Randomised Algorithms

Lecture 3: Concentration Inequalities, Application to Quick-Sort, Extensions

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UNIVERSITY OF CAMBRIDGE

Outline

Application 2: Randomised QuickSort

Extensions of Chernoff Bounds

Applications of Method of Bounded Differences

Appendix: More on Moment Generating Functions (non-examinable)

QuickSort

QUICKSORT (Input $A[1], A[2], \ldots, A[n]$)

Pick an element from the array, the so-called pivot

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    return A

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    Create two subarrays A_1 and A_2 (without the pivot) such that:

        A_1 contains the elements that are smaller than the pivot

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    QUICKSORT(A_1)

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We will now give a proof of this “well-known” result!

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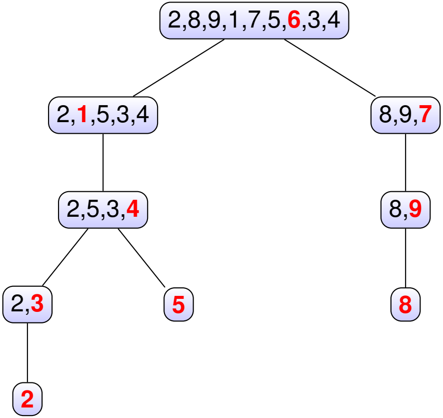

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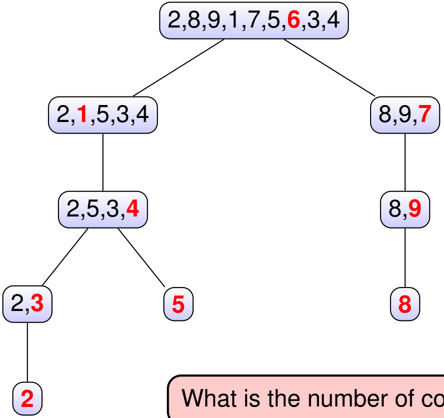

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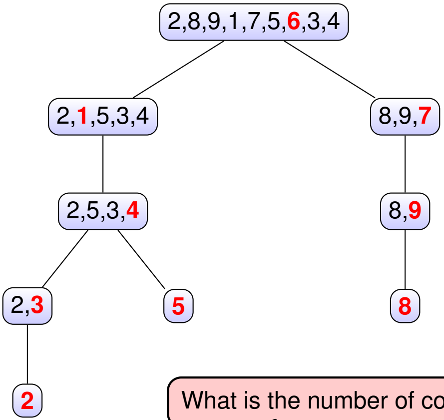

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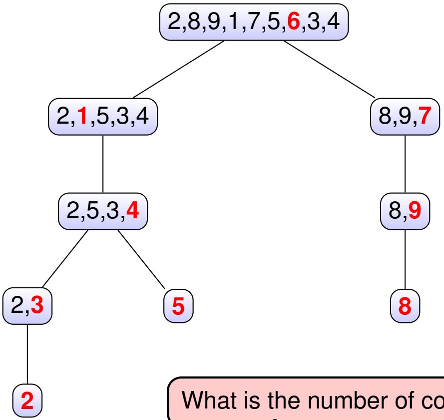

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Note that the number of comparison by QUICKSORT is equivalent to the sum of the depths of all nodes in the tree (why?). In this case:

$$0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 + 4 = 19.$$

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4. Actually, we will prove sth slightly stronger:

$$\mathbf{P}\left[\bigcap_{i=1}^{n} \{ H_i \leq C \log n \} \right] \geq 1 - n^{-1}.$$

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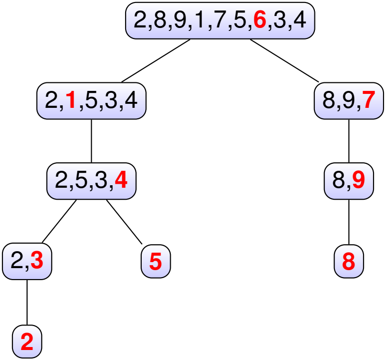

