Quantum Computing (CST Part II)

Lecture 6: Some Applications of Quantum Information

Beam me up, Scotty
Captain Kirk

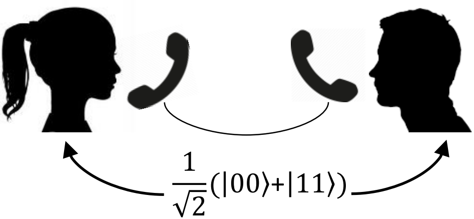
Why look at “some applications of quantum information”?

Before getting into the details of quantum computing proper, we will look at some other aspects of quantum information processing, which have remarkable results that cannot be achieved classically, even in principle. Specifically, we will look at:

- Using *entanglement as a resource*, in teleportation and superdense coding.
- Using quantum phenomena to achieve information theoretically (rather than computationally) secure communications.

Alice and Bob revisited

Alice and Bob once again share an entangled pair ($(1/\sqrt{2})(|00\rangle + |11\rangle)$). Previously we saw that they couldn't use this alone for signalling, so we will also give them a communication channel.

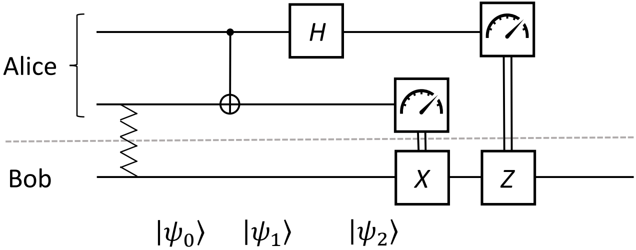


We will now see how they can:

1. Use the shared entanglement and two bits of classical information to transfer one qubit (teleportation).
2. Use the shared entanglement and one qubit of quantum information to transfer two classical bits (superdense coding).

Teleportation

This is teleportation circuit (the zigzag denotes an entangled pair):



$$
\begin{aligned}
|\psi_0\rangle =& \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle) \\
=& \frac{1}{\sqrt{2}}(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)) \\
|\psi_1\rangle =& \frac{1}{\sqrt{2}}(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)) \\
|\psi_2\rangle =& \frac{1}{2}(\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle))
\end{aligned}
$$

Teleportation (cont.)

$$
\begin{aligned}
|\psi_2\rangle = &\frac{1}{2}\left(|00\rangle\left(\alpha|0\rangle+\beta|1\rangle\right)+|01\rangle\left(\alpha|1\rangle+\beta|0\rangle\right)\right. \\
&\left.+|10\rangle\left(\alpha|0\rangle-\beta|1\rangle\right)+|11\rangle\left(\alpha|1\rangle-\beta|0\rangle\right)\right)
\end{aligned}
$$

Alice now measures her two qubits, and sends the results to Bob (this classical transmission is represented in the circuit diagram as the two vertical classical control operations), who uses this classical information to apply a correction to his qubit (qubit 3):

Measurement	**Qubit 3 before**	**Correction**	**Qubit 3 after**
00	$\alpha|0\rangle+\beta|1\rangle$	I	$\alpha|0\rangle+\beta|1\rangle$
01	$\alpha|1\rangle+\beta|0\rangle$	X	$\alpha|0\rangle+\beta|1\rangle$
10	$\alpha|0\rangle-\beta|1\rangle$	Z	$\alpha|0\rangle+\beta|1\rangle$
11	$\alpha|1\rangle-\beta|0\rangle$	ZX	$\alpha|0\rangle+\beta|1\rangle$

So we can see that, regardless of the measurement outcomes, Alice's qubit state has now been realised on qubit 3 (i.e., in Bob's possession). Note that teleportation does not violate the no-cloning principle, as Alice's original qubit has been destroyed in the process.

History of quantum teleportation

- Discovered in 1993
- Experimentally realised in 1997
- The latest reported record distance for quantum teleportation is 1,400 km (870 miles) using the Micius satellite for space-based quantum teleportation

zmescience.com

Micius Satellite

Superdense coding: Alice's transmission

Superdense coding was discovered in 1992, and experimentally realised in 1996, it goes as follows:

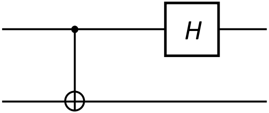
Alice and Bob share an entangled pair, Alice wants to send two bits, i.e., one of 00, 01, 10 or 11. To do so, she applies a single-qubit unitary to **her qubit**:

Initial state	Alice's bitstring	Operation	Final state
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	00	I	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	01	X	$\frac{1}{\sqrt{2}}(\|10\rangle + \|01\rangle)$
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	10	Z	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	11	XZ	$\frac{1}{\sqrt{2}}(\|10\rangle - \|01\rangle)$

Alice then sends her qubit to Bob.

