenvalue 10 : 0.4814019607292904
Stochastic Block Model

\[
G = (V, E) \text{ with clusters } S_1, S_2 \subseteq V, 0 \leq q < p \leq 1
\]

Here:
- \(|S_1| = 80, |S_2| = 120\)
- \(p = 0.08\)
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Number of Vertices: 200
Number of Edges: 919
Eigenvalue 1: -1.1968431479565368e-16
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Eigenvalue 5: 0.4316114413430584
Eigenvalue 6: 0.44379221120189777
Eigenvalue 7: 0.4564011652684181
Eigenvalue 8: 0.4632911204500282
Eigenvalue 9: 0.474638606357877
Eigenvalue 10: 0.4814019607292904
Drawing the 2D-Embedding

12. Clustering © T. Sauerwald
Illustrations of Spectral Clustering and Extension to Non-Regular Graphs
Spectral Clustering
Best Solution found by Spectral Clustering

- Step: 78
- Threshold: −0.027
- Partition Sizes: 78/122
- Cut Edges: 84
- Conductance: 0.145
Clustering induced by Blocks

- Step: 1
- Threshold: 0
- Partition Sizes: 80/120
- Cut Edges: 88
- Conductance: 0.1486

(0, 0)
Additional Example: Stochastic Block Models with 3 Clusters

Graph $G = (V, E)$ with clusters $S_1, S_2, S_3 \subseteq V$; $0 \leq q < p \leq 1$

$$P \left[ \{u, v\} \in E \right] = \begin{cases} p & u, v \in S_i \\ q & u \in S_i, v \in S_j, i \neq j \end{cases}$$
Graph $G = (V, E)$ with clusters $S_1, S_2, S_3 \subseteq V; \quad 0 \leq q < p \leq 1$

$P[\{u, v\} \in E] = \begin{cases} 
  p & u, v \in S_i \\
  q & u \in S_i, v \in S_j, i \neq j 
\end{cases}$

$|V| = 300, |S_i| = 100$
$p = 0.08, q = 0.01.$
Additional Example: Stochastic Block Models with 3 Clusters

Graph $G = (V, E)$ with clusters $S_1, S_2, S_3 \subseteq V$; $0 \leq q < p \leq 1$

$$P \left[ \{u, v\} \in E \right] = \begin{cases} p & u, v \in S_i \\ q & u \in S_i, v \in S_j, i \neq j \end{cases}$$

$|V| = 300$, $|S_i| = 100$
$p = 0.08$, $q = 0.01$.

Spectral embedding
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$|V| = 300, |S_i| = 100$
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How to Choose the Cluster Number $k$

- If $k$ is unknown:
  - small $\lambda_k$ means there exist $k$ sparsely connected subsets in the graph
    (recall: $\lambda_1 = \ldots = \lambda_k = 0$ means there are $k$ connected components)

In the latter example $\lambda = \{0, 0, 0.20, 0.22, 0.43, \ldots\} = \Rightarrow k = 3$.

In the former example $\lambda = \{0, 0.15, 0.37, 0.40, 0.43, \ldots\} = \Rightarrow k = 2$.

For $k = 2$ use sweep-cut extract clusters. For $k \geq 3$ use embedding in $k$-dimensional space and apply $k$-means (geometric clustering).
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- In the latter example \( \lambda = \{0, 0.20, 0.22, 0.43, 0.45, \ldots \} \Rightarrow k = 3. \]
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Another Example

- nodes represent math topics taught within 4 weeks of a Mathcamp
- node colours represent to the week in which they thought
- teachers were asked to assign weights in 0 – 10 indicating how closely related two classes are

(many thanks to Kalina Jasinska)
Summary: Spectral Clustering

Illustration on a (very) small Example

\[
A = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
\]

Sweep: 2
Edge Expansion: 0.666

Clustering Demos T.S.

