Randomised Algorithms

Lecture 4: Markov Chains and Mixing Times

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Lent 2024

UNIVERSITY OF CAMBRIDGE

Outline

Applications of Markov Chains in Computer Science

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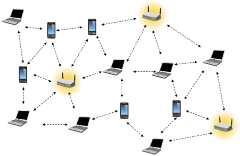
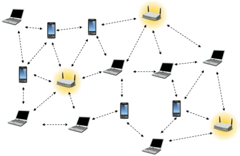


Broadcasting

Applications of Markov Chains in Computer Science



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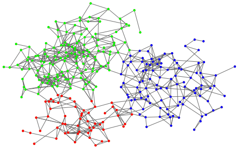
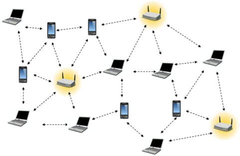


Clustering

Applications of Markov Chains in Computer Science



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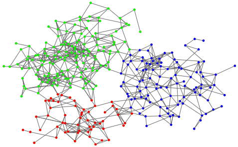

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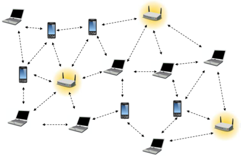

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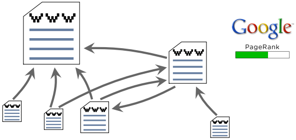

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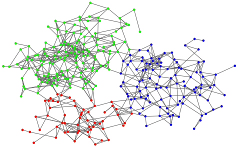

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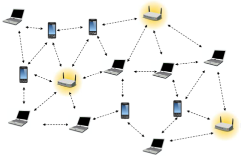

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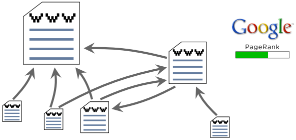

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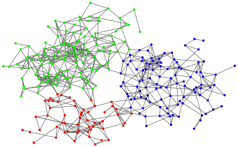

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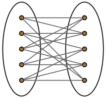
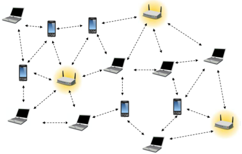


Sampling and Optimisation

Applications of Markov Chains in Computer Science



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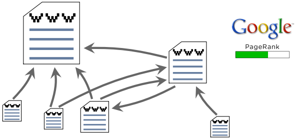

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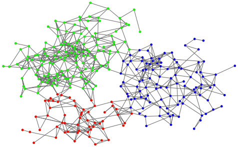

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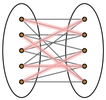

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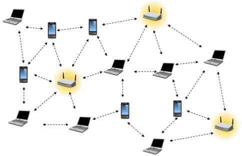

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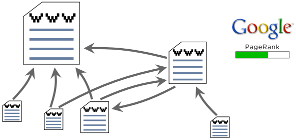

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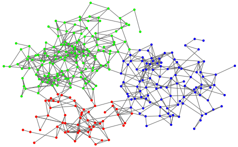

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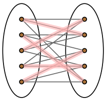

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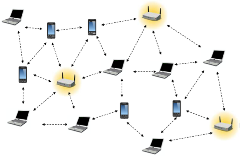

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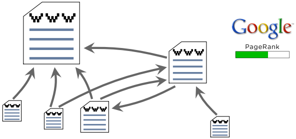

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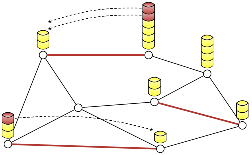
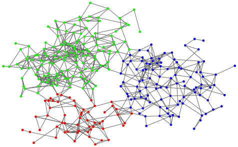


Load Balancing



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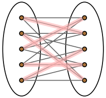

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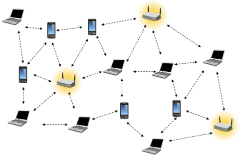

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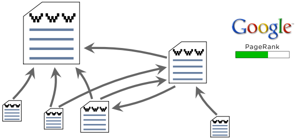

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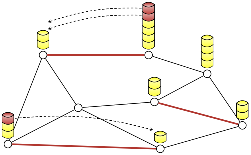

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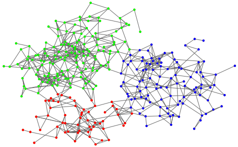

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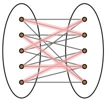

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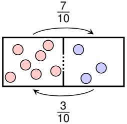

Particle Processes

Markov Chains

Markov Chain (Discrete Time and State, Time Homogeneous)

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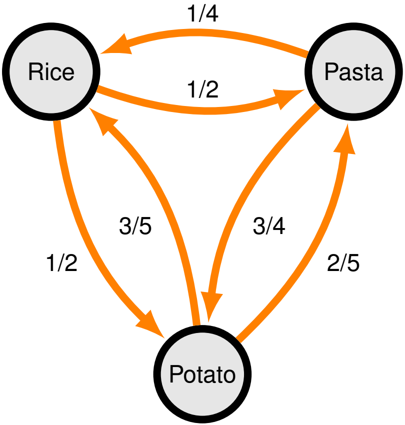
- For all $0 \leq t_1 < t_2, x \in \Omega$,

$$\mathbf{P}[X_{t_2} = x] = \sum_{y \in \Omega} \mathbf{P}[X_{t_2} = x \mid X_{t_1} = y] \cdot \mathbf{P}[X_{t_1} = y].$$

What does a Markov Chain Look Like?

Example: the carbohydrate served with lunch in the college cafeteria.



This has transition matrix:

$$
\begin{array}{c} \\ P = \end{array}
\begin{array}{c}
\begin{array}{ccc} \text{Rice} & \text{Pasta} & \text{Potato} \end{array} \\
\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 3/5 & 2/5 & 0 \end{bmatrix}
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$\Rightarrow$ can replace ρ by any (load) vector and view P as a balancing matrix!

Stopping and Hitting Times

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Example - College Carbs Stopping times:

✓ "We had rice yesterday"

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A non-negative integer random variable τ is a stopping time for $(X_t)_{t\geq 0}$ if for every $s \geq 0$ the event $\{\tau = s\}$ depends only on $X_0, \ldots, X_s$.

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Some distinguish between $\tau_y^+ = \min\{t \geq 1 \colon X_t = y\}$ and $\tau_y = \min\{t \geq 0 \colon X_t = y\}$

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A Useful Identity

Hitting times are the solution to a set of linear equations:

$$h(x, y) \overset{\text{Markov Prop.}}{=} 1 + \sum_{z \in \Omega \setminus \{y\}} P(x, z) \cdot h(z, y) \qquad \forall x \neq y \in \Omega.$$

Outline

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A Markov Chain is irreducible if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$.

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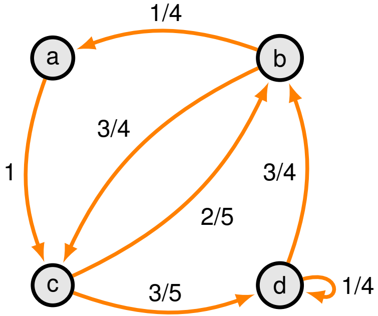

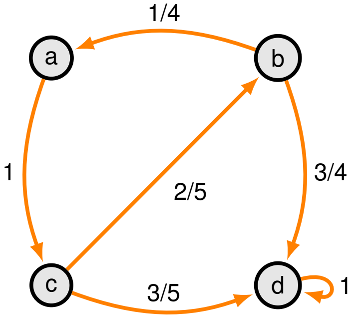

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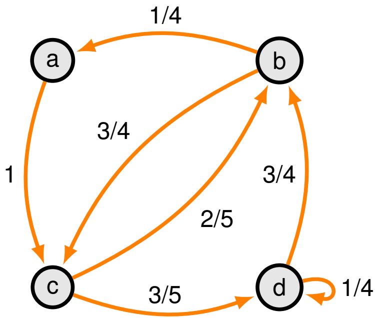

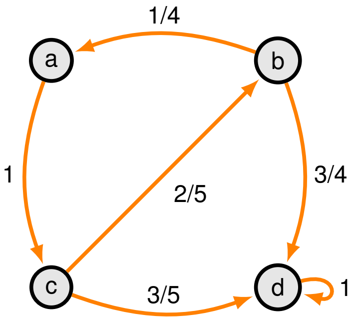

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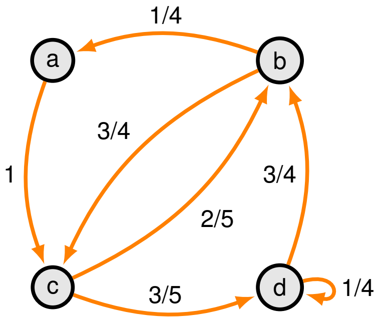

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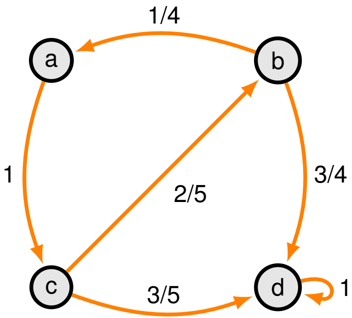

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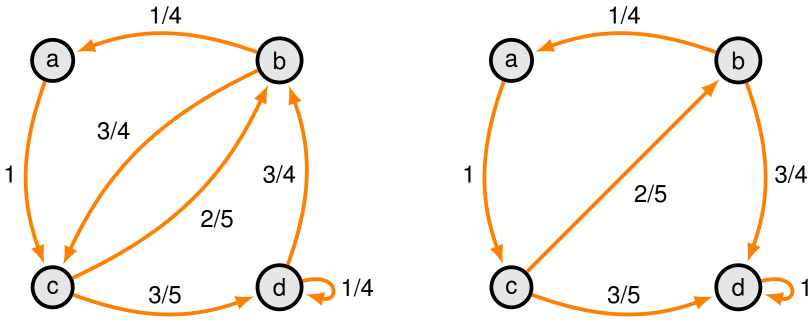

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Finite Hitting Time Theorem

For any states x and y of a finite irreducible Markov Chain $h(x, y) < \infty$.

Stationary Distribution

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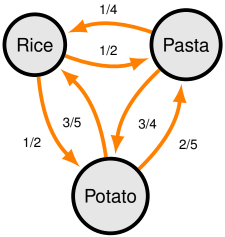

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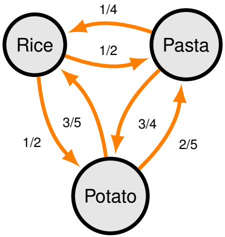

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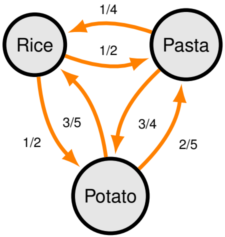

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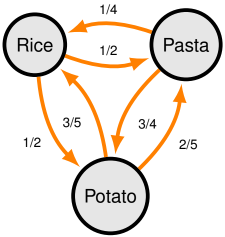

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Existence and Uniqueness of a Positive Stationary Distribution

Let P be finite, irreducible M.C., then there exists a unique probability distribution π on Ω such that $\pi = \pi P$ and $\pi(x) = 1/h(x,x) > 0, \forall x \in \Omega$.

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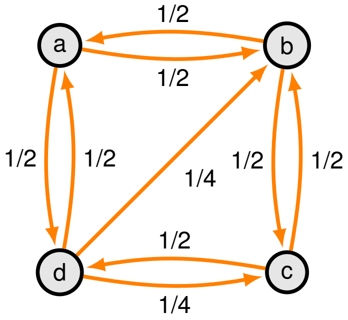

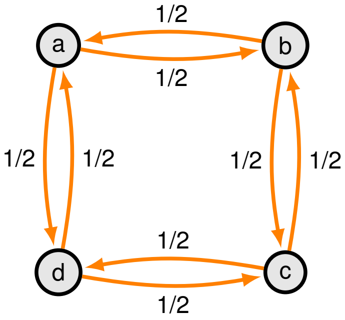

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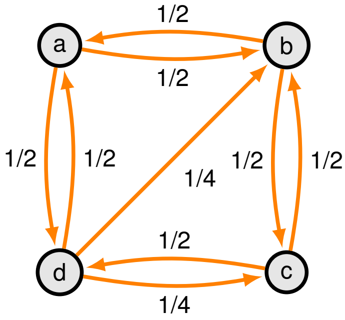

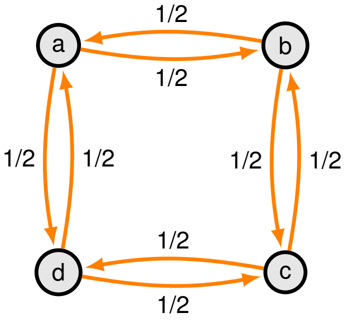

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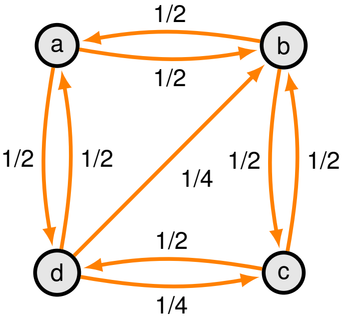

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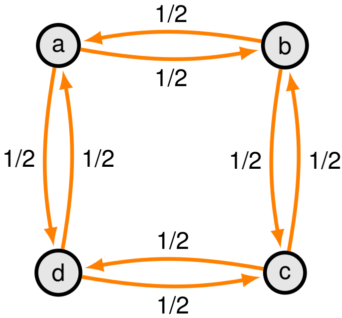

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- We will prove a simpler version of the Convergence Theorem after introducing Spectral Graph Theory.