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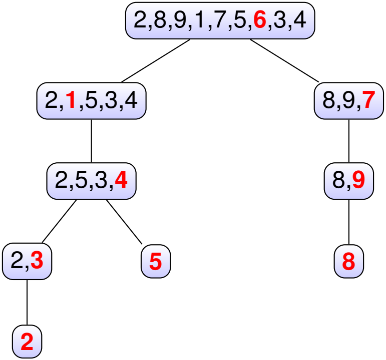

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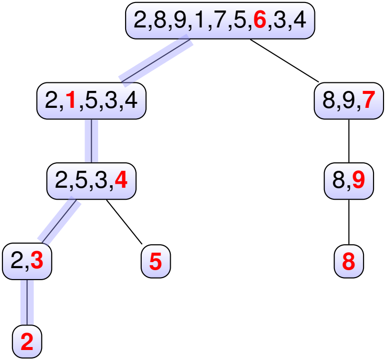

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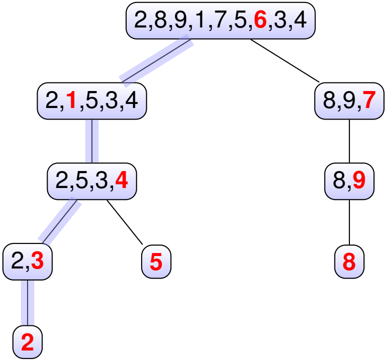

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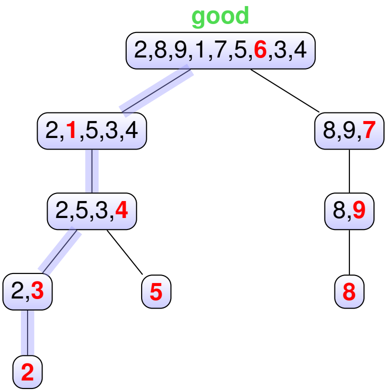

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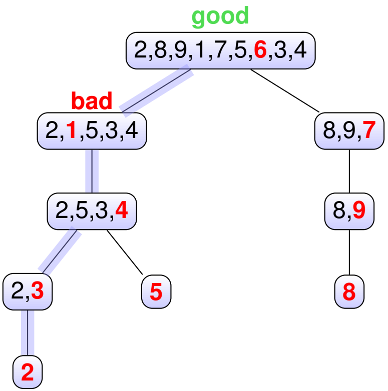

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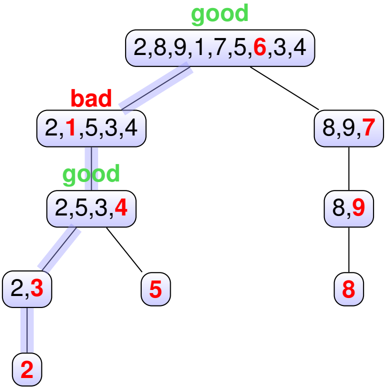

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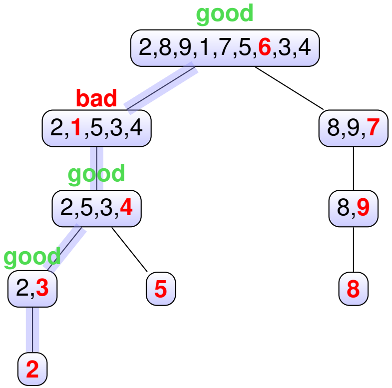

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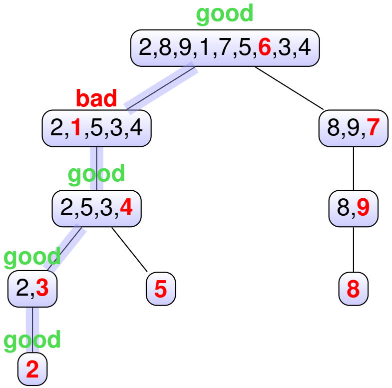

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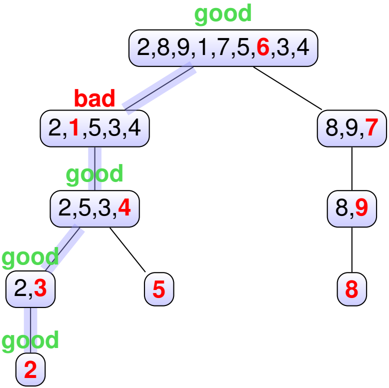

