Introduction to Probability

Lecture 11: Estimators (Part II)

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Outline

Recap

Estimating Population Size (First Version)

Mean Squared Error

Estimating Population Size (Second Version)

Recap: Unbiased Estimators and Bias

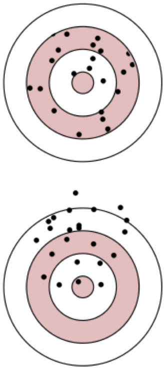
Definition

An estimator T is called an unbiased estimator for a parameter θ if

$$\mathbf{E}[T] = \theta,$$

irrespective of the value θ. The bias is defined as

$$\mathbf{E}[T] - \theta = \mathbf{E}[T - \theta].$$



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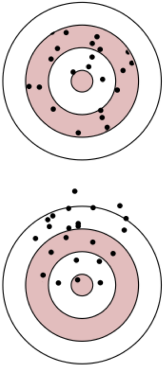

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- How can we measure the accuracy of an estimator? $\rightsquigarrow$ bias and mean-squared error
- If there are several unbiased estimators, which one to choose? $\rightsquigarrow$ mean-squared error (or variance)

An Unbiased Estimator may not always exist

Example 6

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

Answer

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Example 6 (cntd.)

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Answer

- Suppose there exists an unbiased estimator with $\mathbf{E}[\, T(X) \,] = 1/p$.
- Then

$$1 = p \cdot \mathbf{E}[\, T(X) \,]$$
$$= p \cdot \sum_{k=0}^{n} \mathbf{P}[\, X = k \,] \cdot T(k)$$
$$= p \cdot \sum_{k=0}^{n} \binom{n}{k} p^k \cdot (1-p)^{n-k} \cdot T(k)$$

- Last term is a polynomial of degree $n+1$ with constant term zero

  $\Rightarrow p \cdot \mathbf{E}[\, T(X) \,] - 1$ is a (non-zero) polynomial of degree $\leq n+1$

  $\Rightarrow$ this polynomial has at most $n+1$ roots

  $\Rightarrow \mathbf{E}[\, T(X) \,]$ can be equal to $1/p$ for at most $n+1$ values of p, and thus cannot be an unbiased.

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- Achieving unbiasedness alone is not a good strategy
- Improvement: find an estimator which always returns a value at least $\max(X_1, X_2, \ldots, X_n)$

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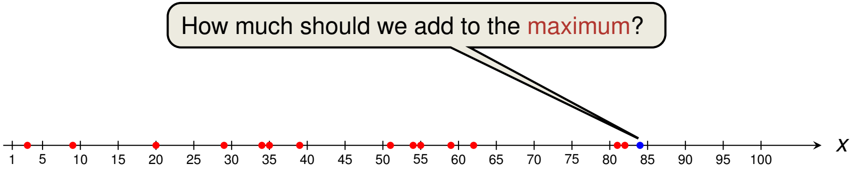

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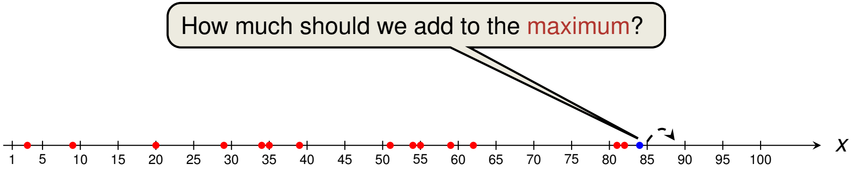

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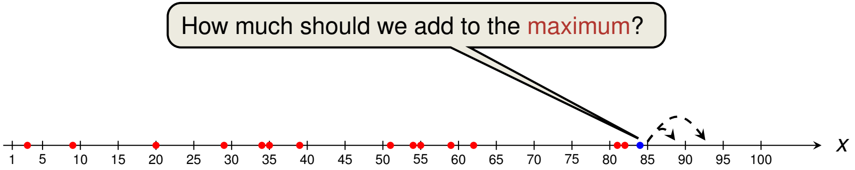