Convergence to Stationarity (Example)

- Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
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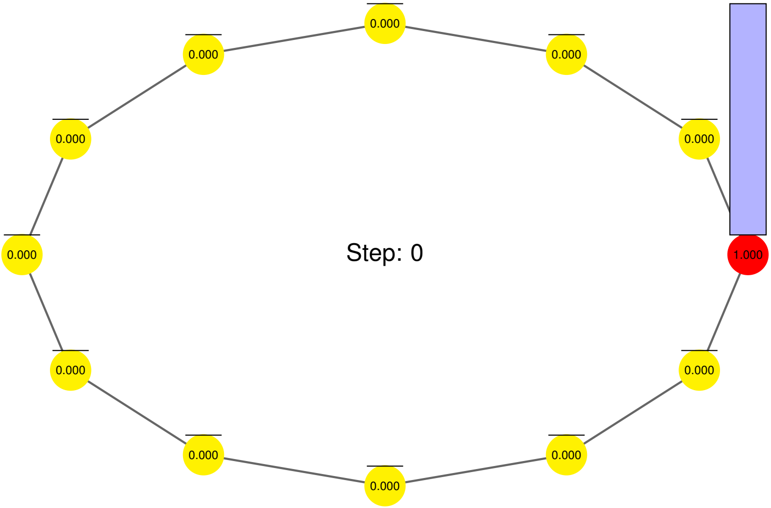

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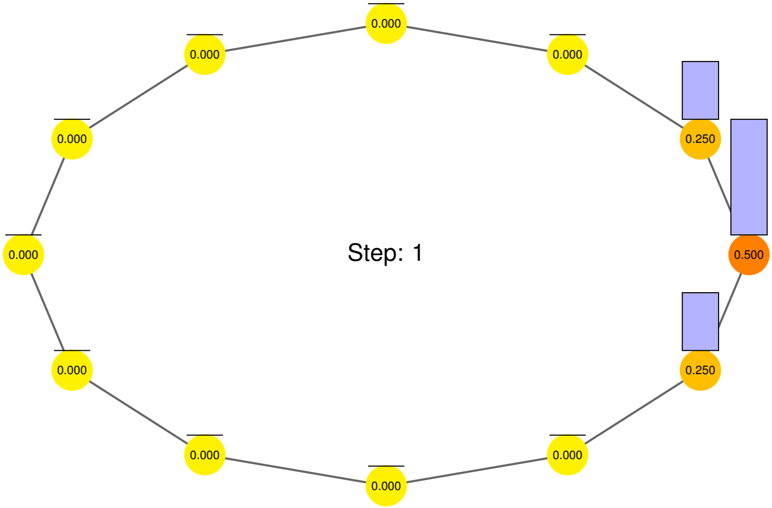

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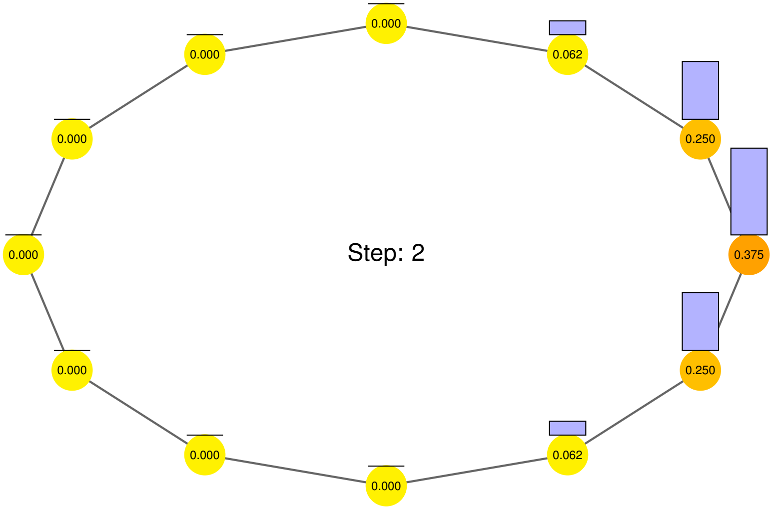

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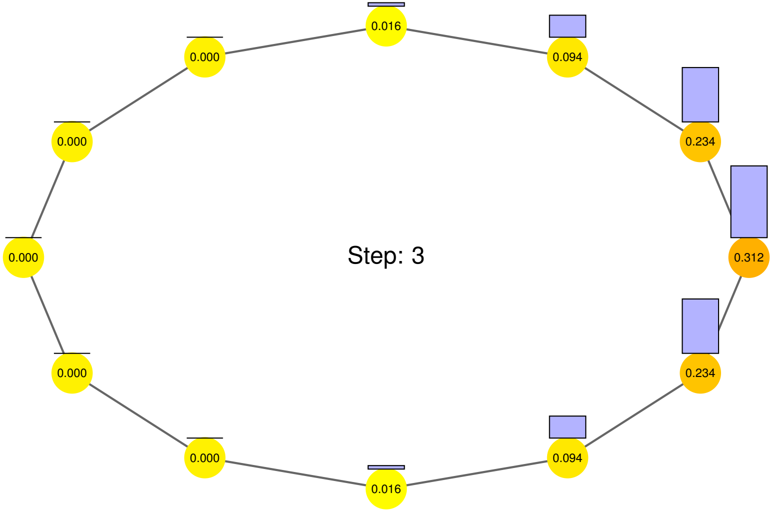

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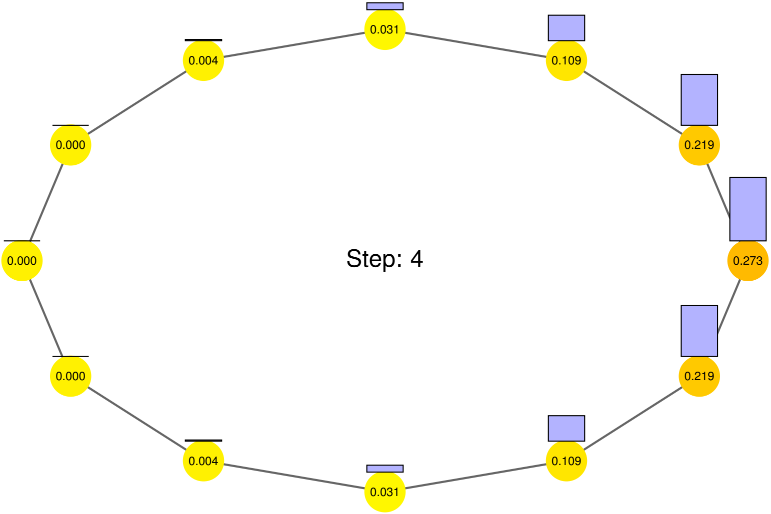

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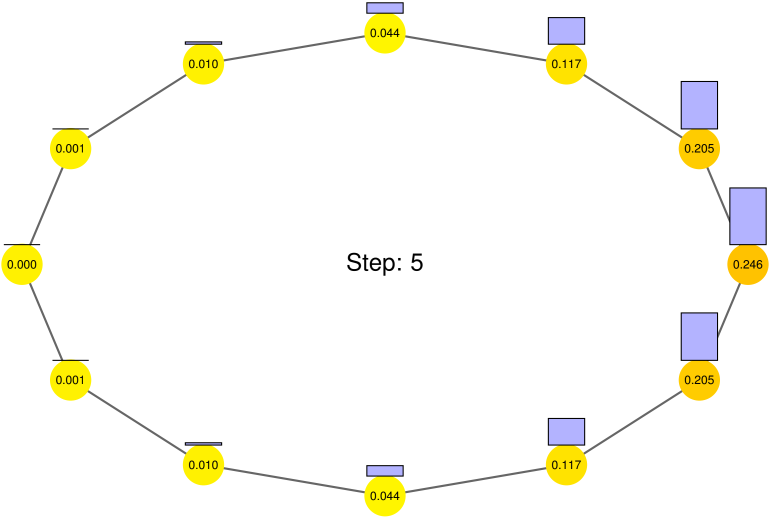

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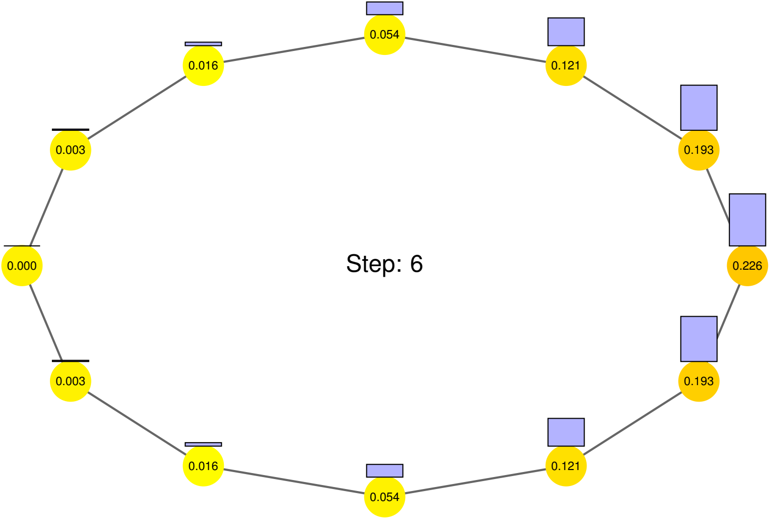

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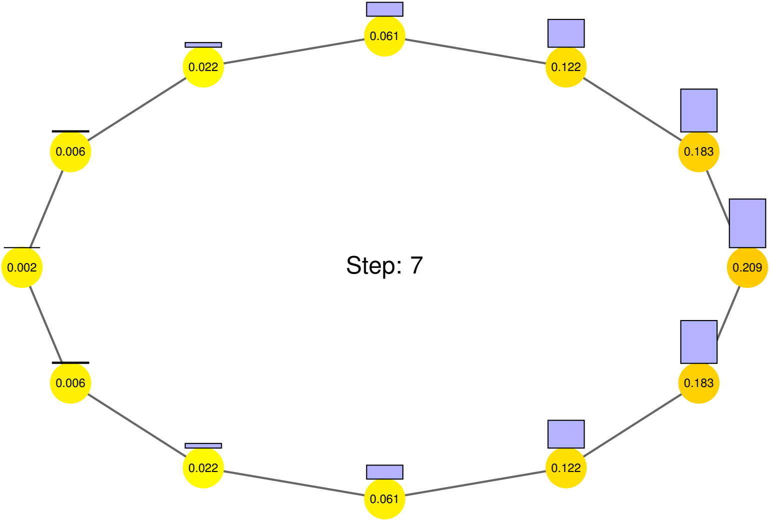

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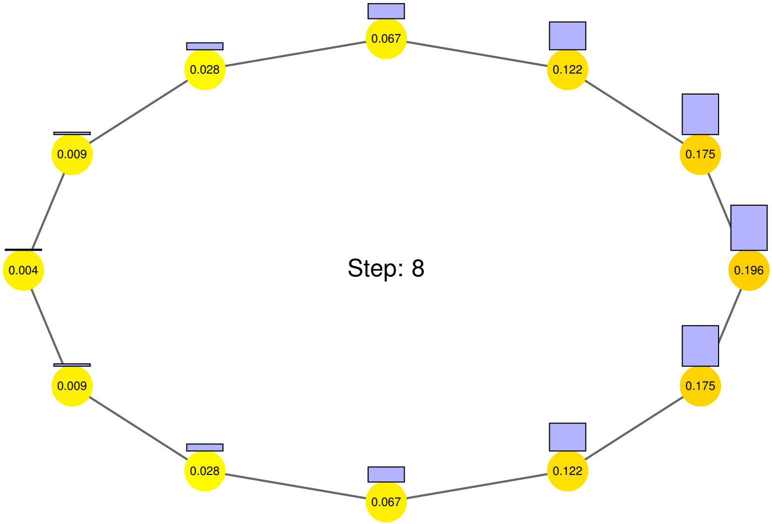

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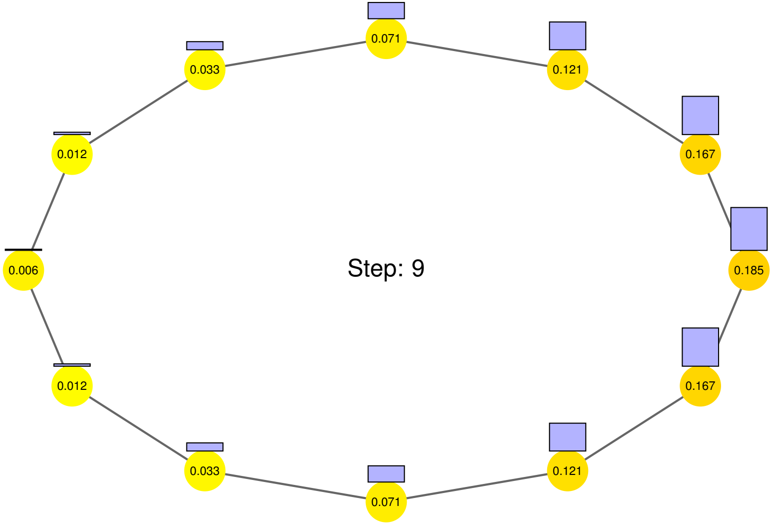

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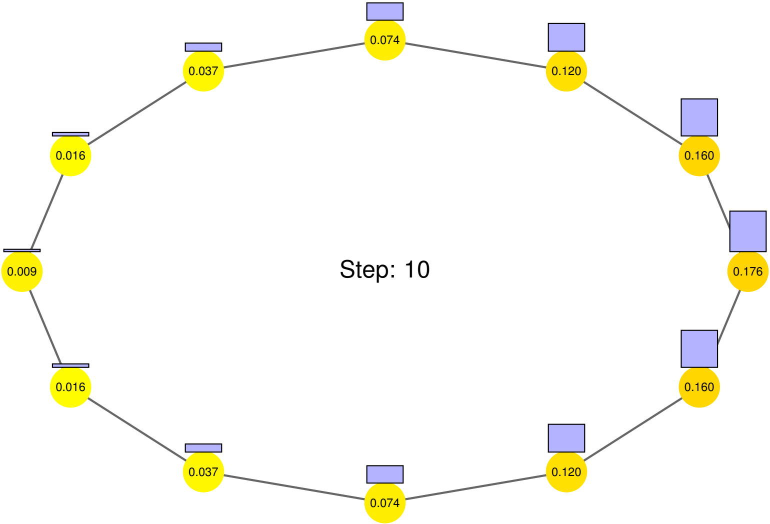

