Lecture 11
Constraint-based analysis

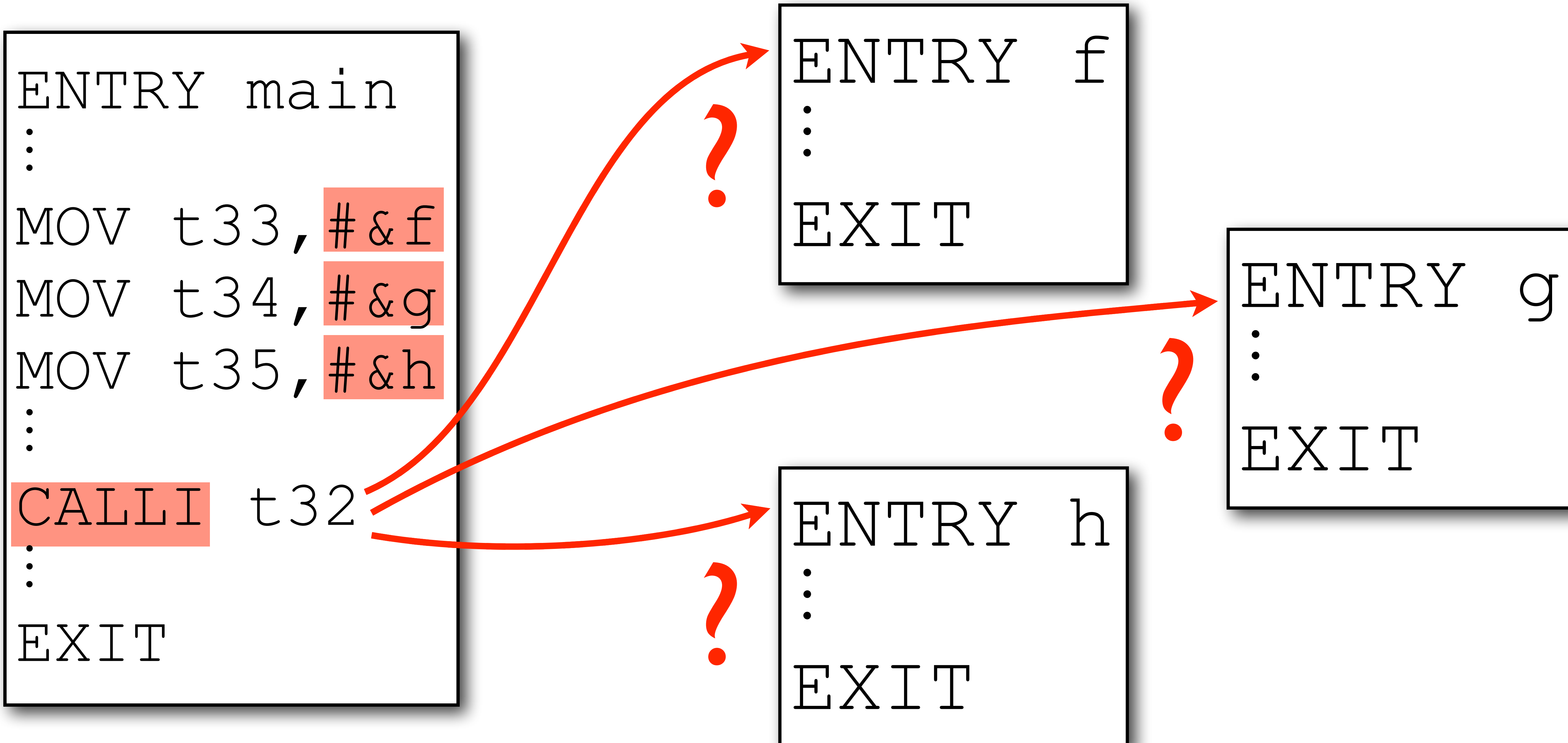
Motivation

Intra-procedural analysis depends upon accurate control-flow information.

In the presence of certain language features (e.g. indirect calls) it is nontrivial to predict accurately how control may flow at execution time — the naïve strategy is very imprecise.

A *constraint-based* analysis called 0CFA can compute a more precise estimate of this information.

Imprecise control flow



Constraint-based analysis

Many of the analyses in this course can be thought of in terms of *solving systems of constraints*.

For example, in LVA, we generate *equality constraints* from each instruction in the program:

$$\mathit{in\text{-}live}(m) = (\mathit{out\text{-}live}(m) \setminus \mathit{def}(m)) \cup \mathit{ref}(m)$$
$$\mathit{out\text{-}live}(m) = \mathit{in\text{-}live}(n) \cup \mathit{in\text{-}live}(o)$$
$$\mathit{in\text{-}live}(n) = (\mathit{out\text{-}live}(n) \setminus \mathit{def}(n)) \cup \mathit{ref}(n)$$
$$\vdots$$

and then iteratively compute their minimal solution.

0CFA

0CFA — “zeroth-order control-flow analysis” — is a constraint-based analysis for discovering which values may reach different places in a program.