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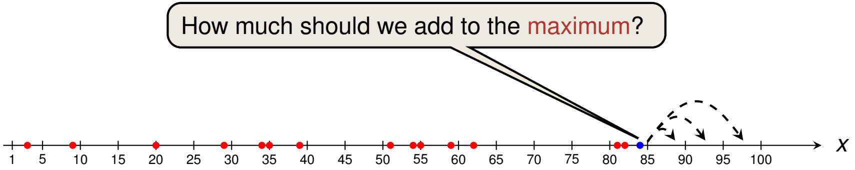

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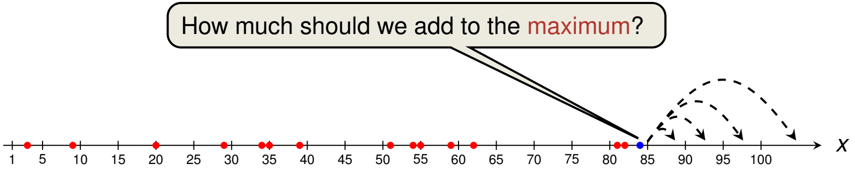

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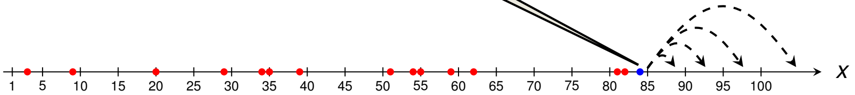
How much should we add to the maximum?



Rearrange the other 14 points equi-spaced between 0 and 84.

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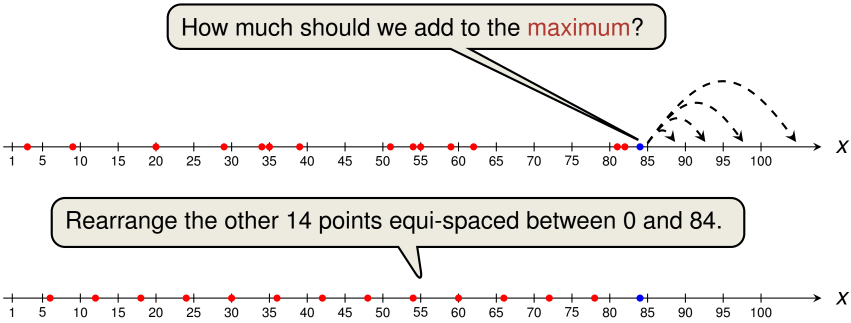

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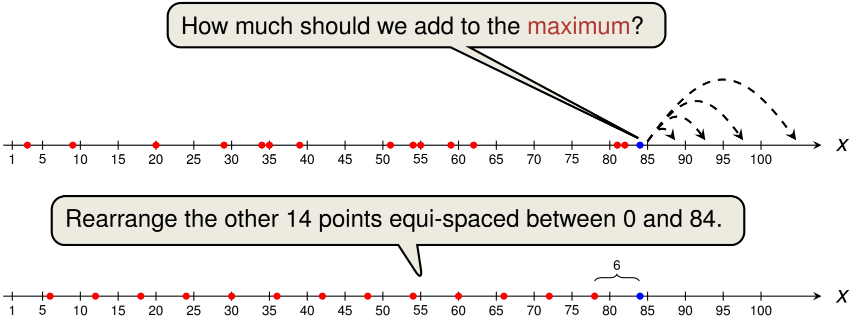

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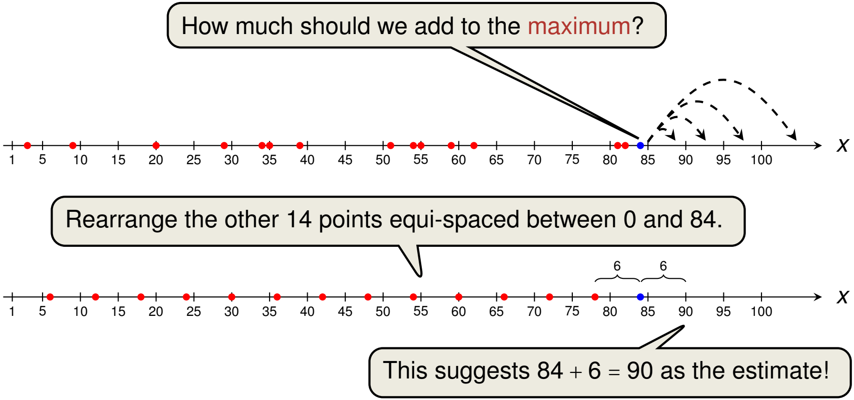