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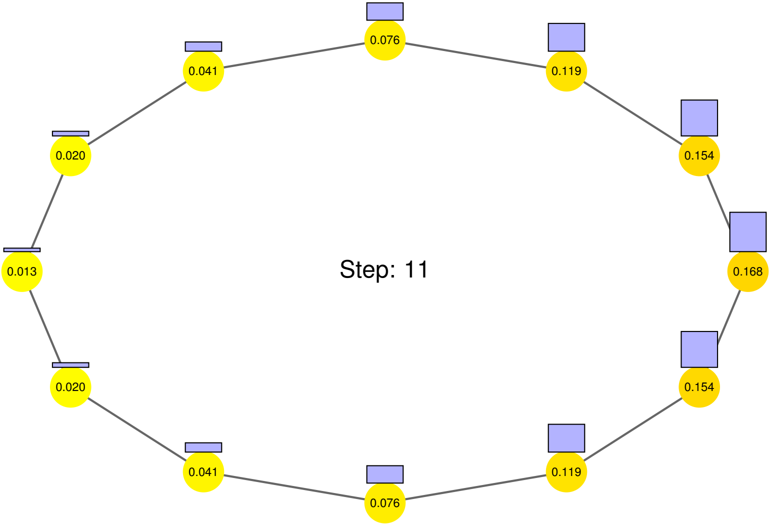

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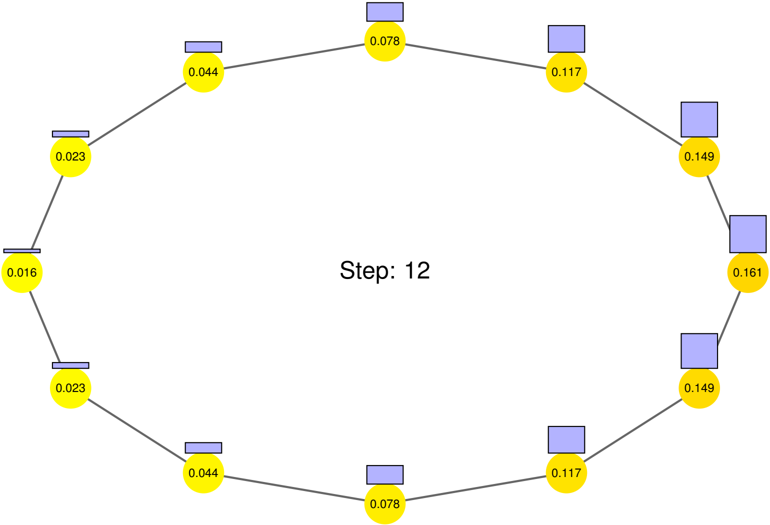

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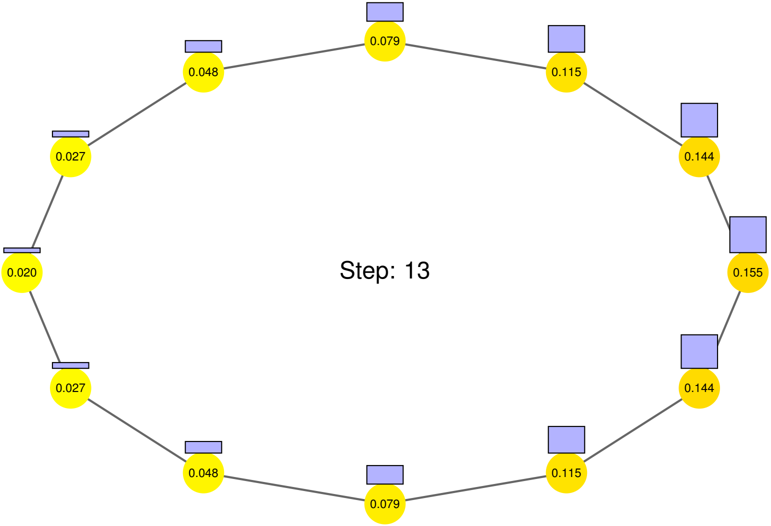

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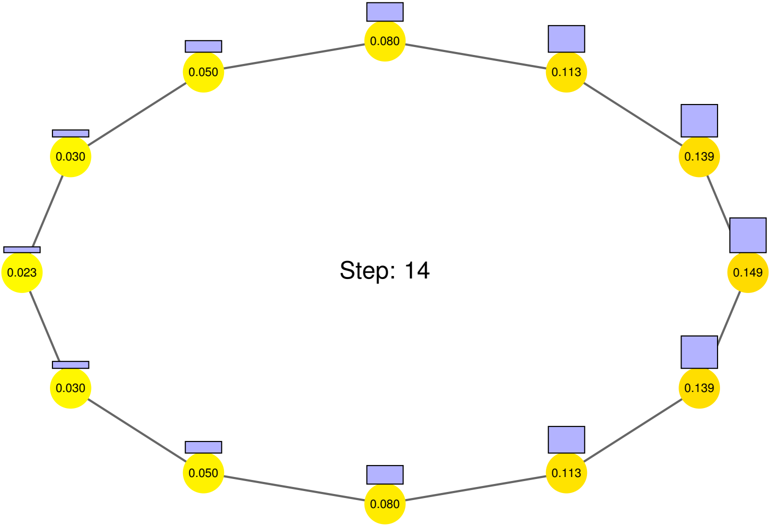

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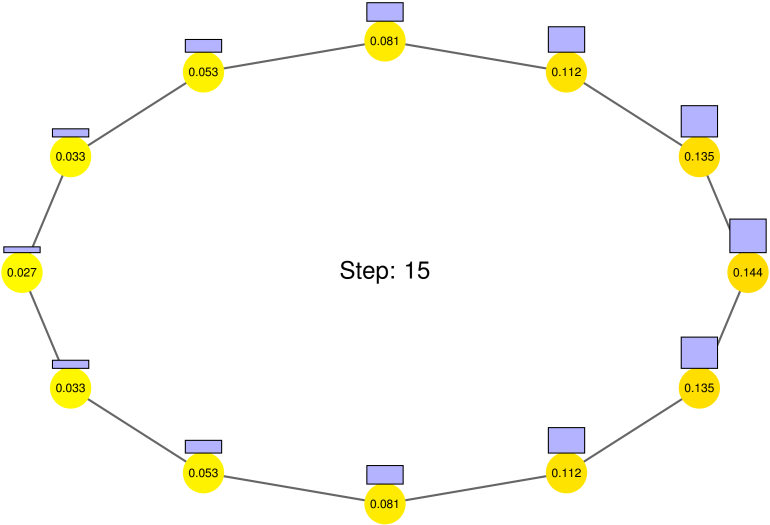

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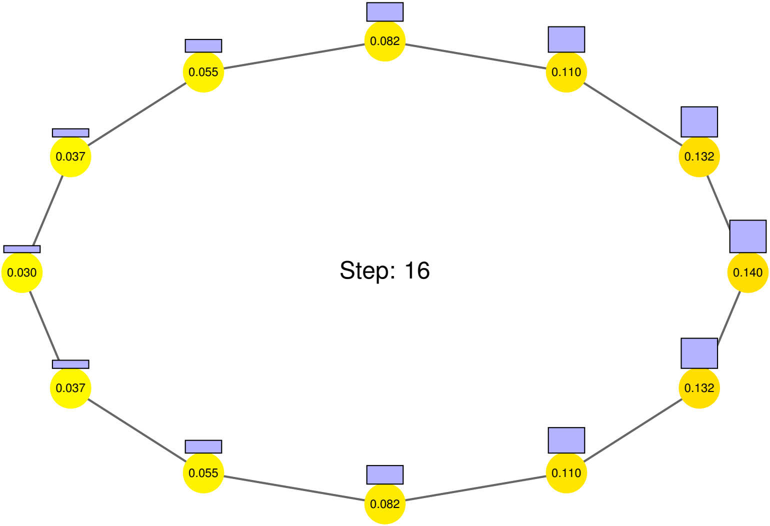

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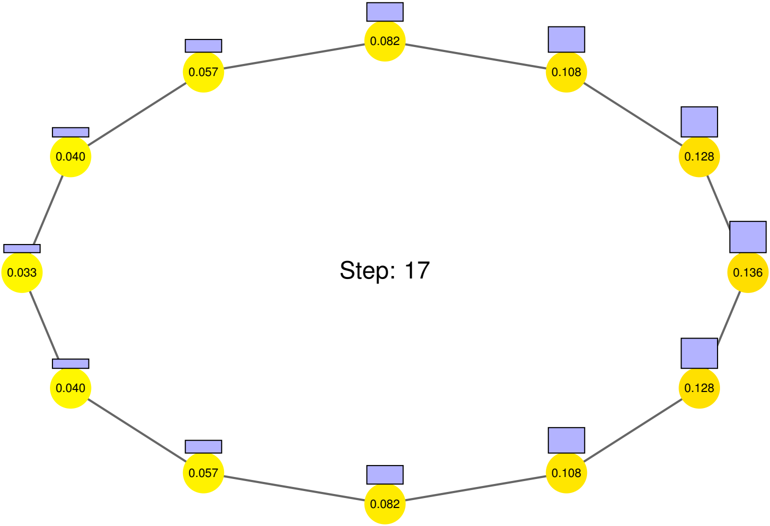

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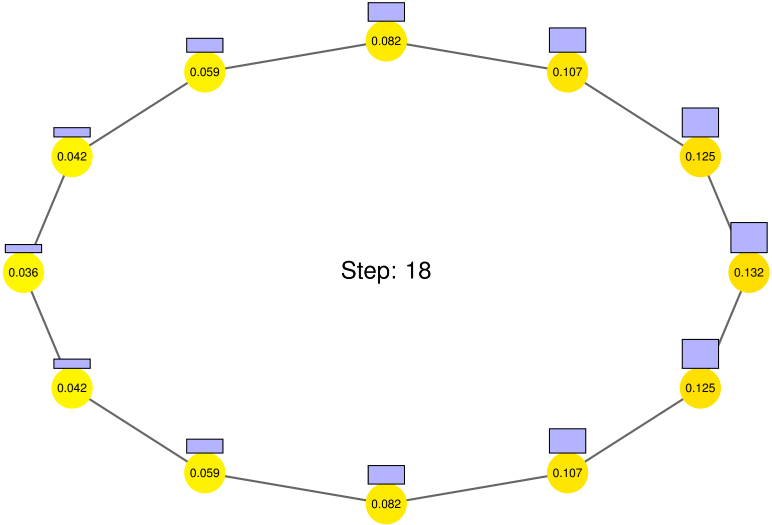

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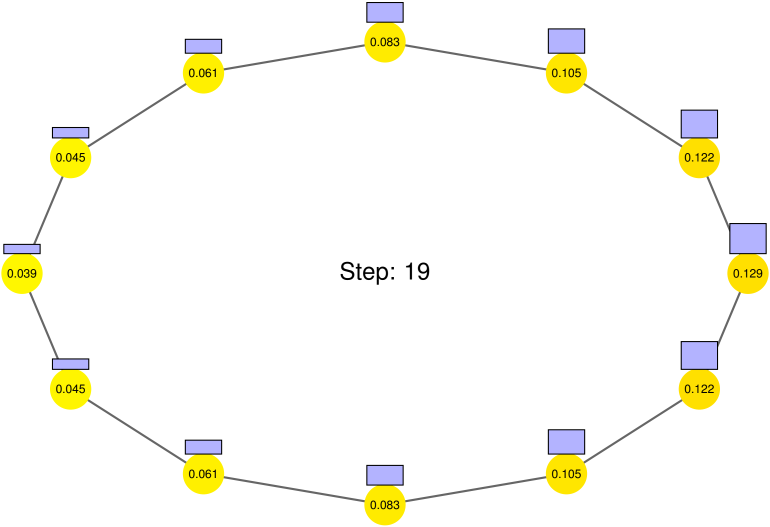

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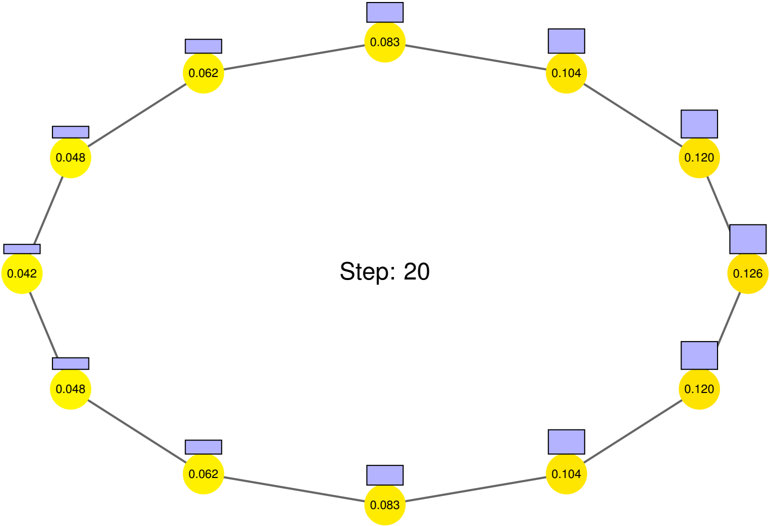