When functions (or pointers to functions) are present, this provides information about which functions may potentially be called at each call site.

We can then build a more precise call graph.

Specimen language

Functional languages are a good candidate for this kind of analysis; they have functions as first-class values, so control flow may be complex.

We will use a minimal syntax for expressions:

$$e ::= x \mid c \mid \lambda x.e \mid e_1 e_2 \mid \text{let } x = e_1 \text{ in } e_2$$

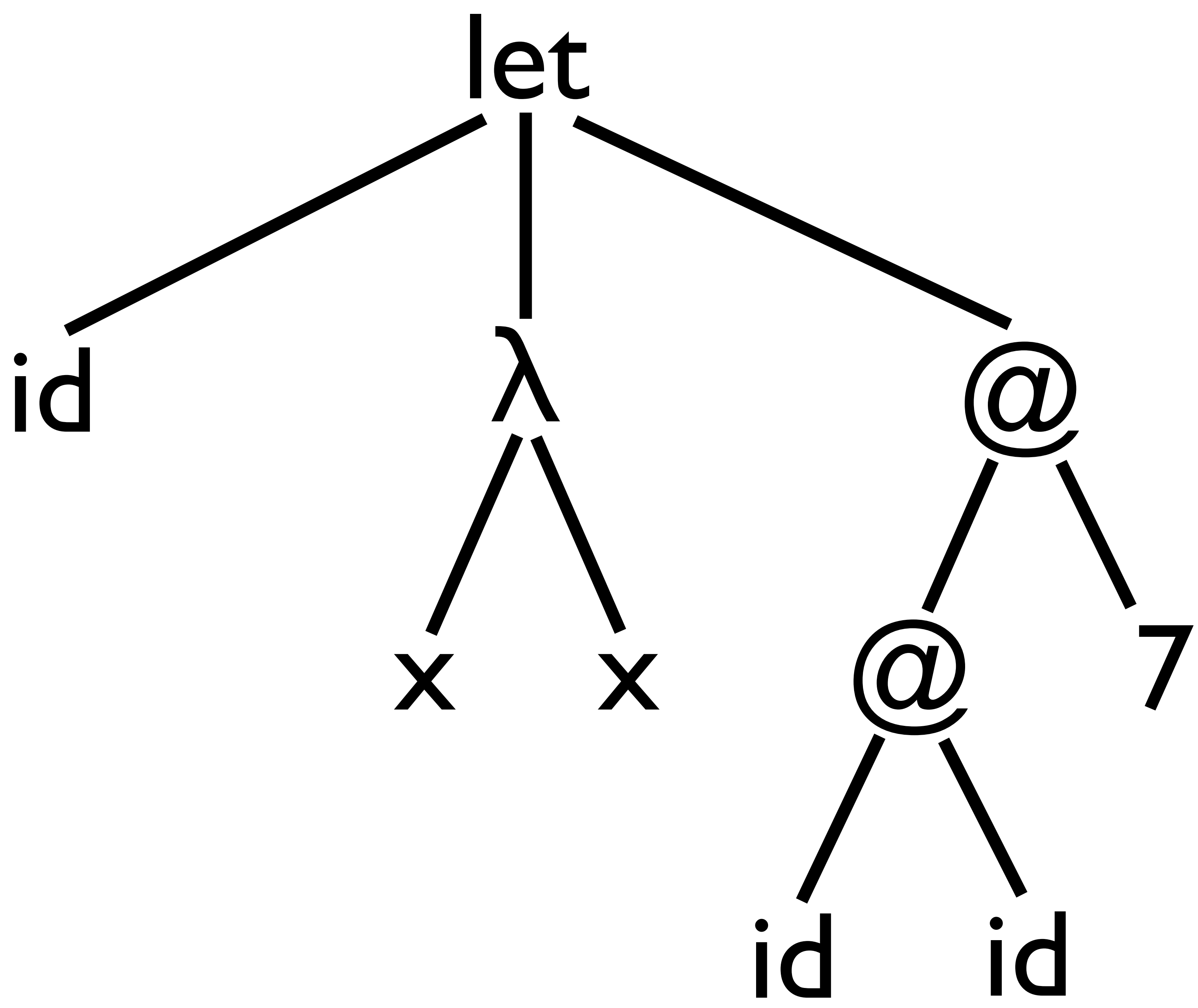
A program in this language is a *closed* expression.