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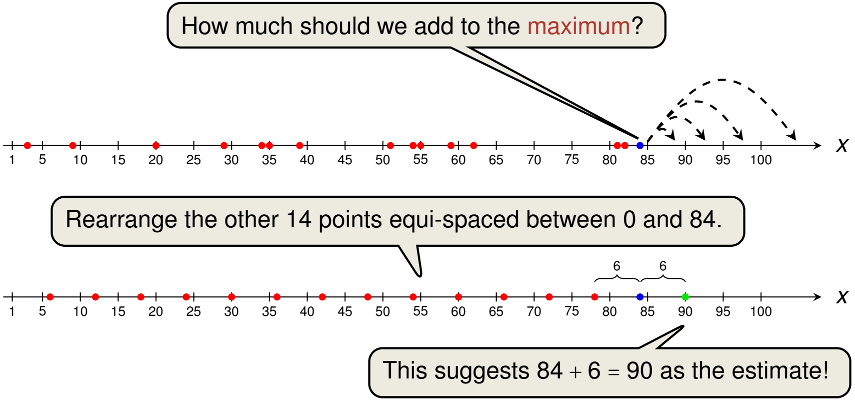

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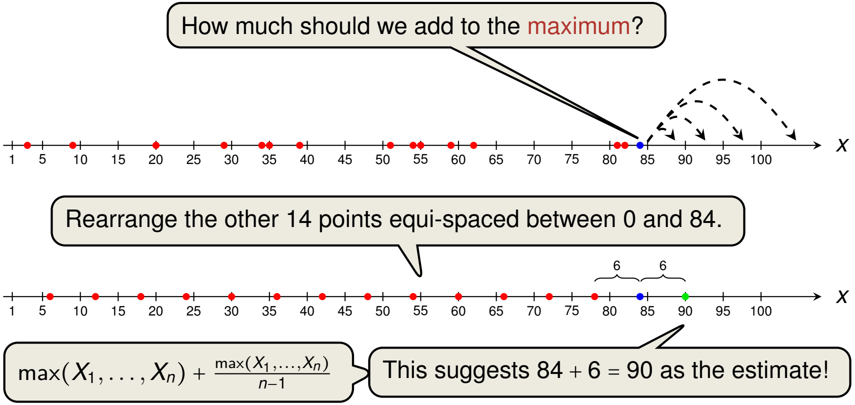

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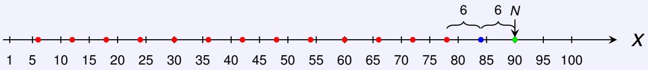

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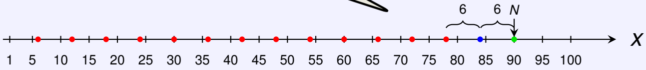

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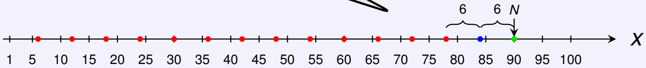

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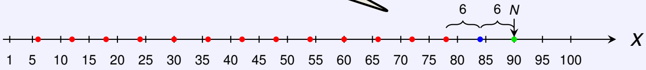

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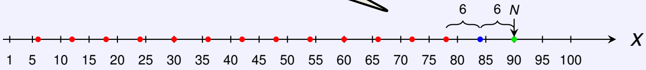