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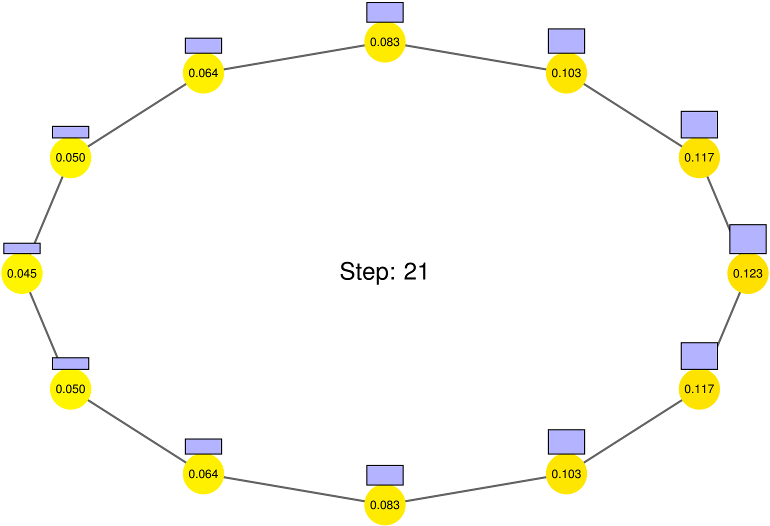

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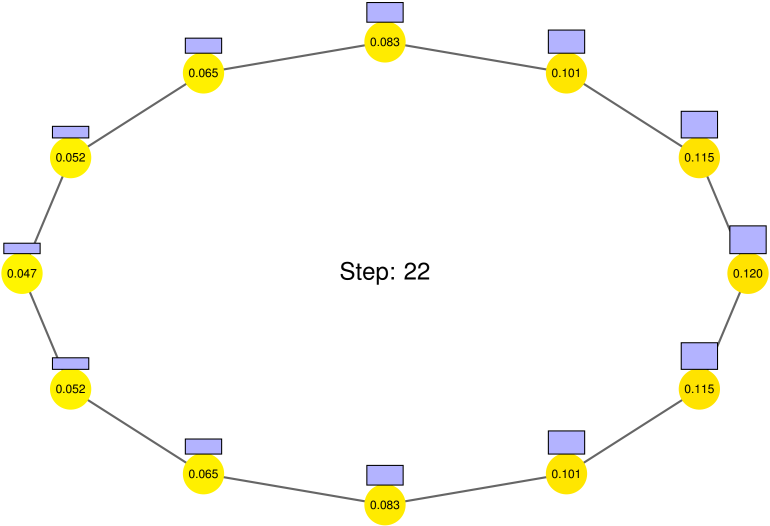

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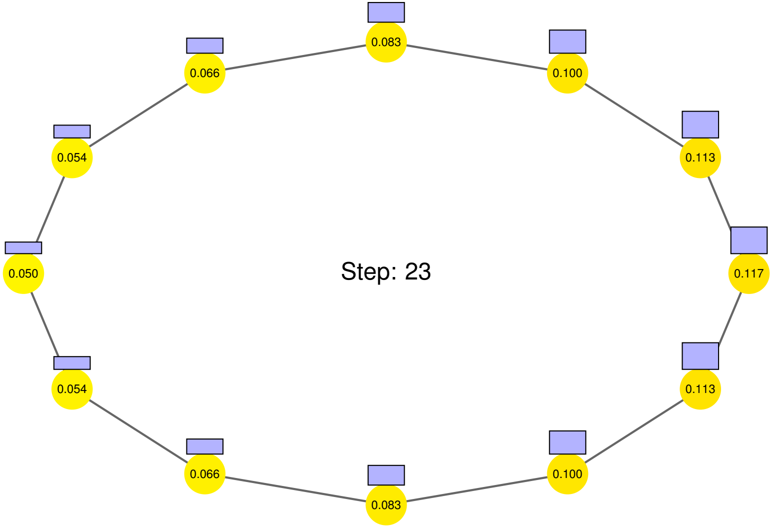

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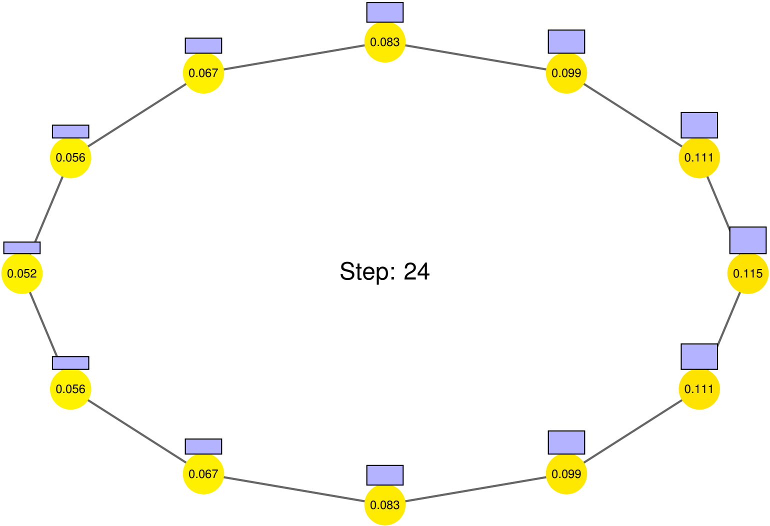

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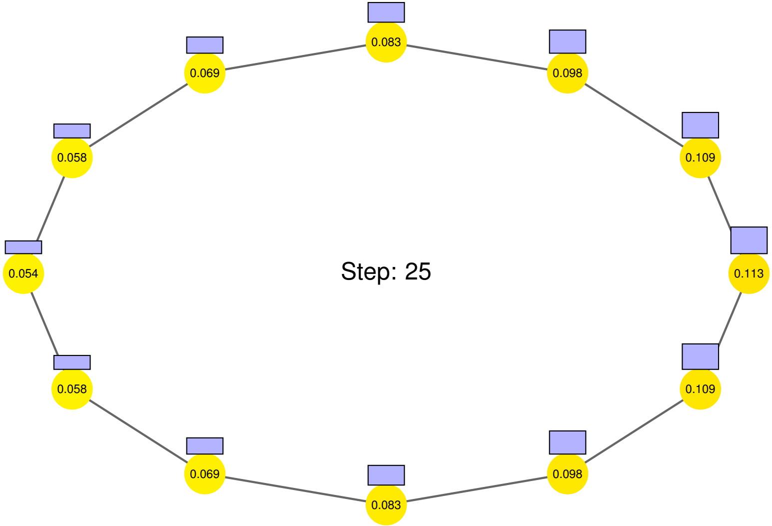

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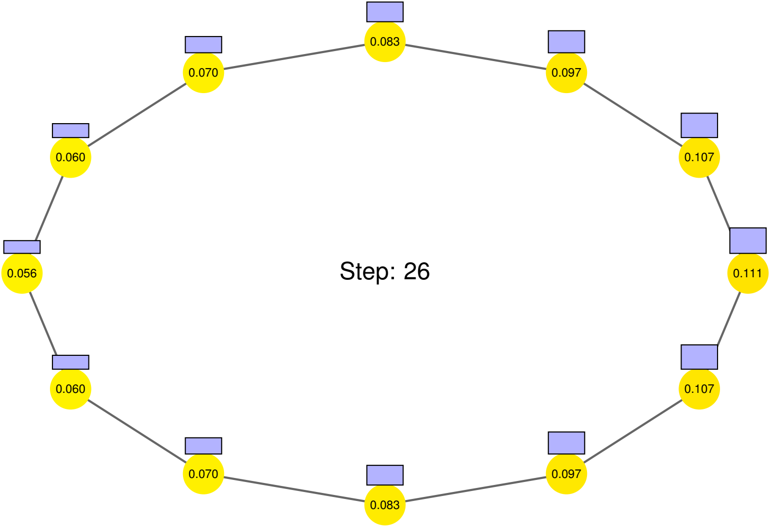

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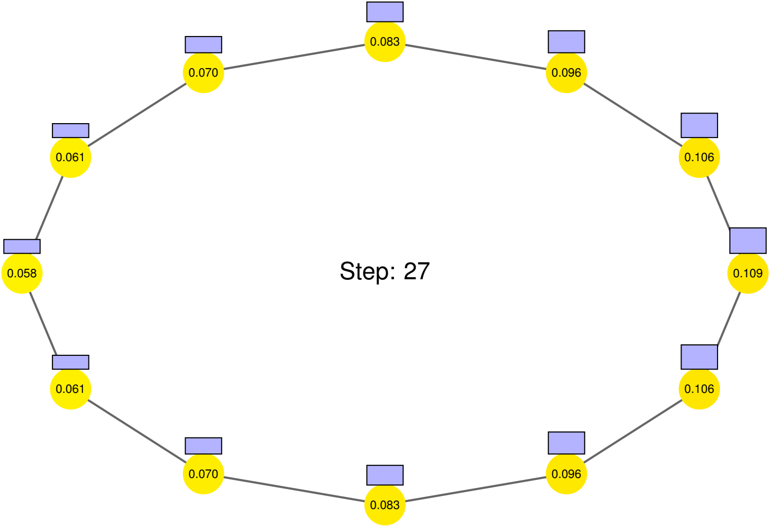

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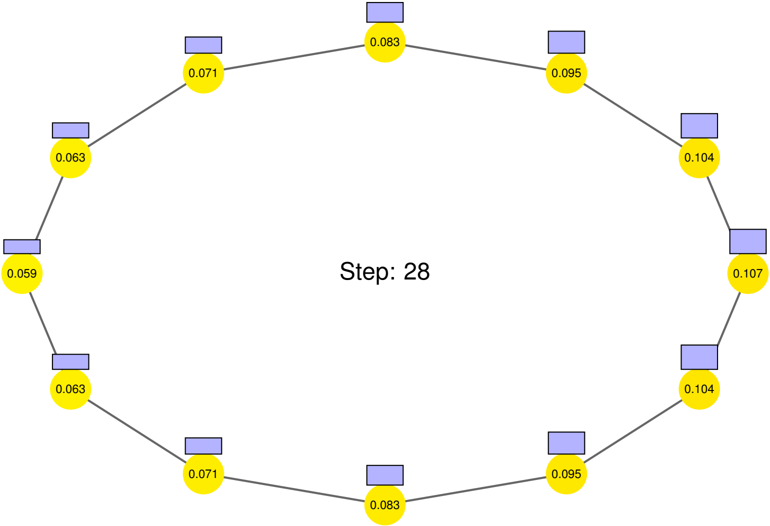

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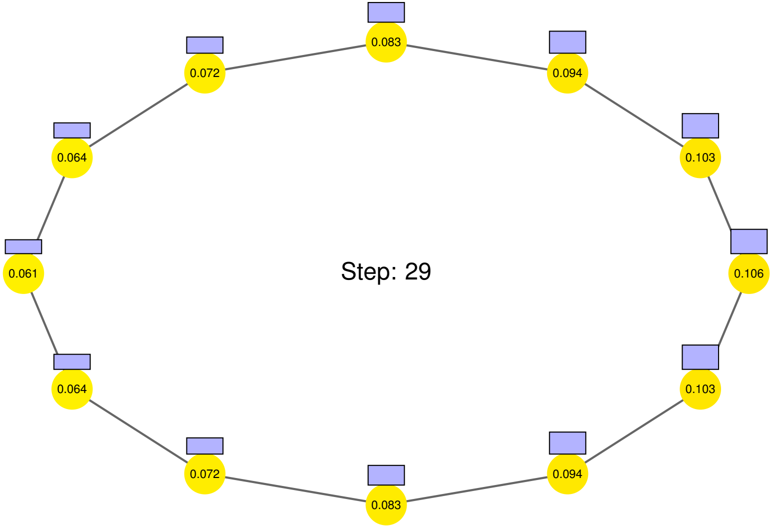

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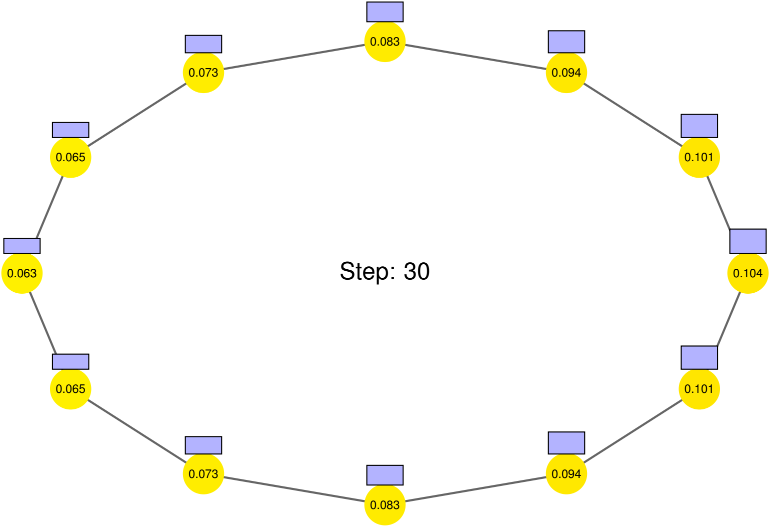

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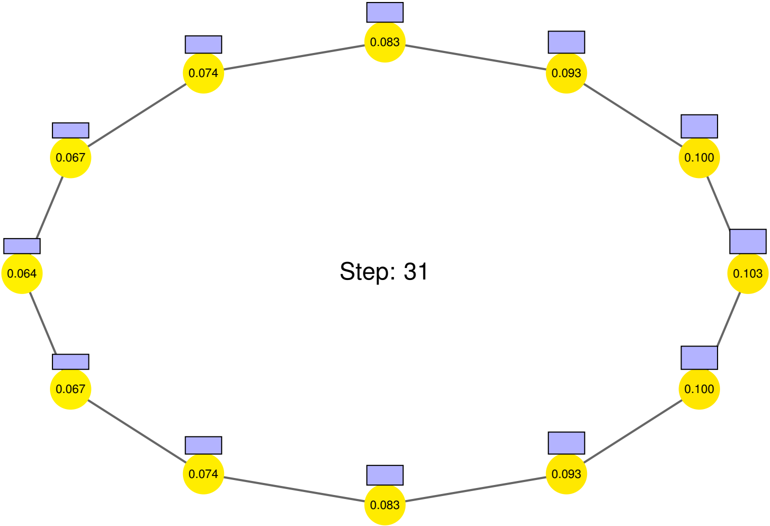