Superdense coding: Bob's correction

Bob then receives Alice's qubit, so now has both qubits, and applies the following circuit to the two:



Which yields (ignoring a global phase factor of -1 after the CNOT in the last line):

Initial state	**After CNOT**	**After H**
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle) = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)\|0\rangle$	$\|00\rangle$
$\frac{1}{\sqrt{2}}(\|10\rangle + \|01\rangle)$	$\frac{1}{\sqrt{2}}(\|11\rangle + \|01\rangle) = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)\|1\rangle$	$\|01\rangle$
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle) = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)\|0\rangle$	$\|10\rangle$
$\frac{1}{\sqrt{2}}(\|10\rangle - \|01\rangle)$	$\frac{1}{\sqrt{2}}(\|11\rangle - \|01\rangle) = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)\|1\rangle$	$\|11\rangle$

The final step is for Bob to perform a computational basis measurements on the two qubits, which will give him Alice's bitstring.

Quantum key distribution

- Quantum key distribution (QKD) was discovered in 1984, and the original protocol (which we will study) is known as BB84 after its discoverers, Charles Bennett and Gilles Brassard .
- It later turned out that QKD had previously been discovered, but not make public, by researchers at GCHQ.
- BB84 does not require entanglement (although some subsequent protocols do).

news.sky.com

GCHQ

- The record bit rate (of exchange of secure keys) is 1 Mbit/s, in a collaboration between this university and Toshiba.

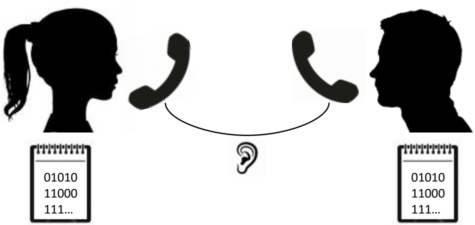
The one-time pad

These days, we are used to public-key cryptography, such as RSA which relies on the one-way nature of some mathematical function (i.e., factoring numbers is hard – or is it?!) to *computationally* guarantee security. A stronger requirement is to absolutely (*information theoretically*) guarantee security. Of which the simplest example is a one-time pad:

- At some date in the future Alice will send Bob an n bit message.
- Before that Alice and Bob meet-up and share a "one-time pad" (or key) – a list of n random bits r.
- When the time comes to send the message m, Alice encodes the message by using her copy of r to send $m \oplus r$.
- Bob receives the message and decodes it by using his copy of r: $(m \oplus r) \oplus r = m$

Alice and Bob then discard r.

Resources required to use a one-time pad



Lets take a more detailed look at the practicalities of using a one-time pad:

1. Alice and Bob must previously meet in person (or communicate at a distance via an absolutely secure channel).
2. Alice sends an encoded message, $m \oplus r$ (that is, $m + r \mod 2$), to Bob via a channel, which in general could be tapped...

...but without access to r, all that an eavesdropper (Eve) would get is a random string of bits.

So the problem here is item 1, that Alice and Bob must meet in advance (or that they must have an absolutely secure channel – in which case they may as well use that for the message transmission).

A one-time pad from quantum key distribution

QKD can be used to generate a one-time pad without Alice and Bob meeting, the resources required to achieve this are:

- An authenticated public classical channel. By “authenticated” we mean that if a transmission purports to be from Bob, then Alice can be absolutely sure it was indeed sent by Bob (and vice versa).
- A quantum channel, which could possibly be eavesdropped.

Additionally,

- **Alice** has a private source of random classical bits.
- **Alice** can produce qubits in states $|0\rangle$, $|1\rangle = X|0\rangle$, $|+\rangle = H|0\rangle$ and $|-\rangle = H|1\rangle$.
- **Bob** can measure qubits in either the computational ($|0\rangle, |1\rangle$) basis, or the $|+\rangle, |-\rangle$ basis.

The BB84 protocol

1. Alice has a bitstring, and for each bit she either encodes $\{0, 1\}$ as $\{|0\rangle, |1\rangle\}$ or $\{|+\rangle, |-\rangle\}$ (chosen at random with equal probability). Alice then sends the qubit to Bob.
2. Bob receives the qubit and either measures in the $\{|0\rangle, |1\rangle\}$ basis or the $\{|+\rangle, |-\rangle\}$ basis.
3. Bob announces over a public channel in which basis he measured the qubit.
4. Alice replies over the public channel whether that was the basis in which the qubit was prepared.
5. If the same basis was indeed used for the preparation and the measurement then Bob's measurement outcome will equal Alice's bit, and they both append this bit to each of their copies of the key, otherwise they discard.