A Larger Example: Sweep Cut
Threshold: 0.00
Partition Sizes: 201 / 200
Cut Edges / Total Edges: 53 / 2601
Edge Expansion: 0.021

Spectral Embedding onto Line
Compute Sweep Cuts

- Given any graph (adjacency matrix)
- Graph Spectrum (computable in poly-time)
  - \( \lambda_2 \) (relates to connectivity)
  - \( \lambda_n \) (relates to bipartiteness)
  - \( \ldots \)

- Cheeger’s Inequality
  - relates \( \lambda_2 \) to conductance
  - unbounded approximation ratio
  - effective in practice

Clustering © T. Sauerwald
Illustrations of Spectral Clustering and Extension to Non-Regular Graphs
Outline

Conductance, Cheeger’s Inequality and Spectral Clustering

Illustrations of Spectral Clustering and Extension to Non-Regular Graphs

Appendix: Relating Spectrum to Mixing Times (non-examinable)
Which graph has a “cluster-structure”?
- Which graph has a “cluster-structure”?
- Which graph mixes faster?
Recall: If the underlying graph $G$ is connected, undirected and $d$-regular, then the random walk converges towards the stationary distribution $\pi = (1/n, \ldots, 1/n)$, which satisfies $\pi \mathbf{P} = \pi$. 

⇒ This implies for $t = O(\log n \log(1/\lambda)) = O(\log n 1/\lambda)$,

$$\|x \mathbf{P}^t - \pi\|_2 \leq \frac{1}{4}.$$
Recall: If the underlying graph $G$ is connected, undirected and $d$-regular, then the random walk converges towards the stationary distribution $\pi = (1/n, \ldots, 1/n)$, which satisfies $\pi P = \pi$.

Here all vector multiplications (including eigenvectors) will always be from the left!
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**Lemma**

Consider a lazy random walk on a connected, undirected and $d$-regular graph. Then for any initial distribution $x$,

$$\left\| xP^t - \pi \right\|_2 \leq \lambda^t,$$

with $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ as eigenvalues and $\lambda := \max\{|\lambda_2|, |\lambda_n|\}$. 

---

12. Clustering © T. Sauerwald  
Appendix: Relating Spectrum to Mixing Times (non-examinable)  
24
Recall: If the underlying graph $G$ is connected, undirected and $d$-regular, then the random walk converges towards the stationary distribution $\pi = (1/n, \ldots, 1/n)$, which satisfies $\pi P = \pi$.

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due to laziness, $\lambda_n \geq 0$
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Lemma

Consider a lazy random walk on a connected, undirected and $d$-regular graph. Then for any initial distribution $x$,

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with $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ as eigenvalues and $\lambda := \max\{|\lambda_2|, |\lambda_n|\}$.

⇒ This implies for $t = \mathcal{O}\left(\frac{\log n}{\log(1/\lambda)}\right) = \mathcal{O}\left(\frac{\log n}{1-\lambda}\right)$,

$$\|xP^t - \pi\|_{tv} \leq \frac{1}{4}.$$

due to laziness, $\lambda_n \geq 0$
Proof of Lemma (non-examinable)

Express $x$ in terms of the orthonormal basis of $P$, $v_1 = \pi, v_2, \ldots, v_n$:

$$x = \sum_{i=1}^{n} \alpha_i v_i.$$  

Since $x$ is a probability vector and all $v_i \geq 2$ are orthogonal to $\pi$, $\alpha_1 = 1$.

$$\|x P - \pi\|^2 \leq \lambda^2 \cdot \|x - \pi\|^2 \leq \lambda^2.$$  

since the $v_i$'s are orthogonal.
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- Express $x$ in terms of the orthonormal basis of $P$, $v_1 = \pi, v_2, \ldots, v_n$:
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\[ \|x - \pi\|_2^2 + \|\pi\|_2^2 \leq \|x\|_2^2 \leq 1 \]
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$$\Rightarrow \quad \|xP - \pi\|_2^2$$
Proof of Lemma (non-examinable)

- Express $x$ in terms of the orthonormal basis of $P$, $v_1 = \pi, v_2, \ldots, v_n$:
  \[ x = \sum_{i=1}^{n} \alpha_i v_i. \]

- Since $x$ is a probability vector and all $v_i \geq 2$ are orthogonal to $\pi$, $\alpha_1 = 1$.

  \[ \|xP - \pi\|_2^2 = \left\| \left( \sum_{i=1}^{n} \alpha_i v_i \right) P - \pi \right\|_2^2 \]
Proof of Lemma (non-examinable)

- Express \( x \) in terms of the orthonormal basis of \( P \), \( v_1 = \pi \), \( v_2, \ldots, v_n \):
  \[
x = \sum_{i=1}^{n} \alpha_i v_i.
  \]

- Since \( x \) is a probability vector and all \( v_i \geq 2 \) are orthogonal to \( \pi \), \( \alpha_1 = 1 \).

