Randomised Algorithms

Lecture 1: Introduction to Course & Introduction to Chernoff Bounds

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UNIVERSITY OF CAMBRIDGE

Outline

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What? Randomised Algorithms utilise random bits to compute their output.

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But often: simple algorithm at the cost of a sophisticated analysis!

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How? This course aims to strengthen your knowledge of probability theory and apply this to analyse examples of randomised algorithms.

What if I (initially) don't care about randomised algorithms?
Many of the techniques in this course (Markov Chains, Concentration of Measure, Spectral Theory) are very relevant to other popular areas of research and employment such as Data Science and Machine Learning.

Some stuff you should know...

In this course we will assume some basic knowledge of probability:

- random variable
- computing expectations and variances
- notions of independence
- "general" idea of how to compute probabilities (manipulating, counting and **estimating**)

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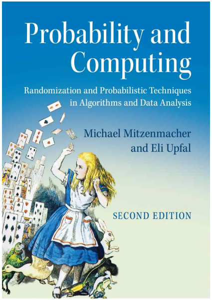
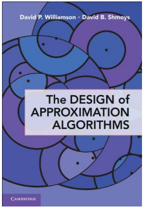
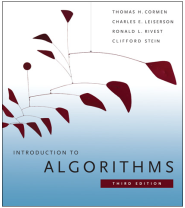
You should also be familiar with basic computer science, mathematics knowledge such as:

- graphs
- basic algorithms (sorting, graph algorithms etc.)
- matrices, norms and vectors

Textbooks







- **($\star$) Michael Mitzenmacher and Eli Upfal. Probability and Computing: Randomized Algorithms and Probabilistic Analysis, Cambridge University Press, 2nd edition, 2017**
- David P. Williamson and David B. Shmoys. The Design of Approximation Algorithms, Cambridge University Press, 2011
- Cormen, T.H., Leiserson, C.D., Rivest, R.L. and Stein, C. Introduction to Algorithms. MIT Press (3rd ed.), 2009
  (We will adopt some of the labels (e.g., Theorem 35.6) from this book in Lectures 6-10)

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- Intro to Randomised Algorithms; Logistics; Recap of Probability; Examples.

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Lectures 2-5 focus on probabilistic tools and techniques.

2–3 **Concentration** (Lectures)

- Concept of Concentration; Recap of Markov and Chebyshev; Chernoff Bounds and Applications; Extensions: Hoeffding's Inequality and Method of Bounded Differences; Applications.

4 **Markov Chains and Mixing Times** (Lecture)

- Recap; Stopping and Hitting Times; Properties of Markov Chains; Convergence to Stationary Distribution; Variation Distance and Mixing Time

5 **Hitting Times and Application to 2-SAT** (Lecture)

- Reversible Markov Chains and Random Walks on Graphs; Cover Times and Hitting Times on Graphs (Example: Paths and Grids); A Randomised Algorithm for 2-SAT Algorithm

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Lectures 6-8 introduce linear programming, a (mostly) deterministic but very powerful technique to solve various optimisation problems.

6–7 Linear Programming (Lectures)

- Introduction to Linear Programming, Applications, Standard and Slack Forms, Simplex Algorithm, Finding an Initial Solution, Fundamental Theorem of Linear Programming

8 Travelling Salesman Problem (Interactive Demo)

- Hardness of the general TSP problem, Formulating TSP as an integer program; Classical TSP instance from 1954; Branch & Bound Technique to solve integer programs using linear programs

We then see how we can efficiently combine linear programming with randomised techniques, in particular, rounding:

9–10 **Randomised Approximation Algorithms** (Lectures)

- MAX-3-CNF and Guessing, Vertex-Cover and Deterministic Rounding of Linear Program, Set-Cover and Randomised Rounding, Concluding Example: MAX-CNF and Hybrid Algorithm

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Lectures 11-12 cover a more advanced topic with ML flavour:

11–12 **Spectral Graph Theory and Spectral Clustering** (Lectures)

- Eigenvalues, Eigenvectors and Spectrum; Visualising Graphs; Expansion; Cheeger's Inequality; Clustering and Examples; Analysing Mixing Times

