1. In the lecture we saw an illustration of a construction to show that acyclicity of graphs is not definable in first-order logic. Write out a proof of this result.

Prove that acyclicity is not definable in Mon.Σ₁. Is it definable in Mon.Π₁?

2. Prove (using Hanf’s theorem or otherwise) that 3-colourability of graphs is not definable in first-order logic.

Graph 3-colourability (and, indeed, 2-colourability) are definable in Mon.Σ₁. Are they definable in Mon.Π₁? Are they definable in universal second-order logic?

3. Prove the lemma stated in the lecture that any formula that is positive in the relation symbol $R$ defines a monotone operator.

4. Prove that the formula of LFP $\neg[lfp_{R,x}\neg\phi(R/\neg R)](x)$, where $\phi(R/\neg R)$ denotes the result of replacing all occurrences of $R$ in $\phi$ by $\neg R$, defines the greatest fixed point of the operator defined by $\phi$.

5. In the lectures, we saw how definitions by simultaneous induction can be replaced by a single application of the $lfp$ operator. In this exercise, you are asked to show the same for nested applications of the $lfp$ operator.

Suppose $\phi(x,y,S,T)$ is a formula in which the relational variables $S$ (of arity $s$) and $T$ (of arity $t$) only appear positively, and $x$ and $y$ are tuples of variables of length $s$ and $t$ respectively. Show that (for any $t$-tuple of terms $t$) the predicate expression

$$[lfp_{S,x}(lfp_{T,y}\phi(t))]$$

is equivalent to an expression with just one application of $lfp$.

6. Consider a vocabulary consisting of two unary relations $P$ and $O$, one binary relation $E$ and two constants $s$ and $t$. We say that a structure $A = (A, P, O, E, s, t)$ in this vocabulary is an arena if $P \cup O = A$ and $P \cap O = \emptyset$. That is, $P$ and $O$ partition the universe into two disjoint sets.

An arena defines the following game played between a player and an opponent. The game involves a token that is initially placed on the element $s$. At each move, if the token is currently on an element of $P$ it is player who plays and if it is on an element of $O$, it is opponent who plays. At each move, if the token is on an element $a$, the one who plays choses an element $b$ such that $(a, b) \in E$ and moves the token from $a$ to $b$. If the token reaches $t$ at any point then player has won the game.
We define GAME to be the class of arenas for which player has a strategy for winning the game. Note that in an arena $A = (A, P, O, E, s, t)$, player has a strategy to win from an element $a$ if either $a \in P$ and there is some move from $a$ so that player still has a strategy to win after that move or $a \in O$ and for every move from $a$, player can win after that move.

(a) Give a sentence of LFP that defines the class of structures GAME.

We say that a collection $C$ of decision problems is closed under logarithmic space reductions if whenever $A \in C$ and $B \leq_L A$ (i.e. $B$ is reducible to $A$ by a logarithmic-space reduction) then $B \in C$.

The class of structures GAME defined above is known to be $P$-complete under logarithmic-space reductions.

(b) Explain why this, together with (a) implies that the class of problems definable in LFP is not closed under logarithmic-space reductions.

7. Give a sentence of LFP that defines the class of linear orders with an even number of elements.

8. The directed graph reachability problem is the problem of deciding, given a structure $(V, E, s, t)$ where $E$ is an arbitrary binary relation on $V$, and $s, t \in V$, whether $(s, t)$ is in the reflexive-transitive closure of $E$. This problem is known to be decidable in $NL$.

Transitive closure logic is the extension of first-order logic with an operator $\text{tc}$ which allows us to form formulae

$$\phi \equiv [\text{tc}_{x,y} \psi](t_1, t_2)$$

where $x$ and $y$ are $k$-tuples of variables and $t_1$ and $t_2$ are $k$-tuples of terms, for some $k$; and all occurrences of variables $x$ and $y$ in $\psi$ are bound in $\phi$. The semantics is given by saying, if $a$ is an interpretation for the free variables of $\phi$, then $A \models \phi[a]$ just in case $(t_{a_1}^1, t_{a_2}^2)$ is in the reflexive-transitive closure of the binary relation defined by $\psi(x, y)$ on $A^k$.

(a) Show that any class of structures definable by a sentence $\phi$, as above, where $\psi$ is first-order, is decidable in $NL$.