Specimen program

$$\text{let id} = \lambda x.\, x \text{ in id id } 7$$

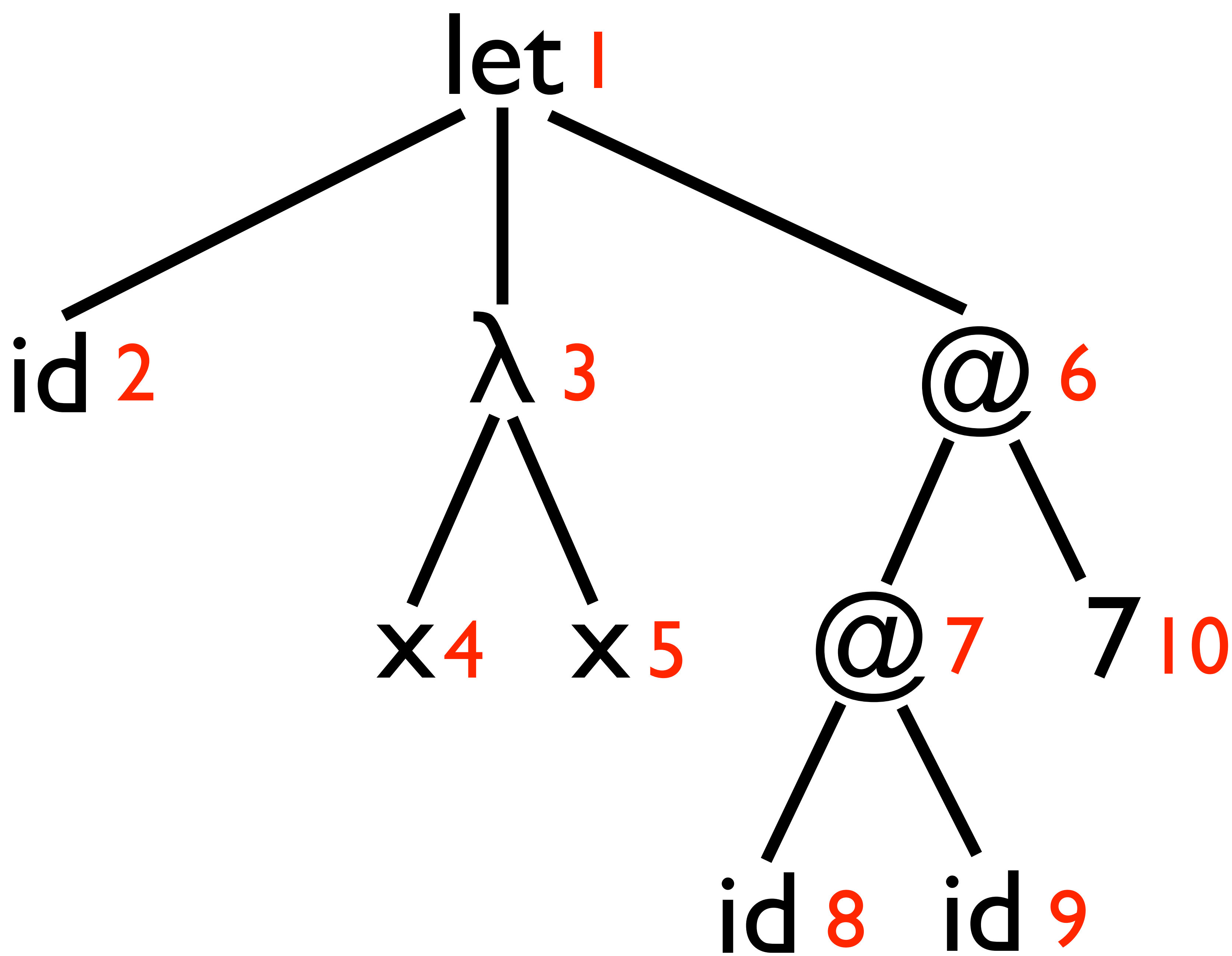
Program points

let id = λx. x in id id 7



Program points

$(\text{let id}^{2} = (\lambda x^{4}.\ x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$



Program points

$$(\text{let } id^{2} = (\lambda x^{4}.\ x^{5})^{3} \text{ in } ((id^{8}\ id^{9})^{7}\ 7^{10})^{6})^{1}$$

Each program point i has an associated *flow variable* α_i.

Each α_i represents the set of *flow values* which may be yielded at program point i during execution.

For this language the flow values are integers and function closures; in this particular program, the only values available are 7^{10} and $(\lambda x^{4}.\ x^{5})^{3}$.

Program points

$$(\text{let id}^{2} = (\lambda x^{4}.\ x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

The precise value of each α_i is undecidable in general, so our analysis will compute a safe overapproximation.

From the structure of the program we can generate a set of constraints on the flow variables, which we can then treat as data-flow inequations and iteratively compute their least solution.