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Recap: Probability Space

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Components of the Probability Space $(\Omega, \Sigma, \mathbf{P})$

- The Sample Space Ω contains all the possible outcomes $\omega_1, \omega_2, \ldots$ of the experiment.
- The Event Space Σ is the power-set of Ω containing events, which are combinations of outcomes (subsets of Ω including $\emptyset$ and Ω).
- The Probability Measure $\mathbf{P}$ is a function from Σ to $\mathbb{R}$ satisfying
  (i) $0 \leq \mathbf{P}[\mathcal{E}] \leq 1$, for all $\mathcal{E} \in \Sigma$
  (ii) $\mathbf{P}[\Omega] = 1$
  (iii) If $\mathcal{E}_1, \mathcal{E}_2, \ldots \in \Sigma$ are pairwise disjoint ($\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for all $i \neq j$) then

$$\mathbf{P}\left[\bigcup_{i=1}^{\infty} \mathcal{E}_i\right] = \sum_{i=1}^{\infty} \mathbf{P}[\mathcal{E}_i].$$

Recap: Random Variables

A random variable X on $(\Omega, \Sigma, \mathbf{P})$ is a function $X : \Omega \to \mathbb{R}$ mapping each sample "outcome" to a real number.

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- The indicator random variable $\mathbf{1}_{\mathcal{E}}$ of an event $\mathcal{E} \in \Sigma$ given by

$$\mathbf{1}_{\mathcal{E}}(\omega) = \begin{cases} 1 & \text{if } \omega \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

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- The number of sixes of two dice throws $X_1, X_2 \in \{1, 2, \ldots, 6\}$ is

$$\mathbf{1}_{X_1=6} + \mathbf{1}_{X_2=6}$$

Recap: Boole's Inequality (Union Bound)

Union Bound

Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be a collection of events in Σ. Then

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3. It is clear that $\mathbf{1}_{\bigcup_{i=1}^{n} \mathcal{E}_i} \leq \sum_{i=1}^{n} \mathbf{1}_{\mathcal{E}_i}$ (**Check this**)
4. Taking expectation completes the proof.

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A Randomised Algorithm for MAX-CUT (1/2)

$E(A, B)$: set of edges with one endpoint in $A \subseteq V$ and the other in $B \subseteq V$.

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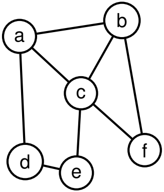

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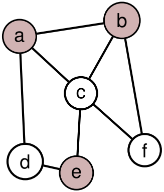

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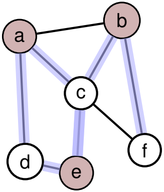

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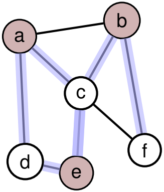

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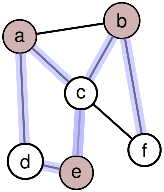
- clustering, statistical physics



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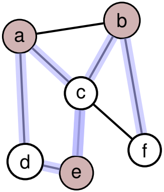
Comments:

- This problem will appear again in the course
- MAX-CUT is NP-hard
- It is different from the clustering problem, where we want to find a sparse cut
- Note that the MIN-CUT problem is solvable in polynomial time!



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A Randomised Algorithm for MAX-CUT (2/2)

RandMaxCut(G)

Start with $S \leftarrow \emptyset$

For each $v \in V$, add v to S with probability $1/2$

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This kind of "random guessing" will appear often in this course!

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Proposition

RANDMAXCUT(G) gives a 2-approximation using time $O(n)$.

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Question:

1. What is the sample space Ω here?
2. Which quantity do we need to analyse?

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More details on approximation algorithms from Lecture 9 onwards!