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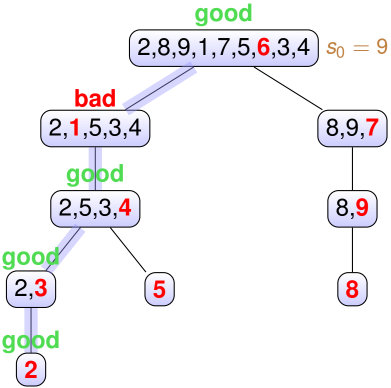

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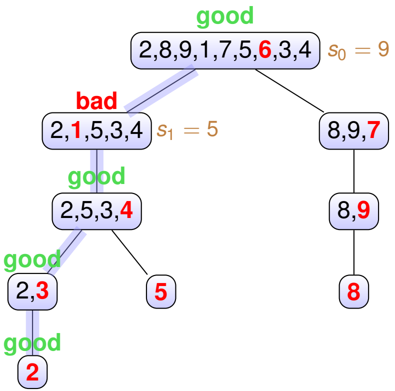

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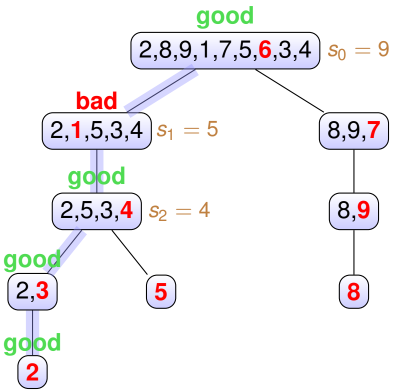

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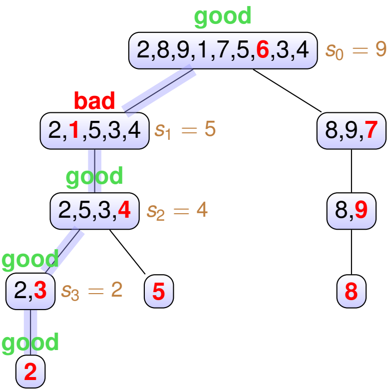

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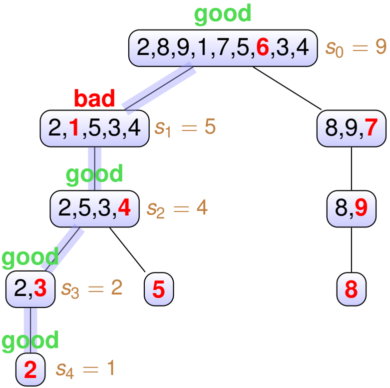

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  $\Rightarrow$ number of **bad** vertices in the first $24 \log n$ levels is more than $21 \log n$.

Randomised QuickSort: Analysis (3/4)

- Consider now any element $i \in \{1, 2, \ldots, n\}$ and construct the path $P = P(i)$ one level by one
- For P to proceed from level k to $k+1$, the condition $s_k > 1$ is necessary

How far could such a path P possibly run until we have $s_k = 1$?

- We start with $s_0 = n$
- **First Case**, **good** node: $s_{k+1} \leq \frac{2}{3} \cdot s_k$.
- **Second Case**, **bad** node: $s_{k+1} \leq s_k$.

This even holds always, i.e., deterministically!

$\Rightarrow$ There are at most $T = \frac{\log n}{\log(3/2)} < 3 \log n$ many **good** nodes on any path P.

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Let us now upper bound the probability that this “bad event” happens!

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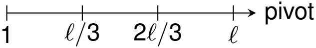

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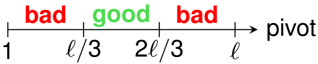

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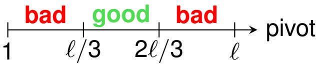

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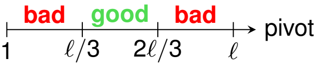

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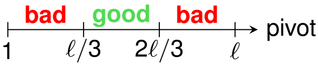

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Answer: We can then simply define X_j as 0 (deterministically).