Generating constraints

$(\text{let id}^{2} = (\lambda x^{4}.\ x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$

Generating constraints

$$(\text{let id}^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

$$c^{a} \rightarrow \alpha_{a} \supseteq \{\, c^{a} \,\}$$

Generating constraints

$$(\text{let id}^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

$$7^{10} \longrightarrow \alpha_{10} \supseteq \{\ 7^{10}\ \}$$

$\alpha_{10} \supseteq \{\ 7^{10}\ \}$

Generating constraints

$$(\text{let } id^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((id^{8}\ id^{9})^{7}\ 7^{10})^{6})^{1}$$

$$(\lambda x^{a}.\, e^{b})^{c} \rightarrow \alpha_{c} \supseteq \{\, (\lambda x^{a}.\, e^{b})^{c} \,\}$$

$$\alpha_{10} \supseteq \{\, 7^{10} \,\}$$

Generating constraints

$$(\text{let id}^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

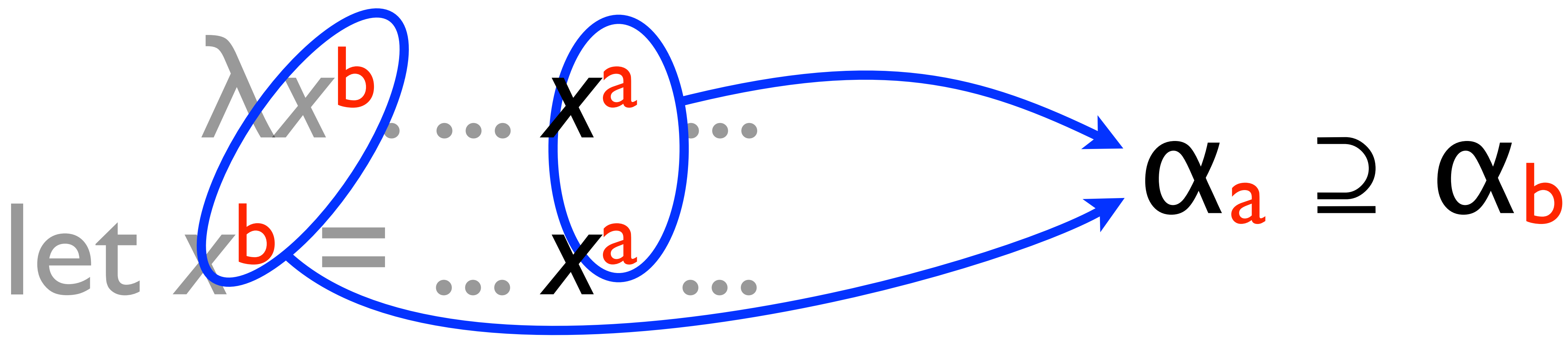
$$(\lambda x^{4}.\, x^{5})^{3} \longrightarrow \alpha_{3} \supseteq \{\, (\lambda x^{4}.\, x^{5})^{3} \,\}$$

$\alpha_{10} \supseteq \{\, 7^{10} \,\}$

$\alpha_{3} \supseteq \{\, (\lambda x^{4}.\, x^{5})^{3} \,\}$

Generating constraints

$(\text{let id}^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$



$\alpha_{10} \supseteq \{\ 7^{10}\ \}$

$\alpha_{3} \supseteq \{\ (\lambda x^{4}.\, x^{5})^{3}\ \}$

Generating constraints

$$(\text{let id}^{2} = (\lambda x^{4}.\ x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

$\lambda x^{4}. \ldots x^{5} \ldots \longrightarrow \alpha_{5} \supseteq \alpha_{4}$

$\text{let id}^{2} = \ldots \text{id}^{8} \ldots \longrightarrow \alpha_{8} \supseteq \alpha_{2}$

$\text{let id}^{2} = \ldots \text{id}^{9} \ldots \longrightarrow \alpha_{9} \supseteq \alpha_{2}$

$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_{3} \supseteq \{ (\lambda x^{4}.\ x^{5})^{3} \}$