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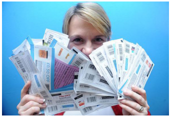

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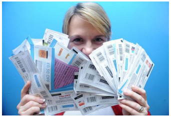

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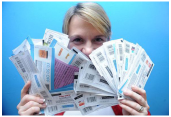

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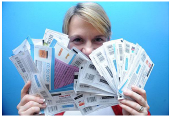

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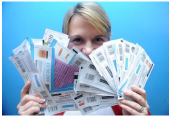

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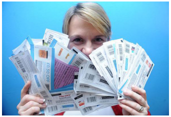

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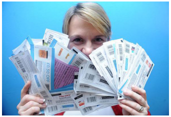

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1. Prove it takes $n \sum_{k=1}^{n} \frac{1}{k} \approx n \log n$ expected boxes to collect all coupons
2. Use Union Bound to prove that the probability it takes more than $n \log n + cn$ boxes to collect all n coupons is $\leq e^{-c}$.

   Hint: It is useful to remember that $1 - x \leq e^{-x}$ for all x

Outline

Introduction

Topics and Syllabus

A (Very) Brief Reminder of Probability Theory

Basic Examples

Introduction to Chernoff Bounds

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  1. **Randomised Algorithms:** Easy to Design and Implement
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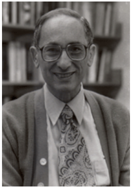

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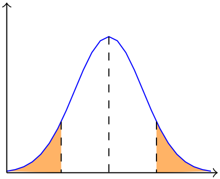

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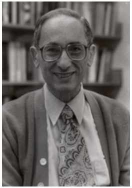

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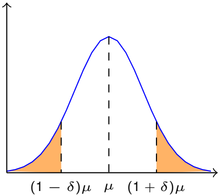

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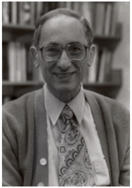
  ⋮



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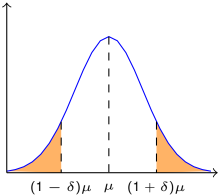

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Chebyshev's inequality (or Markov) can be obtained by chosing $f(X) := (X - \mu)^2$ (or $f(X) := X$, respectively).

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We can consider the first, second, third and more moments! That is the basic idea behind the Chernoff Bounds

Our First Chernoff Bound

Chernoff Bounds (General Form, Upper Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^n p_i$. Then, for any $\delta > 0$ it holds that

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This implies that for any $t > \mu$,

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What about a concrete value of n, say $n = 100$?

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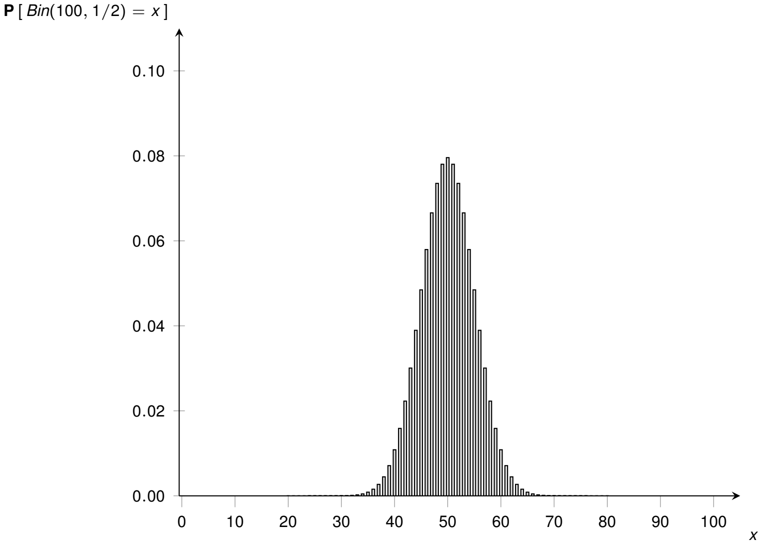

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- Markov's inequality: $\mathbf{E}[X] = 100/2 = 50$.

$$\mathbf{P}[X \geq 3/2 \cdot \mathbf{E}[X]] \leq 2/3 = \mathbf{0.666}.$$

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Consider $n = 100$ independent coin flips. We wish to find an upper bound on the probability that the number of heads is greater or equal than 75.

- Markov's inequality: $\mathbf{E}[X] = 100/2 = 50$.

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  and plugging in $t = 25$ gives an upper bound of $25/25^2 = 1/25 = \mathbf{0.04}$, much better than what we obtained by Markov's inequality.

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Chernoff bound yields a much better result (but needs independence!)