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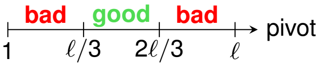

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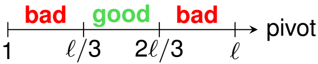

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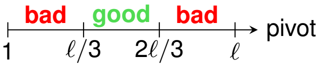

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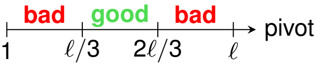

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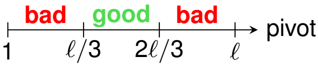

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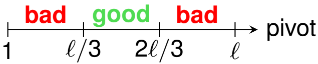

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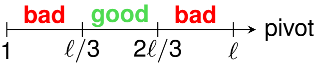

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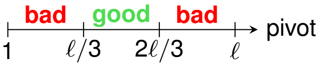

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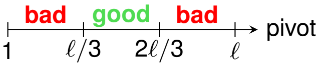

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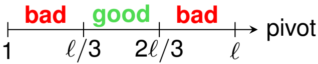

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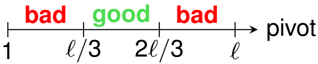

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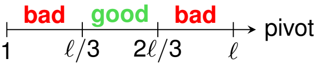

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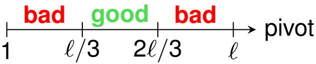

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Exercise: *[Ex 2-3.6]* Our upper bound of $O(n \log n)$ whp also immediately implies a $O(n \log n)$ bound on the expected number of comparisons!

Randomised QuickSort: Final Remarks

- Well-known: any comparison-based sorting algorithm needs $\Omega(n \log n)$
- A classical result: expected number of comparison of randomised QuickSort is $2n \log n + O(n)$ (see, e.g., book by Mitzenmacher & Upfal)

Exercise: *[Ex 2-3.6]* Our upper bound of $O(n \log n)$ whp also immediately implies a $O(n \log n)$ bound on the expected number of comparisons!

- It is possible to deterministically find the best pivot element that divides the array into two subarrays of the same size.
- The latter requires to compute the median of the array in linear time, which is not easy...
- The presented randomised algorithm for QuickSort is much easier to implement!

Outline

Application 2: Randomised QuickSort

Extensions of Chernoff Bounds

Applications of Method of Bounded Differences

Appendix: More on Moment Generating Functions (non-examinable)

Hoeffding's Extension

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Let X be a random variable with mean 0 such that $a \leq X \leq b$. Then for all $\lambda \in \mathbb{R}$,

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We omit the proof of this lemma!

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Hoeffding's Inequality

Let $X_1, \ldots, X_n$ be independent random variable with mean μ_i such that $a_i \leq X_i \leq b_i$. Let $X = X_1 + \ldots + X_n$, and let $\mu = \mathbf{E}[X] = \sum_{i=1}^{n} \mu_i$. Then for any $t > 0$

$$\mathbf{P}[X \geq \mu + t] \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right),$$

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This is not magic! you just need to optimise λ!

Method of Bounded Differences

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Suppose, we have independent random variables $X_1, \ldots, X_n$. We want to study the random variable:

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In all those cases (and more) we can easily prove concentration of $f(X_1, \ldots, X_n)$ around its mean by the so-called **Method of Bounded Differences**.

Method of Bounded Differences

A function f is called Lipschitz with parameters $\mathbf{c} = (c_1, \dots, c_n)$ if for all $i = 1, 2, \dots, n$,

$$|f(x_1, x_2, \dots, x_{i-1}, \boldsymbol{x_i}, x_{i+1}, \dots, x_n) - f(x_1, x_2, \dots, x_{i-1}, \widetilde{\boldsymbol{x_i}}, x_{i+1}, \dots, x_n)| \leq c_i,$$

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- The proof is omitted here (it requires the concept of martingales).