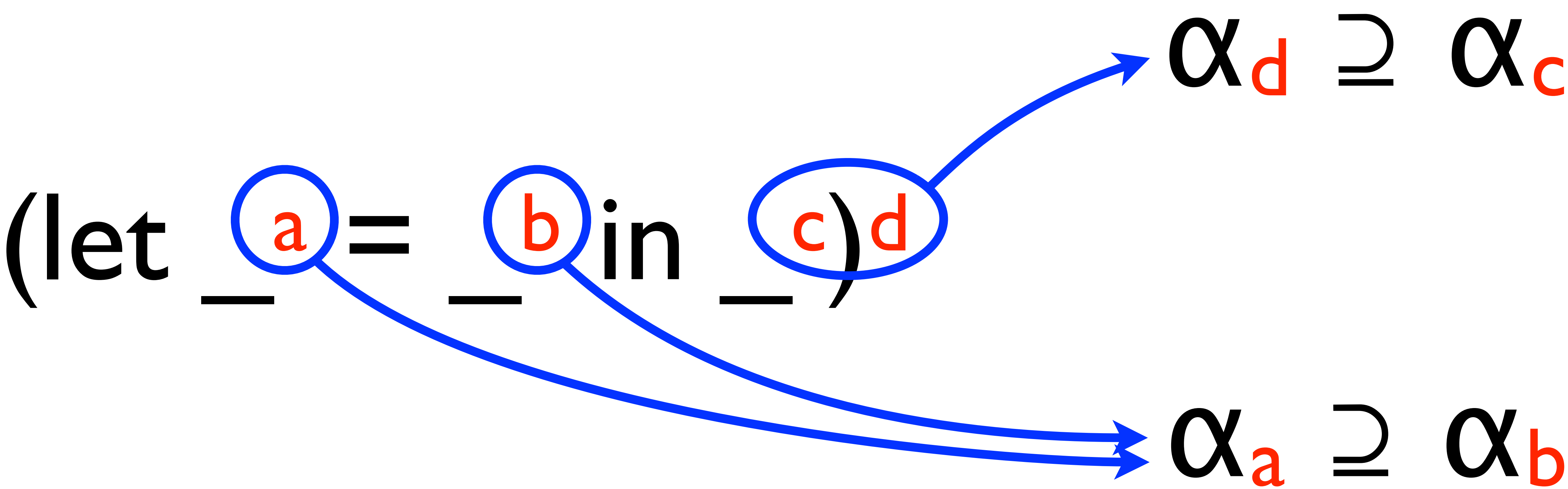
$\alpha_{5} \supseteq \alpha_{4}$

$\alpha_{8} \supseteq \alpha_{2}$

$\alpha_{9} \supseteq \alpha_{2}$

Generating constraints

$$(\text{let id}^2 = (\lambda x^4 . x^5)^3 \text{ in } ((\text{id}^8 \; \text{id}^9)^7 \; 7^{10})^6)^1$$



$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_3 \supseteq \{ (\lambda x^4 . x^5)^3 \}$

$\alpha_5 \supseteq \alpha_4$

$\alpha_8 \supseteq \alpha_2$

$\alpha_9 \supseteq \alpha_2$

Generating constraints

$$(\text{let id}^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

$$(\text{let } \_^{2} = \_^{3} \text{ in } \_^{6})^{1}$$

$$\alpha_{1} \supseteq \alpha_{6}$$

$$\alpha_{2} \supseteq \alpha_{3}$$

$\alpha_{10} \supseteq \{\ 7^{10}\ \}$

$\alpha_{3} \supseteq \{\ (\lambda x^{4}.\, x^{5})^{3}\ \}$

$\alpha_{5} \supseteq \alpha_{4}$

$\alpha_{8} \supseteq \alpha_{2}$

$\alpha_{9} \supseteq \alpha_{2}$

$\alpha_{1} \supseteq \alpha_{6}$

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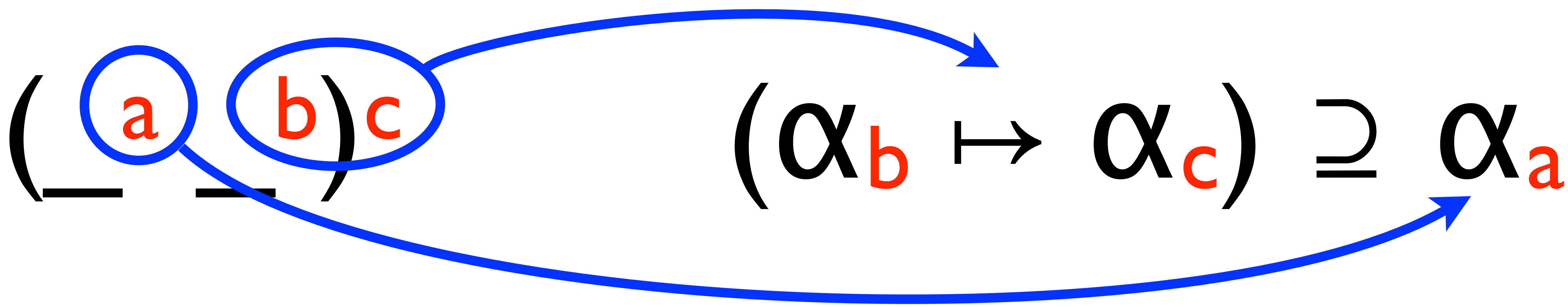
Generating constraints

$$(\text{let } id^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((id^{8}\ id^{9})^{7}\ 7^{10})^{6})^{1}$$



$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_{3} \supseteq \{ (\lambda x^{4}.\, x^{5})^{3} \}$

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$\alpha_{9} \supseteq \alpha_{2}$

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Generating constraints

$$(\text{let id}^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

$$(\_^{8}\ \_^{9})^{7} \longrightarrow (\alpha_{9} \mapsto \alpha_{7}) \supseteq \alpha_{8}$$

$$(\_^{7}\ \_^{10})^{6} \longrightarrow (\alpha_{10} \mapsto \alpha_{6}) \supseteq \alpha_{7}$$

$\alpha_{10} \supseteq \{\ 7^{10}\ \}$	$\alpha_{8} \supseteq \alpha_{2}$	$\alpha_{2} \supseteq \alpha_{3}$
$\alpha_{3} \supseteq \{\ (\lambda x^{4}.\, x^{5})^{3}\ \}$	$\alpha_{9} \supseteq \alpha_{2}$	$(\alpha_{9} \mapsto \alpha_{7}) \supseteq \alpha_{8}$
$\alpha_{5} \supseteq \alpha_{4}$	$\alpha_{1} \supseteq \alpha_{6}$	$(\alpha_{10} \mapsto \alpha_{6}) \supseteq \alpha_{7}$