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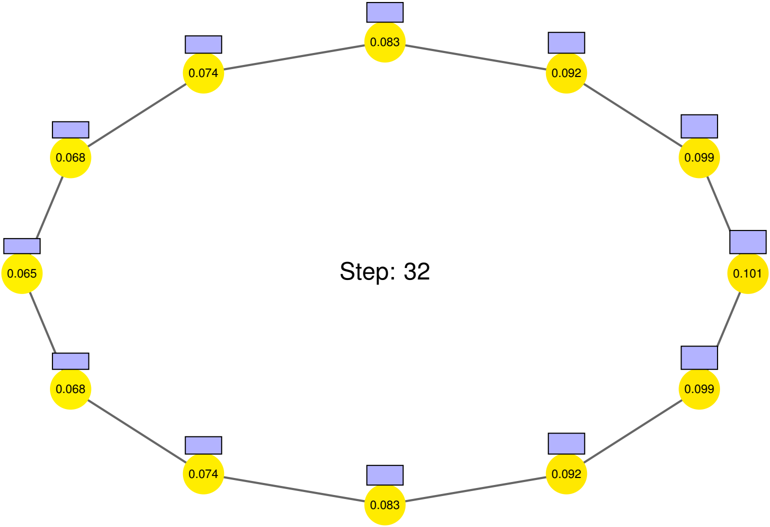

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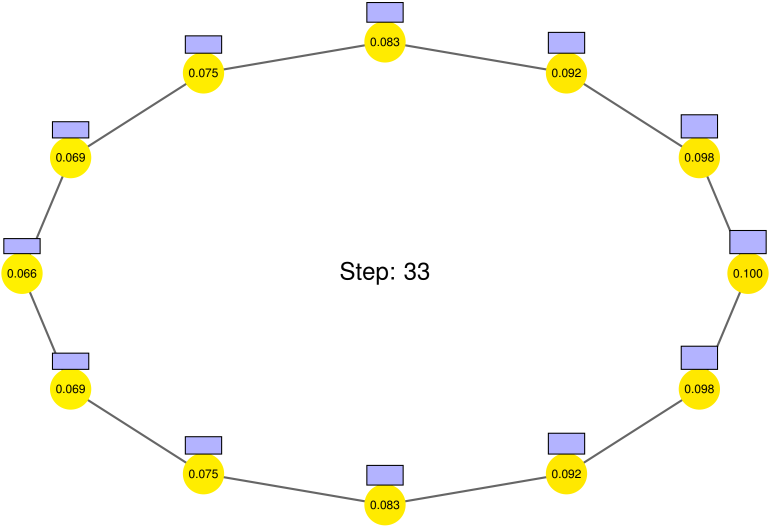

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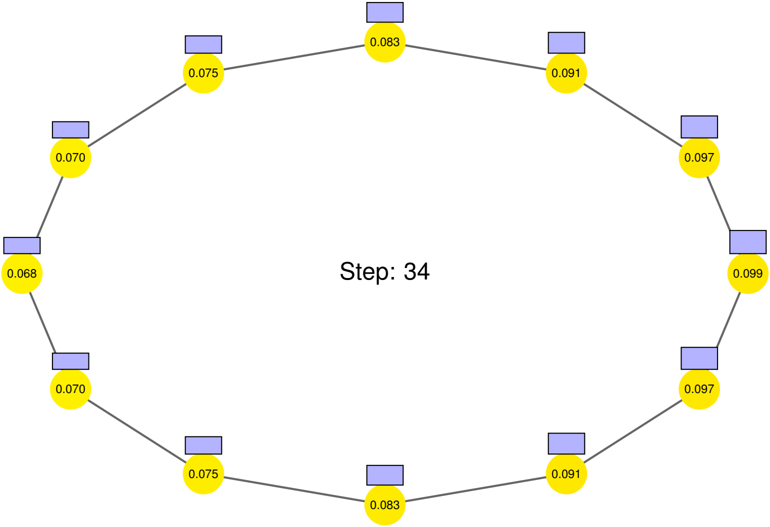

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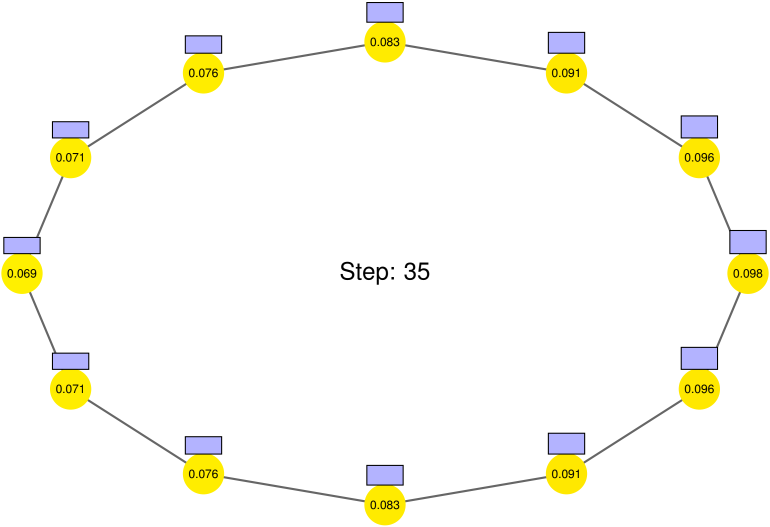

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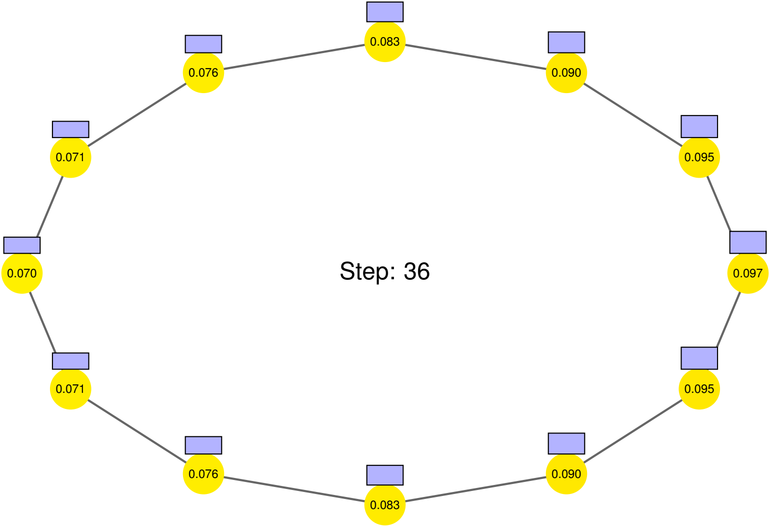

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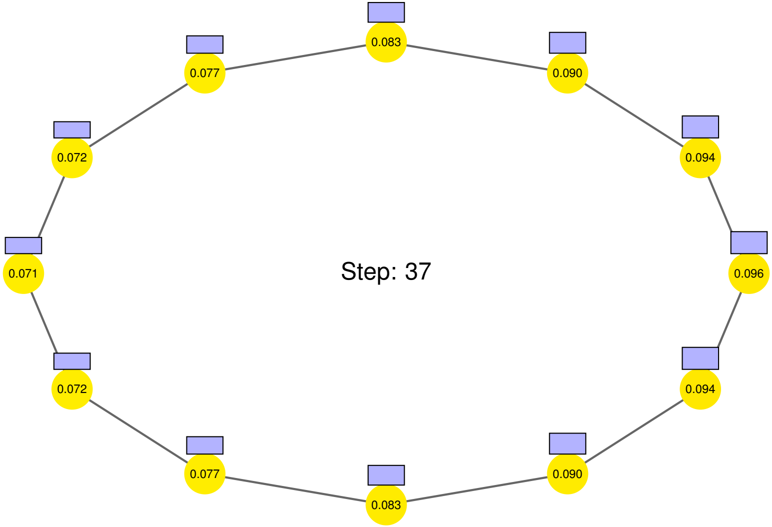

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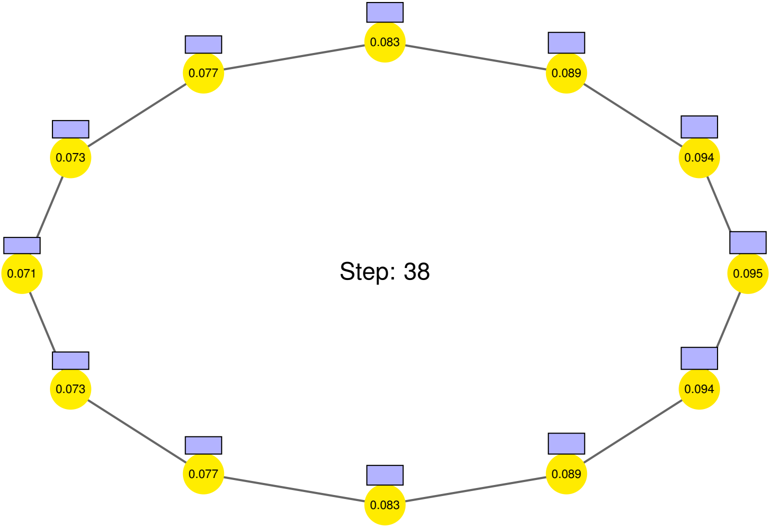

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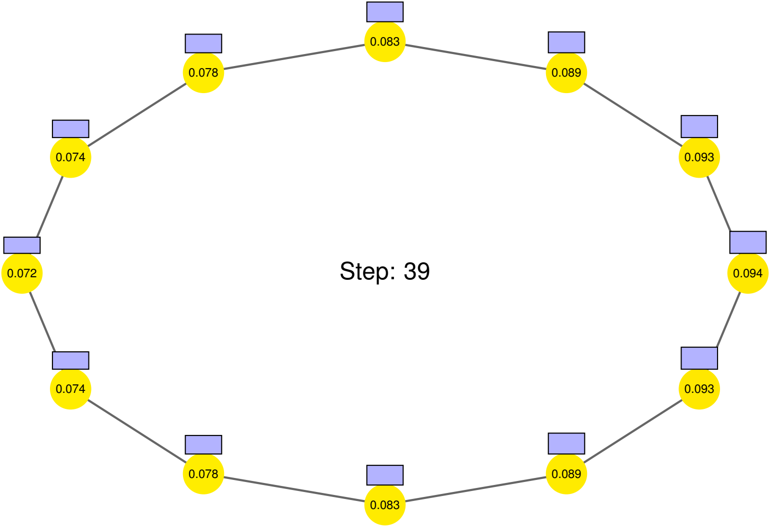

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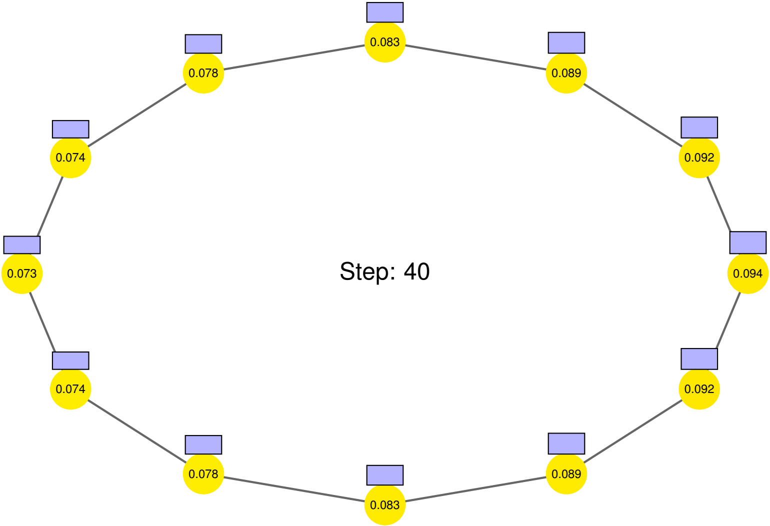

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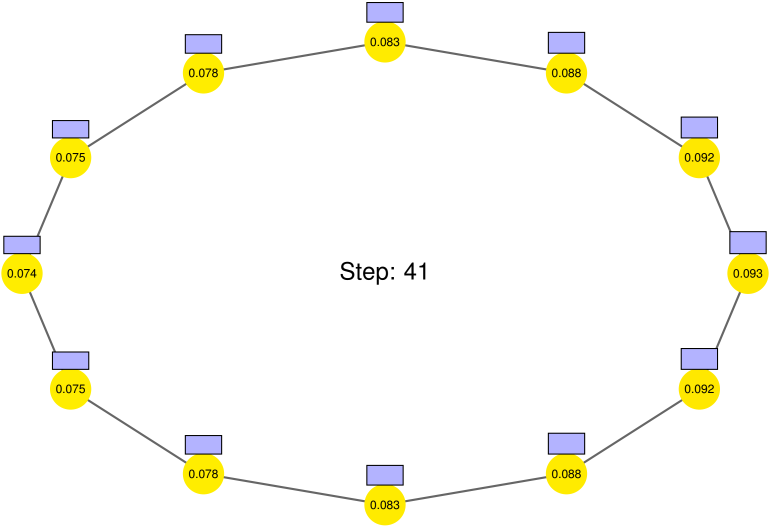

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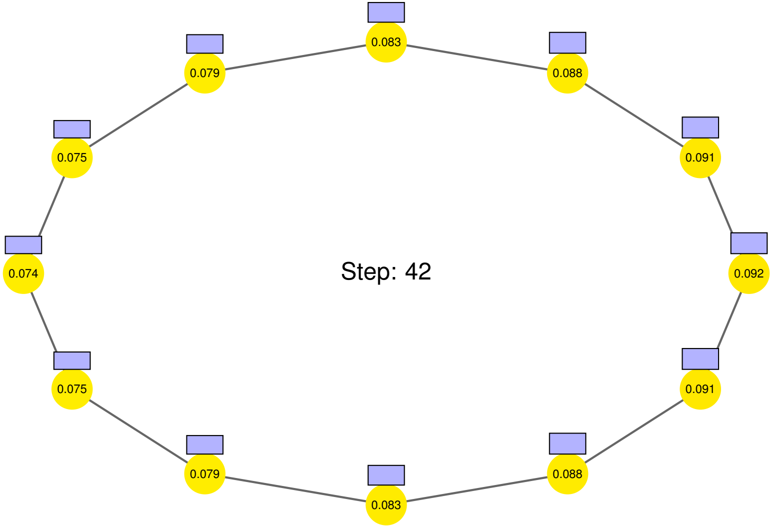