(b) Show that, if $K$ is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in $K$ interprets $<$ as a linear order and

$$\{[A]_< \mid A \in K\}$$

is decidable in $NL$, then there is a sentence of transitive-closure logic that defines $K$.

9. For a binary relation $E$ on a set $A$, define its deterministic transitive closure to be the set of pairs $(a, b)$ for which there are $c_1, \ldots, c_n \in A$ such
that \( a = c_1, b = c_n \) and for each \( i < n, c_{i+1} \) is the unique element of \( A \) with \( (c_i, c_{i+1}) \in E \).

Let \( \text{DTC} \) denote the logic formed by extending first-order logic with an operator \( \text{dtc} \) with syntax analogous to \( \text{tc} \) above, where \( \text{dtc}_{x,y} \psi \) defines the deterministic transitive closure of \( \psi(x,y) \).

(a) Show that every sentence of \( \text{DTC} \) defines a class of structures decidable in \( L \).

(b) Show that, if \( K \) is an isomorphism-closed class of structures in a relational signature including \( < \), such that each structure in \( K \) interprets \( < \) as a linear order and

\[
\{ [A]_< \mid A \in K \}
\]

is decidable in \( L \), then there is a sentence of \( \text{DTC} \) that defines \( K \).

10. On page 47 of Handout 4, the correspondence between \( k \)-pebble games and the equivalence \( \equiv^k \) is stated. This exercise asks you to prove the easy direction of that equivalence. That is, show that if Duplicator has a winning strategy in the \( k \)-pebble game for \( q \) moves starting from position \( (A, a) \) and \( (B, b) \) then \( (A, a) \equiv^k_q (B, b) \).

11. We say that a graph \( G = (V,E) \) has a perfect matching if there is a set \( M \subseteq E \) of edges such that for every vertex \( v \in V \) there is exactly one edge in \( M \) that includes \( v \). Prove that the property of having a perfect matching is not definable in \( \text{LFP} \).

12. Let \( \mathcal{E} \) be the class of equivalence relations. That is, it consists of all structures \( \mathcal{A} = (A, R) \) where \( R \) is a binary relation on \( A \) that is reflexive, symmetric and transitive. Prove that, for any \( k \) there are no more than \( k^k \) equivalence classes of the relation \( \equiv^k \) on \( \mathcal{E} \).

Prove that \( \text{LFP} \) is no more expressive than first-order logic on \( \mathcal{E} \). That is, for any formula \( \phi \) of \( \text{LFP} \), there is a first-order formula \( \psi \) such that, for any \( \mathcal{A} \in \mathcal{E} \), \( \mathcal{A} \models \phi \) if, and only if, \( \mathcal{A} \models \psi \).

13. In Handout 5, pages 10–12, we saw a sketch of the proof that \( \text{CNF-SAT} \) is \( \text{NP} \)-complete under first-order reductions.

Define the problem \( \text{Clique} \) to be the class of structures \( (V,E,U) \) where \( E \) is a binary relation on \( V \) and \( U \) is a unary relation and the graph \( (V,E) \) contains a clique of size \( |U| \). Show that there is a first-order reduction from \( \text{CNF-SAT} \) to \( \text{Clique} \).

14. Give a definition of what it would mean for the complexity class \( L \) to be recursively indexable. Show that it is recursively indexable if, and only if, it has a complete problem under first-order reductions.

The complexity class \( \text{PolyLogSpace} \) is defined to be class of those problems decidable on a deterministic machine in time \( O((\log n)^k) \) for some \( k \). Show that this class has no complete problems under first-order reductions. (Hint: recall the space hierarchy theorem).
15. The aim of this exercise is to show that Hanf’s theorem can be extended to the logic with counting quantifiers. That is, write $A \equiv_C^p B$ to denote that $A$ and $B$ cannot be distinguished by any sentence of first-order logic with counting quantifiers that has quantifier rank at most $p$. Also recall that $A \simeq_r B$ denotes that $A$ and $B$ are Hanf equivalent with radius $r$ (see page 21 of Handout 3). Show that for every vocabulary $\sigma$ and every $p$ there is an $r$ such that if $A$ and $B$ are $\sigma$-structures, then $A \simeq_r B$ implies $A \equiv_C^p B$. 