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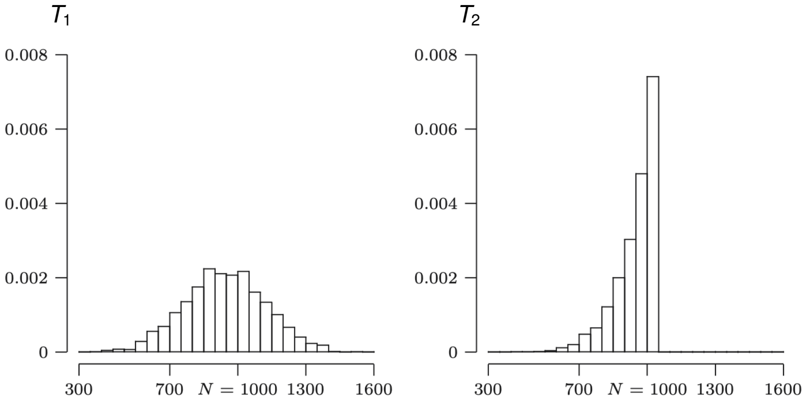
- For our samples before, we get $t_2 = \frac{16}{15} \cdot 84 - 1 = 88.6$.

Empirical Analysis of the two Estimators



Source: Modern Introduction to Statistics

Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

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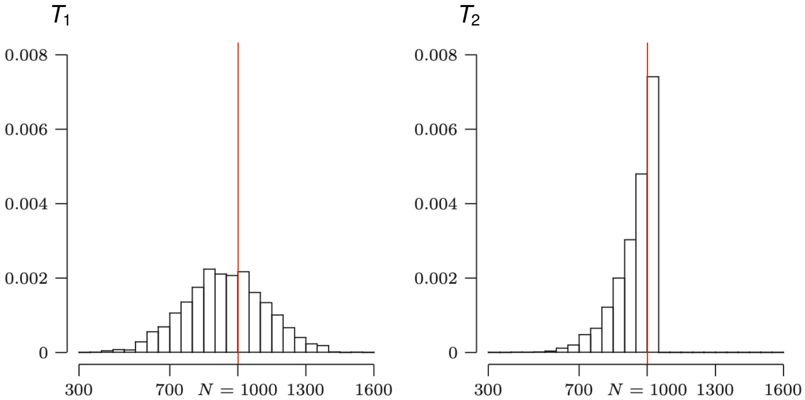

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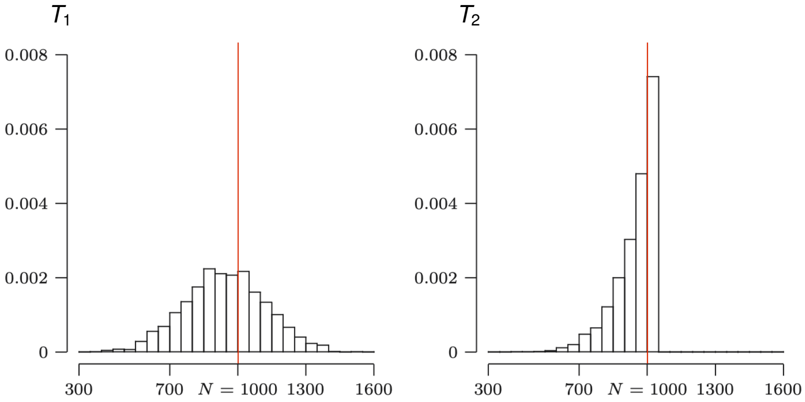

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Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

Can we find a quantity that captures the superiority of T_2 over T_1?

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Bias-Variance Decomposition

The mean squared error can be decomposed into:

$$\mathbf{MSE}[T] = \underbrace{(\mathbf{E}[T]-\theta)^2}_{=\text{Bias}^2} + \underbrace{\mathbf{V}[T]}_{=\text{Variance}}$$

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Bias-Variance Decomposition: Derivation

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We need to prove: $\textbf{MSE}[T] = (\textbf{E}[T] - \theta)^2 + \textbf{V}[T]$.