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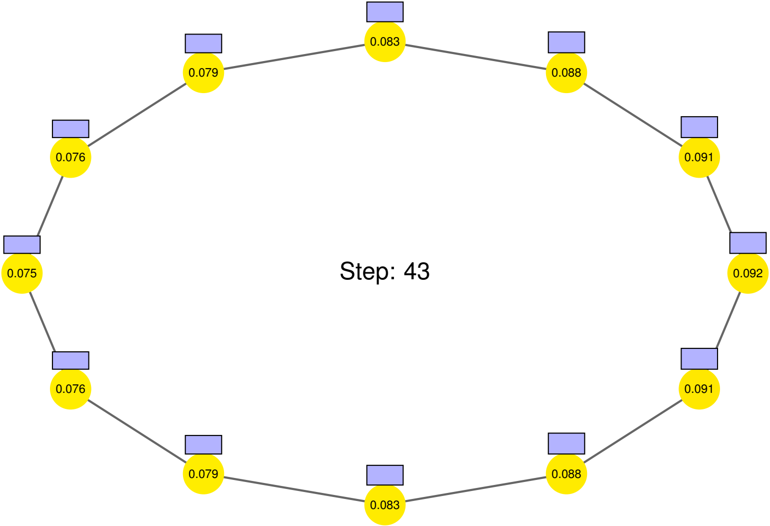

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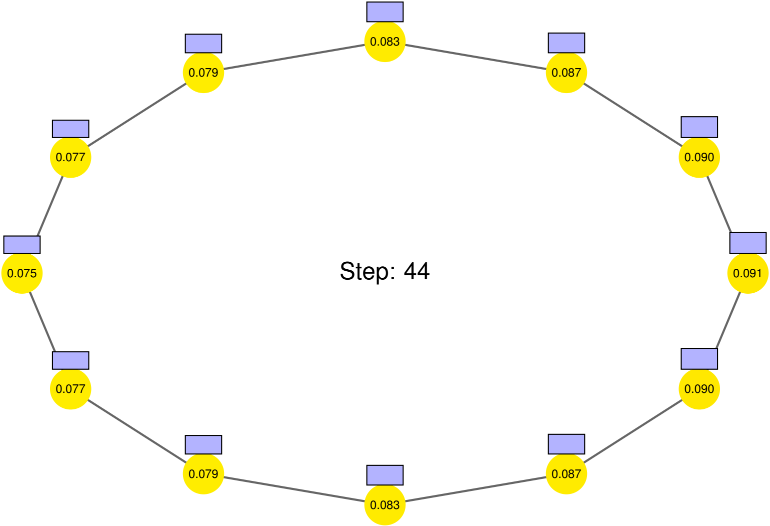

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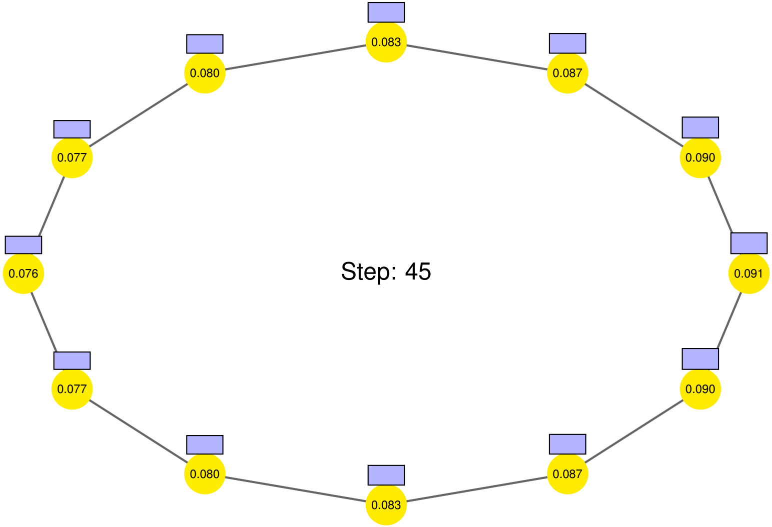

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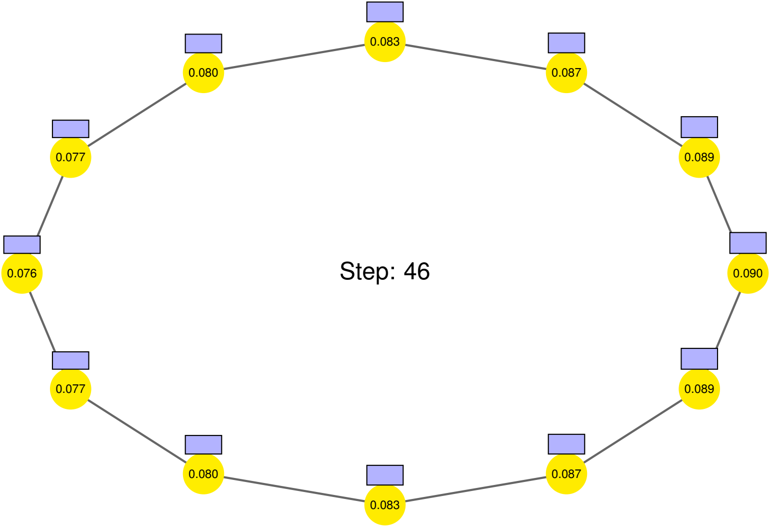

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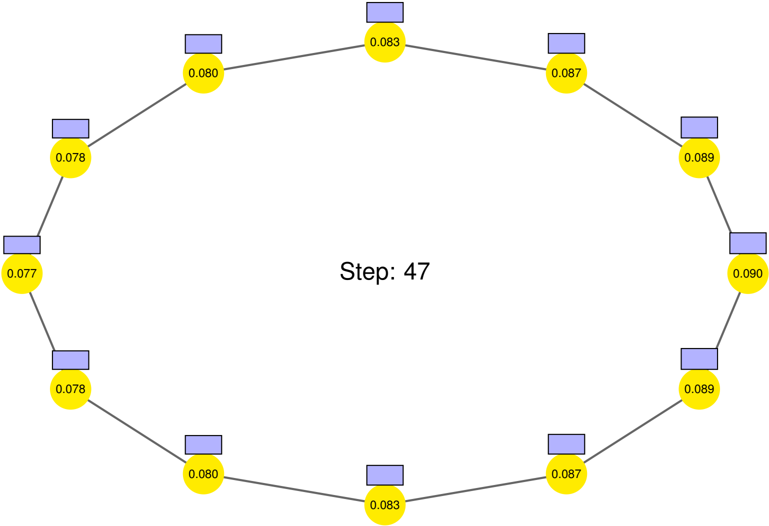

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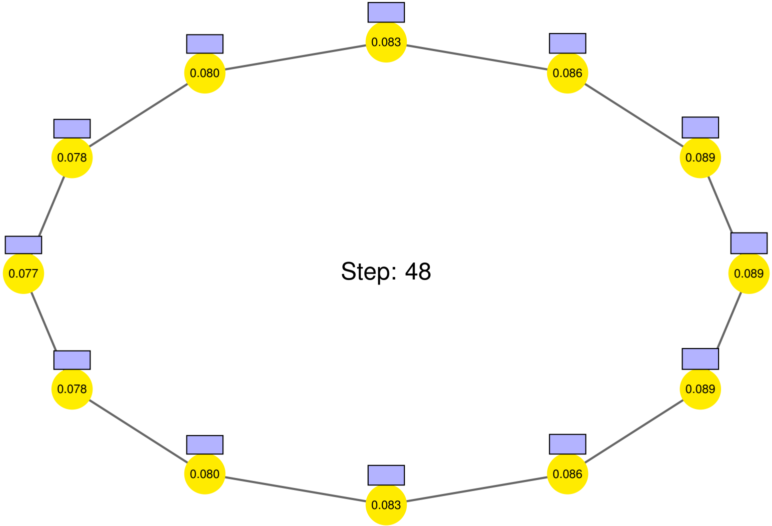

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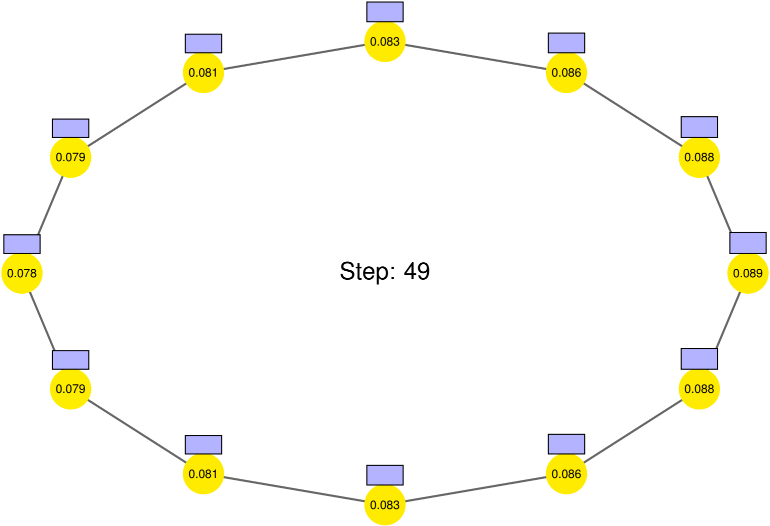

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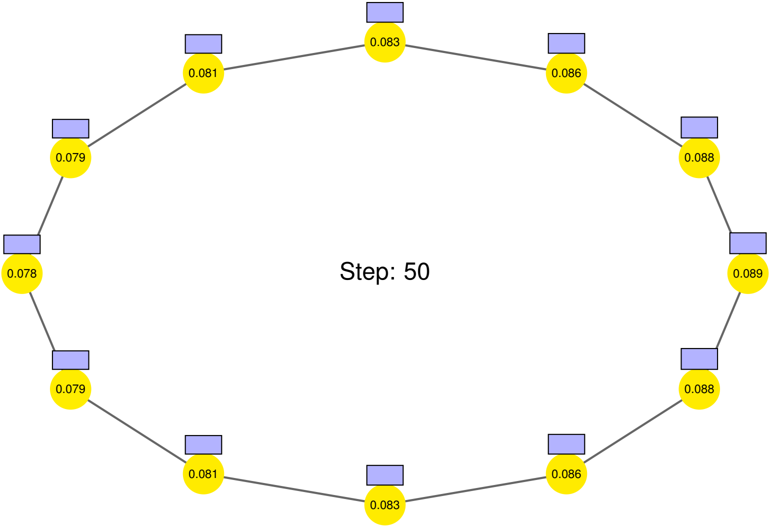

Outline

How Similar are Two Probability Measures?

Loaded Dice

- You are presented three loaded (unfair) dice A, B, C:

x	1	2	3	4	5	6
$\mathbf{P}[A = x]$	1/3	1/12	1/12	1/12	1/12	1/3
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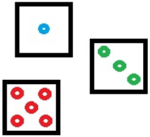
$\mathbf{P}[C = x]$	1/6	1/6	1/8	1/8	1/8	9/24



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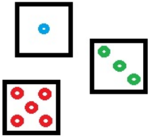

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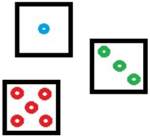

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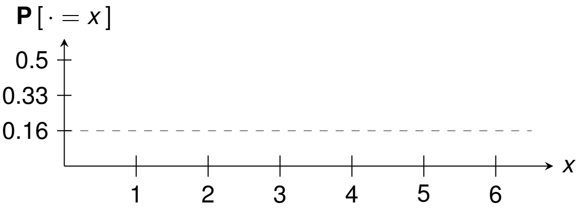

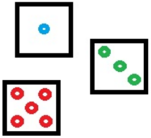

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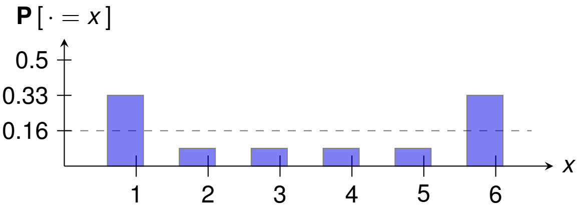

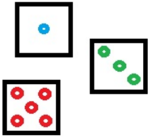

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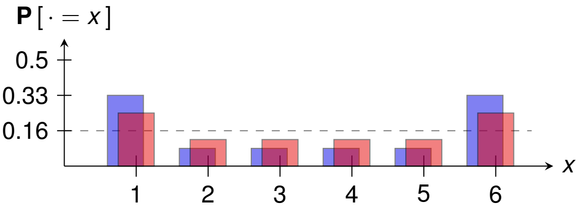

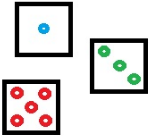

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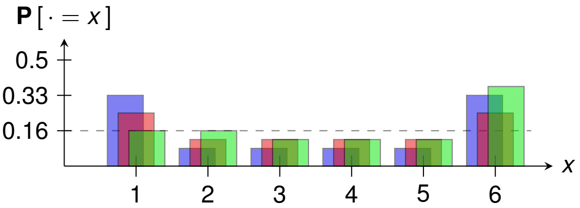

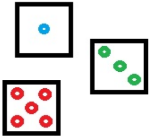

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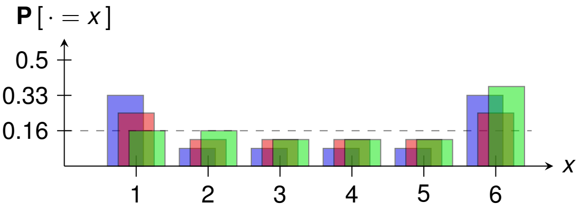

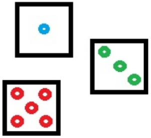

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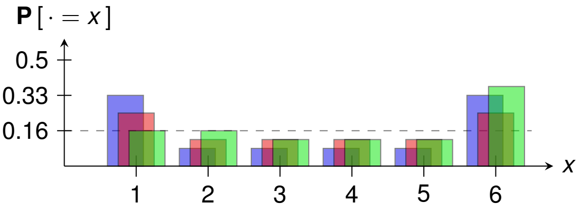

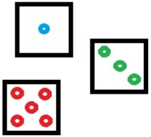

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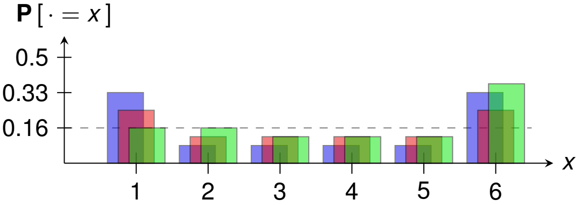

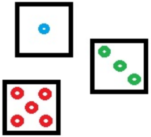

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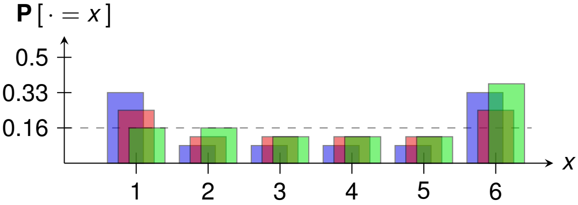
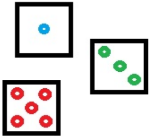
We need a formal "fairness measure" to compare probability distributions!