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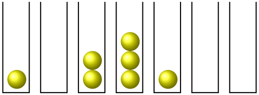

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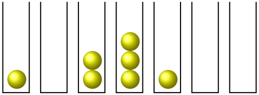

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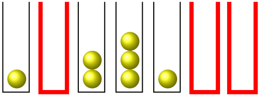

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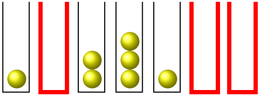

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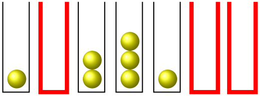

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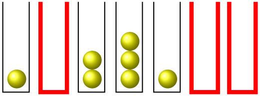

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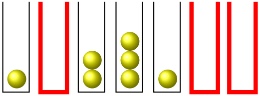

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This is a decent bound, but for some values of m it is far from tight and stronger bounds are possible through a refined analysis.

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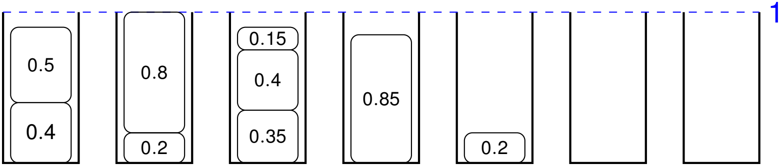

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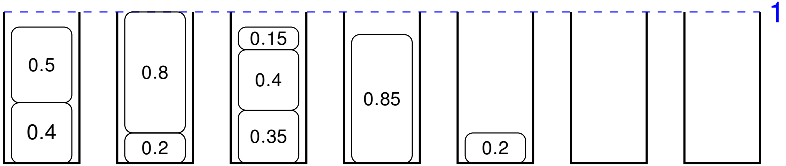

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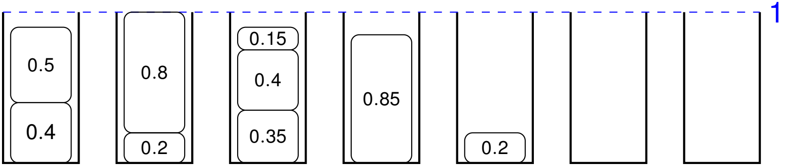

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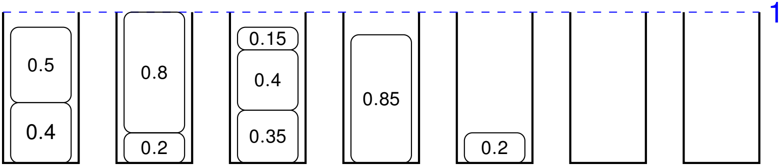

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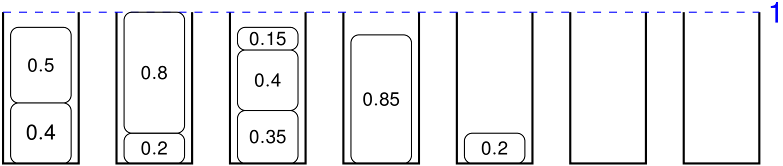

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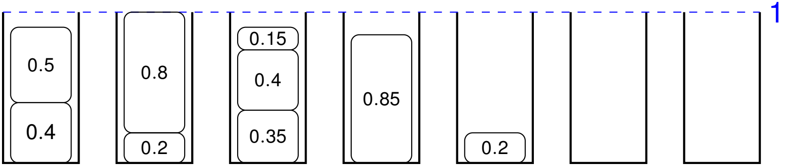

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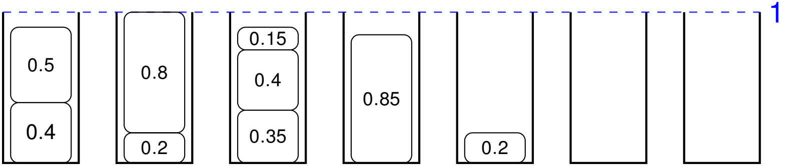

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This is a typical example where proving concentration is much easier than calculating (or estimating) the expectation!

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$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

Proof of 2:

$$M_{X+Y}(t) = \mathbf{E}\left[e^{t(X+Y)}\right] = \mathbf{E}\left[e^{tX} \cdot e^{tY}\right] \stackrel{(!)}{=} \mathbf{E}\left[e^{tX}\right] \cdot \mathbf{E}\left[e^{tY}\right] = M_X(t)M_Y(t) \qquad \square$$