\[\Rightarrow\]

\[
\|x P - \pi\|^2 = \left\| \left( \sum_{i=1}^{n} \alpha_i v_i \right) P - \pi \right\|^2
\]

\[
= \left\| \pi + \sum_{i=2}^{n} \alpha_i \lambda_i v_i - \pi \right\|^2
\]

Hence

\[
\|x P - \pi\|^2 \leq \lambda^2 t \cdot \|x - \pi\|^2
\]
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\[ \Rightarrow \quad \| xP - \pi \|_2^2 = \left\| \left( \sum_{i=1}^{n} \alpha_i v_i \right) P - \pi \right\|_2^2 \]
\[ = \left\| \pi + \sum_{i=2}^{n} \alpha_i \lambda_i v_i - \pi \right\|_2^2 \]
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\[
\Rightarrow \quad \| xP - \pi \|^2_2 = \left\| \left( \sum_{i=1}^{n} \alpha_i v_i \right) P - \pi \right\|^2_2
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since the \( v_i \)'s are orthogonal

\[
= \left\| \sum_{i=2}^{n} \alpha_i \lambda_i v_i \right\|^2_2
\]

\[
= \sum_{i=2}^{n} \| \alpha_i \lambda_i v_i \|^2_2
\]

\[
\leq \lambda_2^2 \sum_{i=2}^{n} \| \alpha_i \lambda_i v_i \|^2_2
\]

\[
\leq \lambda_2^2 \cdot (1)
\]
Proof of Lemma (non-examinable)

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$$= \sum_{i=2}^{n} \|\alpha_i \lambda_i v_i\|^2_2$$

since the $v_i$'s are orthogonal

$$\leq \lambda^2 \sum_{i=2}^{n} \|\alpha_i v_i\|^2_2 = \lambda^2 \left\| \sum_{i=2}^{n} \alpha_i v_i \right\|^2_2$$

Hence $\|xP - \pi\|^2_2 \leq \lambda^2 \cdot 1^2$. Since the $v_i$'s are orthogonal, the $\alpha_i$'s are also orthogonal, and the $\lambda_i$'s are the entries of the matrix $P$. The $\pi$ vector is the first eigenvector of $P$ with eigenvalue 1. Thus, the expression simplifies to

$$\|xP - \pi\|^2_2 = \|x - \pi\|_2^2 = \|\sum_{i=2}^{n} \alpha_i v_i\|^2_2$$

since the $v_i$'s are orthogonal.
Proof of Lemma (non-examinable)

- Express $x$ in terms of the orthonormal basis of $P$, $v_1 = \pi, v_2, \ldots, v_n$:

$$x = \sum_{i=1}^{n} \alpha_i v_i.$$ 

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$$\Rightarrow \quad \|xP - \pi\|_2^2 = \left\|\left(\sum_{i=1}^{n} \alpha_i v_i\right)P - \pi\right\|_2^2$$

$$= \left\|\pi + \sum_{i=2}^{n} \alpha_i \lambda_i v_i - \pi\right\|_2^2$$

\[ \leq \lambda^2 \sum_{i=2}^{n} \|\alpha_i v_i\|_2^2 \]

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\[ \leq \lambda^2 \sum_{i=2}^{n} \|\alpha_i v_i\|_2^2 = \lambda^2 \left\|\sum_{i=2}^{n} \alpha_i v_i\right\|_2^2 = \lambda^2 \|x - \pi\|_2^2 \]
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$$\leq \lambda^2 \sum_{i=2}^{n} \left\| \alpha_i v_i \right\|_2^2 = \lambda^2 \left\| \sum_{i=2}^{n} \alpha_i v_i \right\|_2^2 = \lambda^2 \left\| x - \pi \right\|_2^2$$

- Hence $\|xP^t - \pi\|_2^2$
Proof of Lemma (non-examinable)

- Express $x$ in terms of the orthonormal basis of $P$, $v_1 = \pi, v_2, \ldots, v_n$:

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- Hence $\|xP^t - \pi\|^2_2 \leq \lambda^{2t} \cdot \|x - \pi\|^2_2$
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- Express $x$ in terms of the orthonormal basis of $P$, $v_1 = \pi, v_2, \ldots, v_n$:

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- Hence $\|xP^t - \pi\|_2^2 \leq \lambda^{2t} \cdot \|x - \pi\|_2^2 \leq \lambda^{2t} \cdot 1$.

$$\|x - \pi\|_2^2 + \|\pi\|_2^2 = \|x\|_2^2 \leq 1$$
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Thank you and Best Wishes for the Exam!