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Thus

$$\|D - B\|_{tv} = \|D - C\|_{tv} \quad \text{and} \quad \|D - B\|_{tv}, \|D - C\|_{tv} < \|D - A\|_{tv}.$$

So A is the least “fair”, however B and C are equally “fair” (in TV distance).

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We will see a similar result later after introducing spectral techniques (Lecture 12)!

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See final slides for some comments on why we choose $1/4$.

Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

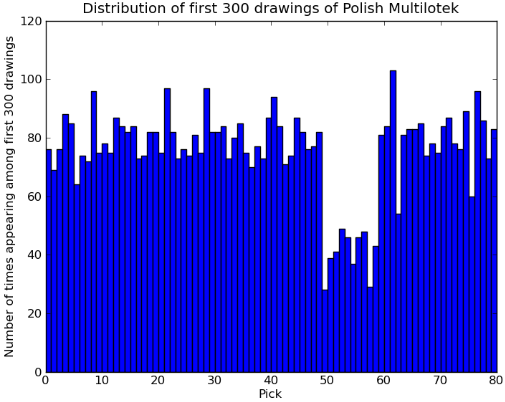
Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Appendix: Remarks on Mixing Time (non-examin.)

Experiment Gone Wrong...



Thanks to Krzysztof Onak (pointer) and Eric Price (graph)

Source: Slides by Ronitt Rubinfeld

What is Card Shuffling?

Source: wikipedia

How long does it take to shuffle a deck of 52 cards?

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The Card Shuffling Markov Chain

TOPTORANDOMSHUFFLE (Input: A pile of n cards)

For $t = 1, 2, \ldots$
    Pick $i \in \{1, 2, \ldots, n\}$ uniformly at random
    Take the top card and insert it behind the i-th card

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TopToRandomShuffle (Input: A pile of n cards)

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For t = 1, 2, ...
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This is a slightly informal definition, so let us look at a small example...

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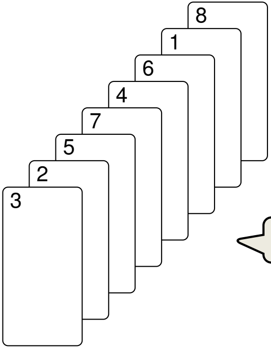
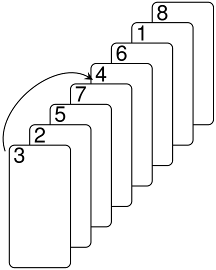


We will focus on this “small” set of cards ($n = 8$)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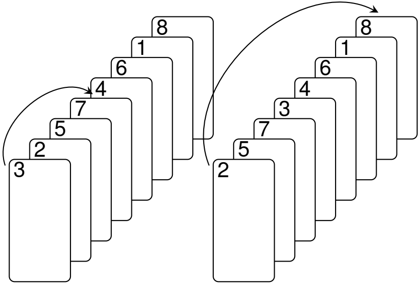
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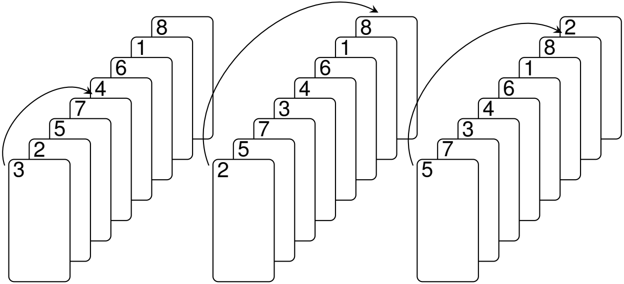
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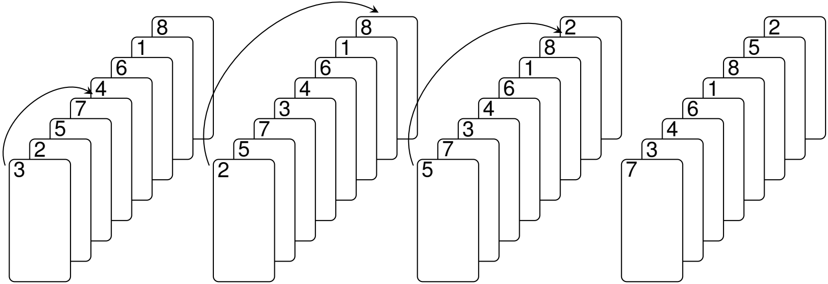
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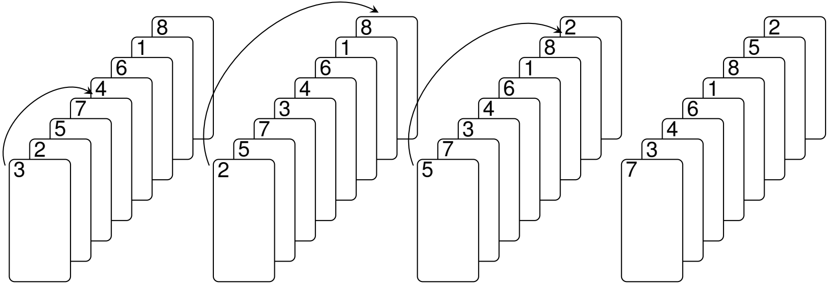

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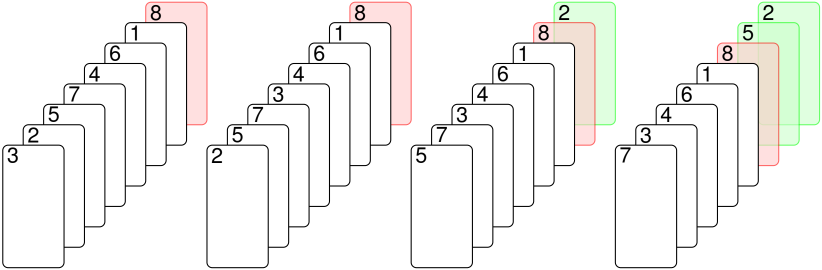

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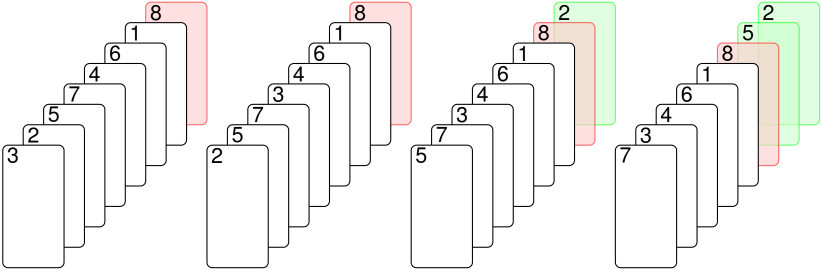

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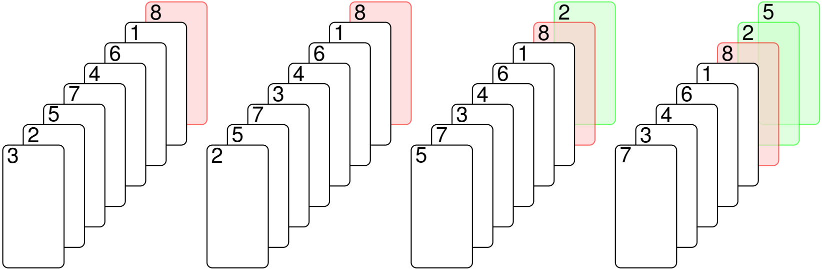

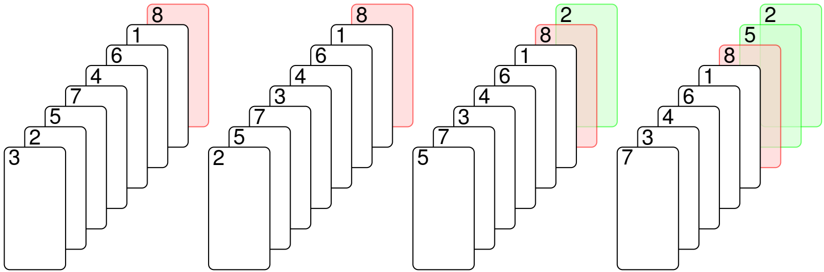

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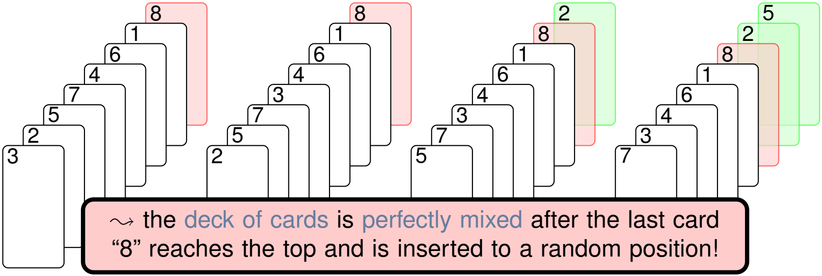

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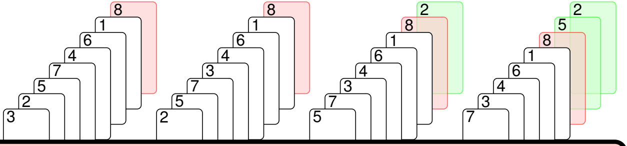
Analysing the Mixing Time (Intuition)



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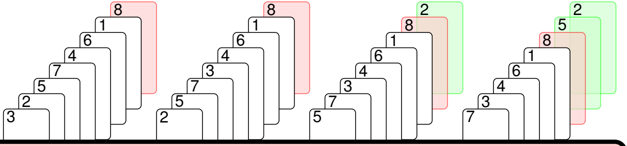

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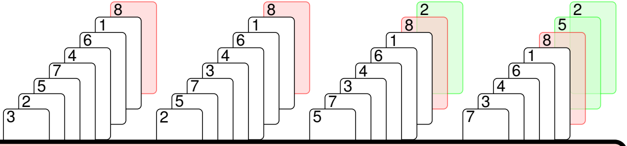

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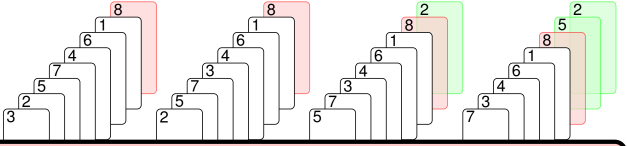

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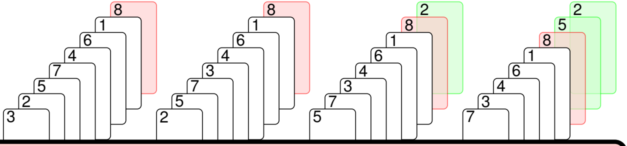

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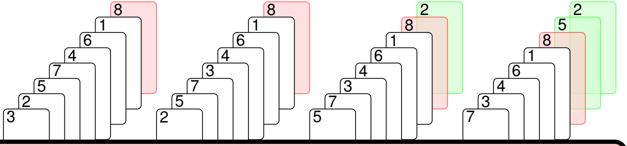

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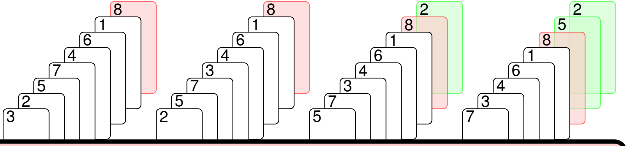

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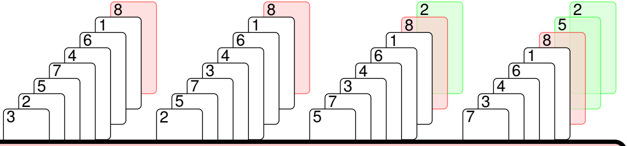
Analysing the Mixing Time (Intuition)



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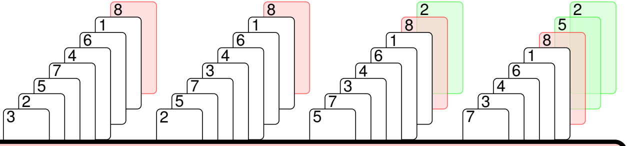

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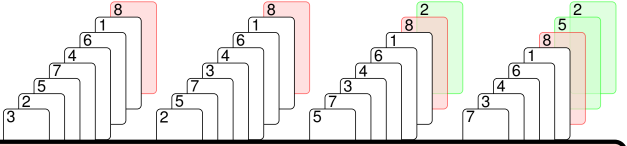

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Using the so-called coupling method, one could prove $t_{mix} \leq n \log n$.