Generating constraints

$$(\text{let id}^{2} = (\lambda x^{4}.\, x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

$$\alpha_{10} \supseteq \{\ 7^{10}\ \}$$

$$\alpha_{3} \supseteq \{\ (\lambda x^{4}.\, x^{5})^{3}\ \}$$

$$\alpha_{5} \supseteq \alpha_{4}$$

$$\alpha_{8} \supseteq \alpha_{2}$$

$$\alpha_{9} \supseteq \alpha_{2}$$

$$\alpha_{1} \supseteq \alpha_{6}$$

$$\alpha_{2} \supseteq \alpha_{3}$$

$$(\alpha_{9} \mapsto \alpha_{7}) \supseteq \alpha_{8}$$

$$(\alpha_{10} \mapsto \alpha_{6}) \supseteq \alpha_{7}$$

Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_3 \supseteq \{ (\lambda x^4 . x^5)^3 \}$

$\alpha_5 \supseteq \alpha_4$

$\alpha_8 \supseteq \alpha_2$

$\alpha_9 \supseteq \alpha_2$

$\alpha_1 \supseteq \alpha_6$

$\alpha_2 \supseteq \alpha_3$

$(\alpha_9 \mapsto \alpha_7) \supseteq \alpha_8$

$(\alpha_{10} \mapsto \alpha_6) \supseteq \alpha_7$

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Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

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$\alpha_{10} = \{ 7^{10} \}$

Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_3 \supseteq \{ (\lambda x^4 . x^5)^3 \}$

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Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

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Solving constraints

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Solving constraints

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Solving constraints

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$\alpha_7 = \{ \}$

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$\alpha_{10} = \{ 7^{10} \}$

Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_{3} \supseteq \{ (\lambda x^{4}. x^{5})^{3} \}$

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$\alpha_{4} \supseteq \alpha_{9}$

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$$\alpha_{1} = \{ \}$$

$$\alpha_{2} = \{ (\lambda x^{4}. x^{5})^{3} \}$$

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$$\alpha_{5} = \{ (\lambda x^{4}. x^{5})^{3} \}$$

$$\alpha_{6} = \{ \}$$

$$\alpha_{7} = \{ (\lambda x^{4}. x^{5})^{3} \}$$

$$\alpha_{8} = \{ (\lambda x^{4}. x^{5})^{3} \}$$

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$$\alpha_{10} = \{ 7^{10} \}$$

Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

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$\alpha_{1} = \{ \}$

$\alpha_{2} = \{ (\lambda x^{4}. x^{5})^{3} \}$

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$\alpha_{4} = \{ (\lambda x^{4}. x^{5})^{3} \}$

$\alpha_{5} = \{ (\lambda x^{4}. x^{5})^{3} \}$

$\alpha_{6} = \{ \}$

$\alpha_{7} = \{ (\lambda x^{4}. x^{5})^{3} \}$

$\alpha_{8} = \{ (\lambda x^{4}. x^{5})^{3} \}$

$\alpha_{9} = \{ (\lambda x^{4}. x^{5})^{3} \}$

$\alpha_{10} = \{ 7^{10} \}$

Solving constraints

$\alpha_{10} \supseteq \{\, 7^{10} \,\}$ $\quad$ $\alpha_8 \supseteq \alpha_2$ $\quad$ $\alpha_2 \supseteq \alpha_3$

$\alpha_3 \supseteq \{\, (\lambda x^4.\, x^5)^3 \,\}$ $\quad$ $\alpha_9 \supseteq \alpha_2$ $\quad$ $(\alpha_9 \mapsto \alpha_7) \supseteq \alpha_8$

$\alpha_5 \supseteq \alpha_4$ $\quad$ $\alpha_1 \supseteq \alpha_6$ $\quad$ $(\alpha_{10} \mapsto \alpha_6) \supseteq \alpha_7$

$\alpha_4 \supseteq \alpha_9$ $\quad$ $\alpha_7 \supseteq \alpha_5$ $\quad$ $\alpha_4 \supseteq \alpha_{10}$ $\quad$ $\alpha_6 \supseteq \alpha_5$

$$\alpha_1 = \{\,\}$$
$$\alpha_2 = \{\, (\lambda x^4.\, x^5)^3 \,\}$$
$$\alpha_3 = \{\, (\lambda x^4.\, x^5)^3 \,\}$$
$$\alpha_4 = \{\, (\lambda x^4.\, x^5)^3, 7^{10} \,\}$$
$$\alpha_5 = \{\, (\lambda x^4.\, x^5)^3 \,\}$$
$$\alpha_6 = \{\,\}$$
$$\alpha_7 = \{\, (\lambda x^4.\, x^5)^3 \,\}$$
$$\alpha_8 = \{\, (\lambda x^4.\, x^5)^3 \,\}$$
$$\alpha_9 = \{\, (\lambda x^4.\, x^5)^3 \,\}$$
$$\alpha_{10} = \{\, 7^{10} \,\}$$

Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_3 \supseteq \{ (\lambda x^4. x^5)^3 \}$

$\alpha_5 \supseteq \alpha_4$

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$\alpha_7 \supseteq \alpha_5$

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$\alpha_9 \supseteq \alpha_2$

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$\alpha_4 \supseteq \alpha_{10}$

$\alpha_2 \supseteq \alpha_3$

$(\alpha_9 \mapsto \alpha_7) \supseteq \alpha_8$

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$$\alpha_4 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$$

$$\alpha_5 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_6 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_7 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_8 = \{ (\lambda x^4. x^5)^3 \}$$

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$$\alpha_{10} = \{ 7^{10} \}$$

Solving constraints

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$\alpha_4 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$

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$\alpha_6 = \{ (\lambda x^4. x^5)^3 \}$

$\alpha_7 = \{ (\lambda x^4. x^5)^3 \}$

$\alpha_8 = \{ (\lambda x^4. x^5)^3 \}$

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Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

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$\alpha_8 \supseteq \alpha_2$

$\alpha_9 \supseteq \alpha_2$

$\alpha_1 \supseteq \alpha_6$

$\alpha_4 \supseteq \alpha_{10}$

$\alpha_2 \supseteq \alpha_3$

$(\alpha_9 \mapsto \alpha_7) \supseteq \alpha_8$

$(\alpha_{10} \mapsto \alpha_6) \supseteq \alpha_7$

$\alpha_6 \supseteq \alpha_5$

$\alpha_1 = \{ \}$

$\alpha_2 = \{ (\lambda x^4 . x^5)^3 \}$

$\alpha_3 = \{ (\lambda x^4 . x^5)^3 \}$

$\alpha_4 = \{ (\lambda x^4 . x^5)^3, 7^{10} \}$

$\alpha_5 = \{ (\lambda x^4 . x^5)^3, 7^{10} \}$

$\alpha_6 = \{ (\lambda x^4 . x^5)^3 \}$

$\alpha_7 = \{ (\lambda x^4 . x^5)^3, 7^{10} \}$

$\alpha_8 = \{ (\lambda x^4 . x^5)^3 \}$

$\alpha_9 = \{ (\lambda x^4 . x^5)^3 \}$

$\alpha_{10} = \{ 7^{10} \}$

Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_3 \supseteq \{ (\lambda x^4. x^5)^3 \}$

$\alpha_5 \supseteq \alpha_4$

$\alpha_4 \supseteq \alpha_9$

$\alpha_7 \supseteq \alpha_5$

$\alpha_8 \supseteq \alpha_2$

$\alpha_9 \supseteq \alpha_2$

$\alpha_1 \supseteq \alpha_6$

$\alpha_4 \supseteq \alpha_{10}$

$\alpha_2 \supseteq \alpha_3$

$(\alpha_9 \mapsto \alpha_7) \supseteq \alpha_8$

$(\alpha_{10} \mapsto \alpha_6) \supseteq \alpha_7$

$\alpha_6 \supseteq \alpha_5$

$$\alpha_1 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_2 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_3 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_4 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$$

$$\alpha_5 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$$

$$\alpha_6 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_7 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$$

$$\alpha_8 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_9 = \{ (\lambda x^4. x^5)^3 \}$$

$$\alpha_{10} = \{ 7^{10} \}$$

Solving constraints

$\alpha_{10} \supseteq \{\, 7^{10} \,\}$

$\alpha_3 \supseteq \{\, (\lambda x^4.\, x^5)^3 \,\}$

$\alpha_5 \supseteq \alpha_4$

$\alpha_4 \supseteq \alpha_9$

$\alpha_7 \supseteq \alpha_5$

$\alpha_8 \supseteq \alpha_2$

$\alpha_9 \supseteq \alpha_2$

$\alpha_1 \supseteq \alpha_6$

$\alpha_4 \supseteq \alpha_{10}$

$\alpha_2 \supseteq \alpha_3$

$(\alpha_9 \mapsto \alpha_7) \supseteq \alpha_8$

$(\alpha_{10} \mapsto \alpha_6) \supseteq \alpha_7$

$\alpha_6 \supseteq \alpha_5$

$\alpha_1 = \{\, (\lambda x^4.\, x^5)^3 \,\}$

$\alpha_2 = \{\, (\lambda x^4.\, x^5)^3 \,\}$

$\alpha_3 = \{\, (\lambda x^4.\, x^5)^3 \,\}$

$\alpha_4 = \{\, (\lambda x^4.\, x^5)^3, 7^{10} \,\}$

$\alpha_5 = \{\, (\lambda x^4.\, x^5)^3, 7^{10} \,\}$

$\alpha_6 = \{\, (\lambda x^4.\, x^5)^3, 7^{10} \,\}$

$\alpha_7 = \{\, (\lambda x^4.\, x^5)^3, 7^{10} \,\}$

$\alpha_8 = \{\, (\lambda x^4.\, x^5)^3 \,\}$

$\alpha_9 = \{\, (\lambda x^4.\, x^5)^3 \,\}$

$\alpha_{10} = \{\, 7^{10} \,\}$

Solving constraints

$\alpha_{10} \supseteq \{ 7^{10} \}$ $\alpha_8 \supseteq \alpha_2$ $\alpha_2 \supseteq \alpha_3$