On average, Alice and Bob will discard half of their bits. In the following worked example we will see that this does indeed yield a shared key, and furthermore we will see that it is private in the sense that any attempt by a third-party to discover the key will lead to a detectable change.

BB84: worked example

A bit	A basis	Qubit	B basis	B bit
0	$\|0\rangle, \|1\rangle$	$\|0\rangle$	$\|0\rangle, \|1\rangle$	0
1	$\|+\rangle, \|-\rangle$	$\|-\rangle$	$\|+\rangle, \|-\rangle$	1
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$
0	$\|0\rangle, \|1\rangle$	$\|0\rangle$	$\|0\rangle, \|1\rangle$	0
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|0\rangle, \|1\rangle$	1
0	$\|+\rangle, \|-\rangle$	$\|+\rangle$	$\|+\rangle, \|-\rangle$	0
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$
1	$\|+\rangle, \|-\rangle$	$\|-\rangle$	$\|0\rangle, \|1\rangle$	$\{0, 1\}$

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1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$
0	$\|0\rangle, \|1\rangle$	$\|0\rangle$	$\|0\rangle, \|1\rangle$	0
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|0\rangle, \|1\rangle$	1
0	$\|+\rangle, \|-\rangle$	$\|+\rangle$	$\|+\rangle, \|-\rangle$	0
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$
1	$\|+\rangle, \|-\rangle$	$\|-\rangle$	$\|0\rangle, \|1\rangle$	$\{0, 1\}$

The shared key (one-time pad) is the bits for which A and B agree on the measurement basis.

BB84 attack : intercept, measure and retransmit

Which of the bits Alice and Bob transmitted / measured in the same basis is a matter of public record. However, it is also possible that an eavesdropper could “tap” the quantum channel to try to discover the key. The first option is for Eve to intercept, measure and retransmit.

However, owing to the fact that the protocol specifies that it is Bob who announces his measurement basis and Alice who then replies, it is necessary that Eve forwards on to Bob the intercepted qubit before it is made public which basis Alice transmitted in. So it follows that Eve would have to make a random decision about a basis to measure in. Moreover, the agreement about which qubits would be used in the key remains between Alice and Bob alone.

BB84: worked example with eavesdropping

Consider the same example as before, but now with an eavesdropper between Alice and Bob.

A bit	A basis	Qubit	E basis	E bit	Qubit	B basis	B bit
0	$\|0\rangle, \|1\rangle$	$\|0\rangle$	$\|0\rangle, \|1\rangle$	0	$\|0\rangle$	$\|0\rangle, \|1\rangle$	0
1	$\|+\rangle, \|-\rangle$	$\|-\rangle$	$\|0\rangle, \|1\rangle$	$\{0, 1\}$	$\{\|0\rangle, \|1\rangle\}$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$	$\{\|+\rangle, \|-\rangle\}$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$
0	$\|0\rangle, \|1\rangle$	$\|0\rangle$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$	$\{\|+\rangle, \|-\rangle\}$	$\|0\rangle, \|1\rangle$	$\{0, 1\}$
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|0\rangle, \|1\rangle$	1	$\|1\rangle$	$\|0\rangle, \|1\rangle$	1
0	$\|+\rangle, \|-\rangle$	$\|+\rangle$	$\|+\rangle, \|-\rangle$	0	$\|+\rangle$	$\|+\rangle, \|-\rangle$	0
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|0\rangle, \|1\rangle$	1	$\|1\rangle$	$\|+\rangle, \|-\rangle$	$\{0, 1\}$
1	$\|+\rangle, \|-\rangle$	$\|-\rangle$	$\|0\rangle, \|1\rangle$	$\{0, 1\}$	$\{\|0\rangle, \|1\rangle\}$	$\|0\rangle, \|1\rangle$	$\{0, 1\}$

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0	$\|0\rangle, \|1\rangle$	$\|0\rangle$	$\|0\rangle, \|1\rangle$	0	$\|0\rangle$	$\|0\rangle, \|1\rangle$	0
1	$\|+\rangle, \|-\rangle$	$\|-\rangle$	$\|0\rangle, \|1\rangle$	$\{0,1\}$	$\{\|0\rangle, \|1\rangle\}$	$\|+\rangle, \|-\rangle$	$\{0,1\}$
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|+\rangle, \|-\rangle$	$\{0,1\}$	$\{\|+\rangle, \|-\rangle\}$	$\|+\rangle, \|-\rangle$	$\{0,1\}$
0	$\|0\rangle, \|1\rangle$	$\|0\rangle$	$\|+\rangle, \|-\rangle$	$\{0,1\}$	$\{\|+\rangle, \|-\rangle\}$	$\|0\rangle, \|1\rangle$	$\{0,1\}$
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|0\rangle, \|1\rangle$	1	$\|1\rangle$	$\|0\rangle, \|1\rangle$	1
0	$\|+\rangle, \|-\rangle$	$\|+\rangle$	$\|+\rangle, \|-\rangle$	0	$\|+\rangle$	$\|+\rangle, \|-\rangle$	0
1	$\|0\rangle, \|1\rangle$	$\|1\rangle$	$\|0\rangle, \|1\rangle$	1	$\|1\rangle$	$\|+\rangle, \|-\rangle$	$\{0,1\}$
1	$\|+\rangle, \|-\rangle$	$\|-\rangle$	$\|0\rangle, \|1\rangle$	$\{0,1\}$	$\{\|0\rangle, \|1\rangle\}$	$\|0\rangle, \|1\rangle$	$\{0,1\}$