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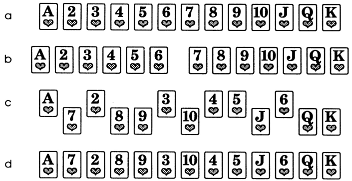

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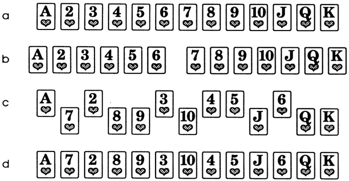

t	1	2	3	4	5	6	7	8	9	10
$\|\|P^t - \pi\|\|_{tv}$	1.000	1.000	1.000	1.000	0.924	0.614	0.334	0.167	0.085	0.043

Figure: Total Variation Distance for t riffle shuffles of 52 cards.

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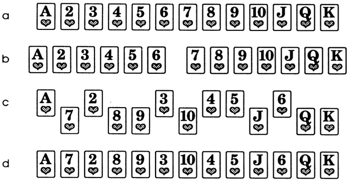

The Annals of Applied Probability
1992, Vol. 2, No. 2, 294–313

TRAILING THE DOVETAIL SHUFFLE TO ITS LAIR

By Dave Bayer[1] and Persi Diaconis[2]

Columbia University and Harvard University

We analyze the most commonly used method for shuffling cards. The main result is a simple expression for the chance of any arrangement after any number of shuffles. This is used to give sharp bounds on the approach to randomness: $\frac{3}{2}\log_2 n + \theta$ shuffles are necessary and sufficient to mix up n cards.

Key ingredients are the analysis of a card trick and the determination of the idempotents of a natural commutative subalgebra in the symmetric group algebra.

t	1	2	3	4	5	6	7	8	9	10
$\lVert P^t - \pi \rVert_{tv}$	1.000	1.000	1.000	1.000	0.924	0.614	0.334	0.167	0.085	0.043

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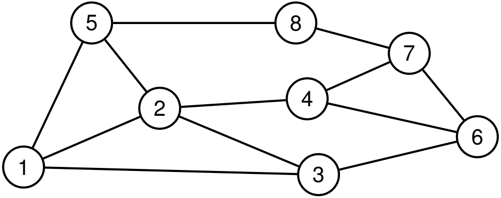
Outline

Markov Chain for Sampling Independent Sets (1/2) (non-examin.)



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Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$.

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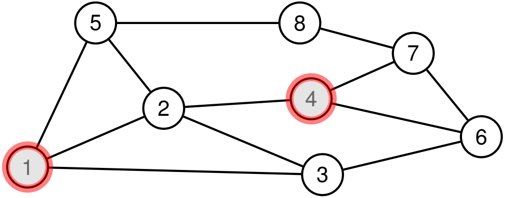

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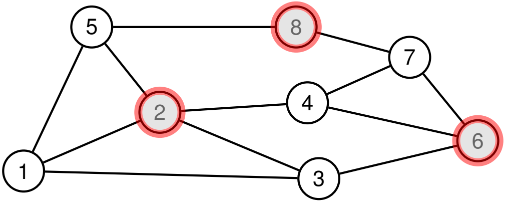

$S = \{2, 6, 8\}$ is an independent set ✓

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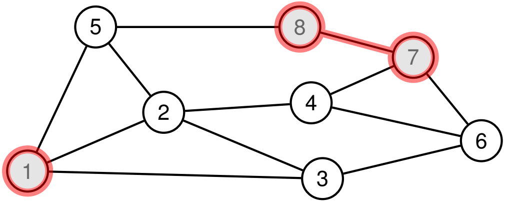

$S = \{1, 7, 8\}$ is **not** an independent set ×

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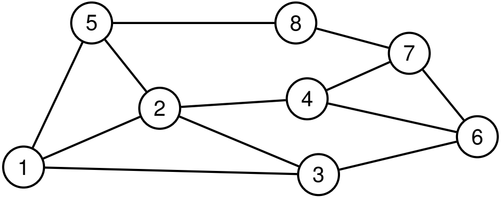

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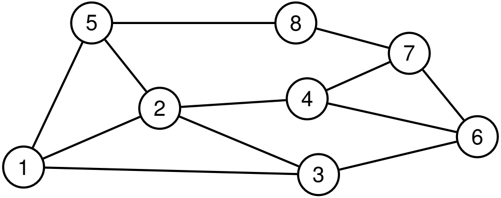

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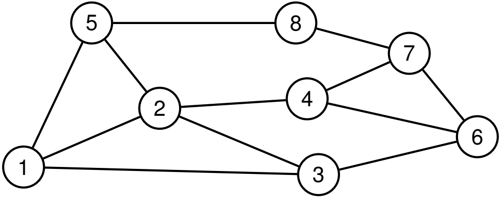

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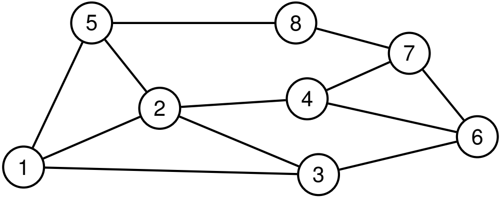

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We can use a generic Markov Chain Monte Carlo approach to tackle this problem!

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INDEPENDENTSETSAMPLER

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Let $X_0$ be an arbitrary independent set in $G$
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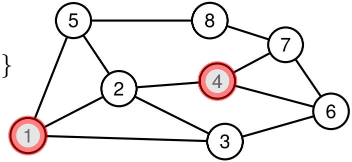
$X_0 = \{1, 4\}$



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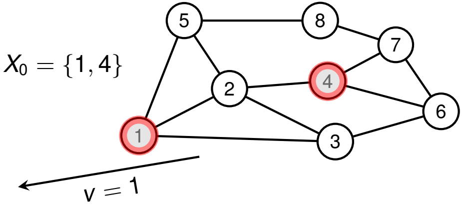

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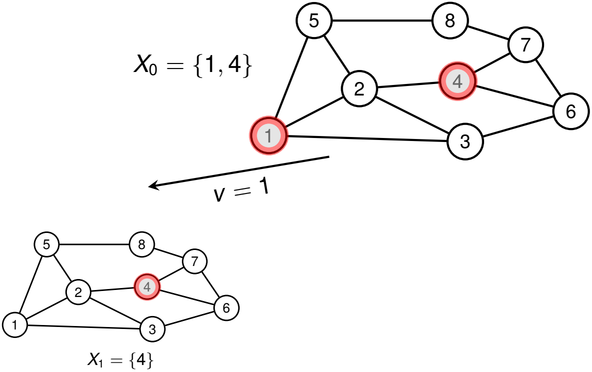

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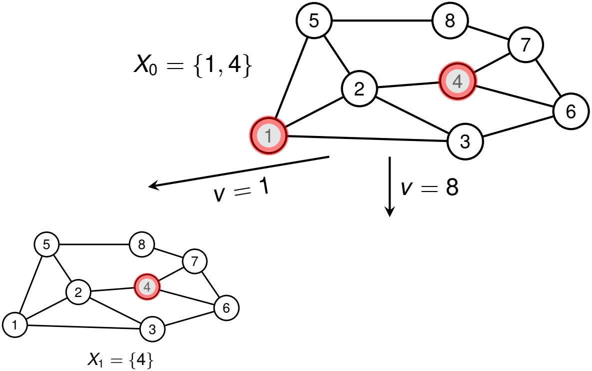

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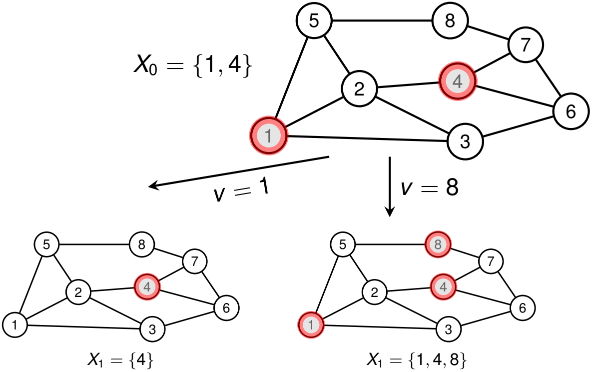

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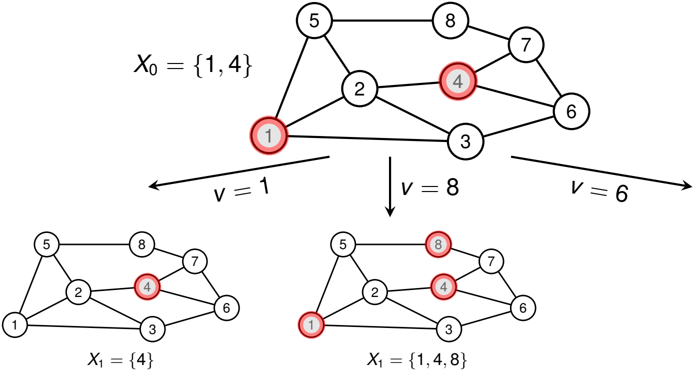

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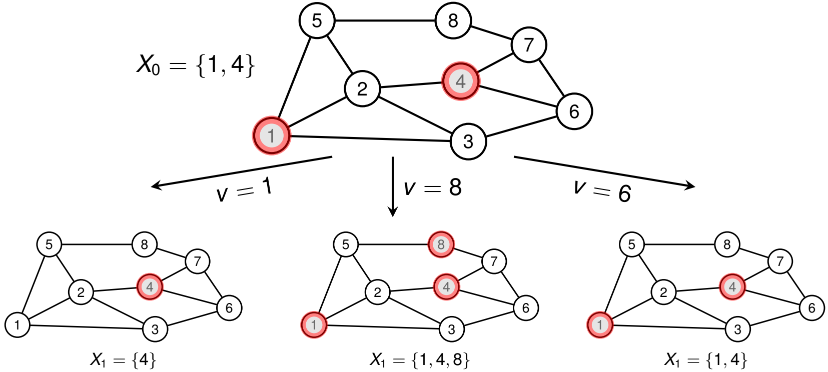

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not covered here, see the textbook by Mitzenmacher and Upfal

Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Appendix: Remarks on Mixing Time (non-examin.)

Further Remarks on the Mixing Time (non-examin.)

- One can prove $\max_x \left\| P_x^t - \pi \right\|_{tv}$ is non-increasing in t (this means if the chain is "ϵ-mixed" at step t, then this also holds in future steps) *[Mitzenmacher, Upfal, 12.3]*

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Remark: This freedom on how to pick ϵ relies on the sub-multiplicative property of a (version) of the variation distance. First, let

$$d(t) := \max_x \left\| P_x^t - \pi \right\|_{tv}$$

be the variation distance after t steps when starting from the worst state. Further, define

$$\overline{d}(t) := \max_{\mu,\nu} \left\| P_\mu^t - P_\nu^t \right\|_{tv}.$$

These quantities are related by the following double inequality

$$d(t) \leq \overline{d}(t) \leq 2d(t).$$

Further, $\overline{d}(t)$ is sub-multiplicative, that is for any $s, t \geq 1$,

$$\overline{d}(s+t) \leq \overline{d}(s) \cdot \overline{d}(t).$$

Hence for any fixed $0 < \epsilon < \delta < 1/2$ it follows from the above that

$$\tau(\epsilon) \leq \left\lceil \frac{\ln \epsilon}{\ln(2\delta)} \right\rceil \tau(\delta).$$

In particular, for any $\epsilon < 1/4$

$$\tau(\epsilon) \leq \left\lceil \log_2 \epsilon^{-1} \right\rceil \tau(1/4).$$

Further Remarks on the Mixing Time (non-examin.)

- One can prove $\max_x \left\| P_x^t - \pi \right\|_{tv}$ is non-increasing in t (this means if the chain is "ϵ-mixed" at step t, then this also holds in future steps) *[Mitzenmacher, Upfal, 12.3]*
- We chose $t_{mix} := \tau(1/4)$, but other choices of ϵ are perfectly fine too (e.g, $t_{mix} := \tau(1/e)$ is often used); in fact, any constant $\epsilon \in (0, 1/2)$ is possible.

Remark: This freedom on how to pick ϵ relies on the sub-multiplicative property of a (version) of the variation distance. First, let

$$d(t) := \max_x \left\| P_x^t - \pi \right\|_{tv}$$

be the variation distance after t steps when starting from the worst state. Further, define

$$\overline{d}(t) := \max_{\mu,\nu} \left\| P_\mu^t - P_\nu^t \right\|_{tv}.$$

These quantities are related by the following double inequality

$$d(t) \leq \overline{d}(t) \leq 2d(t).$$

This 2 is the reason why we ultimately need $\epsilon < 1/2$ in this derivation. On the other hand, see *[Exercise (4/5).8]* why $\epsilon < 1/2$ is also necessary.

Further, $\overline{d}(t)$ is sub-multiplicative, that is for any $s, t \geq 1$,

$$\overline{d}(s+t) \leq \overline{d}(s) \cdot \overline{d}(t).$$

Hence for any fixed $0 < \epsilon < \delta < 1/2$ it follows from the above that

$$\tau(\epsilon) \leq \left\lceil \frac{\ln \epsilon}{\ln(2\delta)} \right\rceil \tau(\delta).$$

In particular, for any $\epsilon < 1/4$

$$\tau(\epsilon) \leq \left\lceil \log_2 \epsilon^{-1} \right\rceil \tau(1/4).$$

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Hence smaller constants $\epsilon < 1/4$ only increase the mixing time by some constant factor.