$\alpha_3 \supseteq \{ (\lambda x^4. x^5)^3 \}$ $\alpha_9 \supseteq \alpha_2$ $(\alpha_9 \mapsto \alpha_7) \supseteq \alpha_8$

$\alpha_5 \supseteq \alpha_4$ $\alpha_1 \supseteq \alpha_6$ $(\alpha_{10} \mapsto \alpha_6) \supseteq \alpha_7$

$\alpha_4 \supseteq \alpha_9$ $\alpha_7 \supseteq \alpha_5$ $\alpha_4 \supseteq \alpha_{10}$ $\alpha_6 \supseteq \alpha_5$

$\alpha_1 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$ $\alpha_6 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$

$\alpha_2 = \{ (\lambda x^4. x^5)^3 \}$ $\alpha_7 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$

$\alpha_3 = \{ (\lambda x^4. x^5)^3 \}$ $\alpha_8 = \{ (\lambda x^4. x^5)^3 \}$

$\alpha_4 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$ $\alpha_9 = \{ (\lambda x^4. x^5)^3 \}$

$\alpha_5 = \{ (\lambda x^4. x^5)^3, 7^{10} \}$ $\alpha_{10} = \{ 7^{10} \}$

Using solutions

$$\alpha_{10} = \{\ 7^{10}\ \}$$

$$\alpha_{2}, \alpha_{3}, \alpha_{8}, \alpha_{9} = \{\ (\lambda x^{4}.\ x^{5})^{3}\ \}$$

$$\alpha_{1}, \alpha_{4}, \alpha_{5}, \alpha_{6}, \alpha_{7} = \{\ (\lambda x^{4}.\ x^{5})^{3}, 7^{10}\ \}$$

$$(\text{let id}^{2} = (\lambda x^{4}.\ x^{5})^{3} \text{ in } ((\text{id}^{8}\ \text{id}^{9})^{7}\ 7^{10})^{6})^{1}$$

Limitations

0CFA is still imprecise because it is *monovariant*: each expression has only one flow variable associated with it, so multiple calls to the same function allow multiple values into the single flow variable for the function body, and these values "leak out" at all potential call sites.

Limitations

$\alpha_{10} \supseteq \{ 7^{10} \}$

$\alpha_3 \supseteq \{ (\lambda x^4 . x^5)^3 \}$

$\alpha_5 \supseteq \alpha_4$

$\alpha_4 \supseteq \alpha_9$

$\alpha_7 \supseteq \alpha_5$

$\alpha_8 \supseteq \alpha_2$

$\alpha_9 \supseteq \alpha_2$

$\alpha_1 \supseteq \alpha_6$

$\alpha_4 \supseteq \alpha_{10}$

$\alpha_2 \supseteq \alpha_3$

$(\alpha_9 \mapsto \alpha_7) \supseteq \alpha_8$

$(\alpha_{10} \mapsto \alpha_6) \supseteq \alpha_7$

$\alpha_6 \supseteq \alpha_5$

$$\alpha_7 = \{ (\lambda x^4 . x^5)^3 \}$$

$$\alpha_8 = \{ (\lambda x^4 . x^5)^3 \}$$

$$\alpha_9 = \{ (\lambda x^4 . x^5)^3 \}$$

$$\alpha_{10} = \{ 7^{10} \}$$

$$(\text{let id}^2 = (\lambda x^4 . x^5)^3 \text{ in } ((\text{id}^8\ \text{id}^9)^7\ 7^{10})^6)^1$$

1CFA

0CFA is still imprecise because it is *monovariant*: each expression has only one flow variable associated with it, so multiple calls to the same function allow multiple values into the single flow variable for the function body, and these values "leak out" at all potential call sites.

A better approximation is given by 1CFA ("first-order..."), in which a function has a separate flow variable for each call site in the program; this isolates separate calls to the same function, and so produces a more precise result.

1CFA

1CFA is a *polyvariant* approach.

Another alternative is to use a *polymorphic* approach, in which the values themselves are enriched to support specialisation at different call sites (cf. ML polymorphic types).

It's unclear which approach is "best".

Summary

- Many analyses can be formulated using constraints
- 0CFA is a constraint-based analysis
- Inequality constraints are generated from the syntax of a program
- A minimal solution to the constraints provides a safe approximation to dynamic control-flow behaviour
- Polyvariant (as in 1CFA) and polymorphic approaches may improve precision