As before, Alice and Bob expect to use the bits for which they agree on the measurement basis as the shared key.

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Consider the same example as before, but now with an eavesdropper between Alice and Bob.

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1	$\vert +\rangle, \vert -\rangle$	$\vert -\rangle$	$\vert 0\rangle, \vert 1\rangle$	$\{0,1\}$	$\{\vert 0\rangle, \vert 1\rangle\}$	$\vert +\rangle, \vert -\rangle$	$\{0,1\}$
1	$\vert 0\rangle, \vert 1\rangle$	$\vert 1\rangle$	$\vert +\rangle, \vert -\rangle$	$\{0,1\}$	$\{\vert +\rangle, \vert -\rangle\}$	$\vert +\rangle, \vert -\rangle$	$\{0,1\}$
0	$\vert 0\rangle, \vert 1\rangle$	$\vert 0\rangle$	$\vert +\rangle, \vert -\rangle$	$\{0,1\}$	$\{\vert +\rangle, \vert -\rangle\}$	$\vert 0\rangle, \vert 1\rangle$	$\{0,1\}$
1	$\vert 0\rangle, \vert 1\rangle$	$\vert 1\rangle$	$\vert 0\rangle, \vert 1\rangle$	1	$\vert 1\rangle$	$\vert 0\rangle, \vert 1\rangle$	1
0	$\vert +\rangle, \vert -\rangle$	$\vert +\rangle$	$\vert +\rangle, \vert -\rangle$	0	$\vert +\rangle$	$\vert +\rangle, \vert -\rangle$	0
1	$\vert 0\rangle, \vert 1\rangle$	$\vert 1\rangle$	$\vert 0\rangle, \vert 1\rangle$	1	$\vert 1\rangle$	$\vert +\rangle, \vert -\rangle$	$\{0,1\}$
1	$\vert +\rangle, \vert -\rangle$	$\vert -\rangle$	$\vert 0\rangle, \vert 1\rangle$	$\{0,1\}$	$\{\vert 0\rangle, \vert 1\rangle\}$	$\vert 0\rangle, \vert 1\rangle$	$\{0,1\}$

As before, Alice and Bob expect to use the bits for which they agree on the measurement basis as the shared key.

However, the presence of the eavesdropper has now randomised some of Bob's measurement outcomes. It follows that a comparison between Alice and Bob's measurement outcomes for *some* of the bits where they have the same measurement basis suffices (statistically) to detect the eavesdropper, and thus indicates that the rest of the bits have been compromised, and cannot be used as the key.

Eavesdropping – other factors

As seen in the worked example, eavesdropping disturbs the shared key – thus whilst Alice and Bob can rest assured that Eve hasn't discovered their key, they do need to set aside a subset of the bits to compare on the public channel, to check whether their key has been disturbed by eavesdropping.

Eve may try to avoid the problem by using a more sophisticated intercept, copy, retransmit attack, where she would keep a copy of the qubit, and only measure it once Alice and Bob had shared on the public channel the bases they agreed on – but this would violate the no-cloning principle.

Another attack could involve Eve storing (not copying) the intercepted qubit, and forwarding on a pre-prepared random qubit to Bob. Bob and Alice would then publicly share their measurement bases, and so Eve could indeed use this information to perform a measurement on the intercepted qubit and thus discover the key. However, as Bob would simply have a load of random bits, the public comparison of measurement outcomes that Alice and Bob conduct on some of the bits for which they agree the measurement basis would reveal the presence of the eavesdropper.

Summary

In this lecture we have looked at three applications of quantum information:

- **Teleportation**: using shared entanglement as a resource that allows a qubit to be transmitted using two bits.
- **Superdense coding**: using shared entanglement as a resource that allows two bits to be transmitted using a single qubit.
- **Quantum key distribution**: creating a one-time pad without Alice and Bob meeting or sharing an absolutely secure communication channel.