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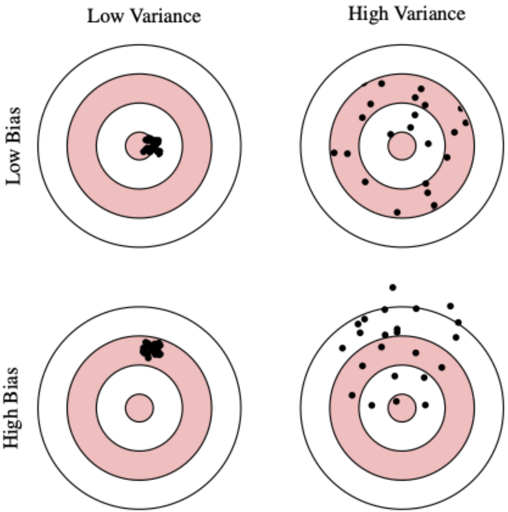
Bias-Variance Decomposition: Illustration



Source: Edwin Leuven (Point Estimation)

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It holds that $\mathbf{MSE}[T_1] = \Theta\left(\frac{N^2}{n}\right)$, where $T_1 = 2 \cdot \overline{X}_n - 1$.

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- Rearranging and simplifying gives

$$\mathbf{V}[T_1] = \frac{(N+1)(N-n)}{3n}.$$

Analysis of the MSE for T_2 (non-examinable)

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It holds that **MSE** $[\, T_2 \,] = \Theta\left(\frac{N^2}{n^2}\right)$, where $T_2 = \frac{n+1}{n} \cdot \max(X_1, \ldots, X_n) - 1$.

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- One can prove:

For details see Dekking et al.

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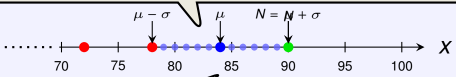

Maximum could have equally likely taken any value between 79 and 90

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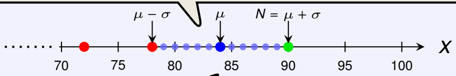

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- $\mathbf{MSE}[T_2]$ is much lower than $\mathbf{MSE}[T_1] = \Theta\left(\frac{N^2}{n}\right)$, i.e., $\frac{\mathbf{MSE}[T_1]}{\mathbf{MSE}[T_2]} = \frac{n+2}{3}$
- $\Rightarrow$ confirms simulations suggesting that T_2 is better than T_1!
- can be shown T_2 is the best unbiased estimator, i.e., it minimises MSE.

Outline

Recap

Estimating Population Size (First Version)

Mean Squared Error

Estimating Population Size (Second Version)

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- Population/ID space $S = \{1, 2, \ldots, N\}$
- We take uniform samples from S without replacement
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Similar idea applies to situations where elements are not labelled before we see them first time (Mark & Recapture Method)

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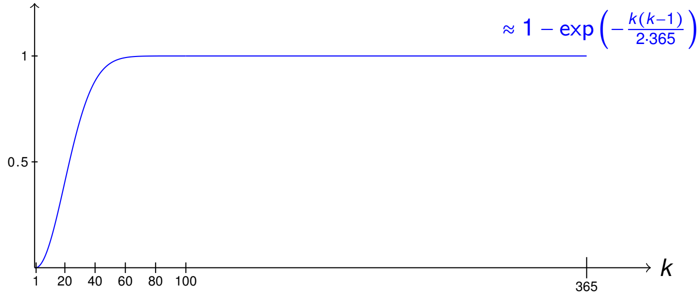

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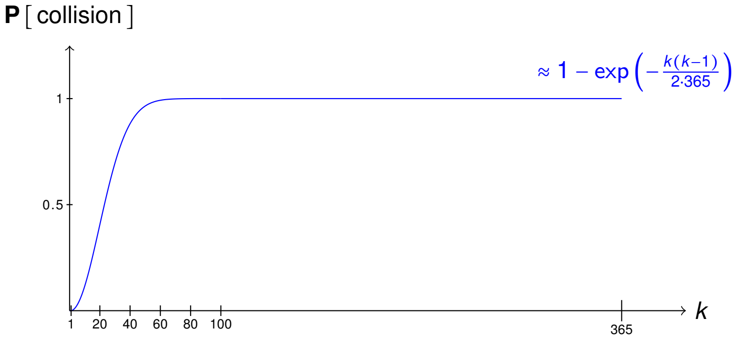

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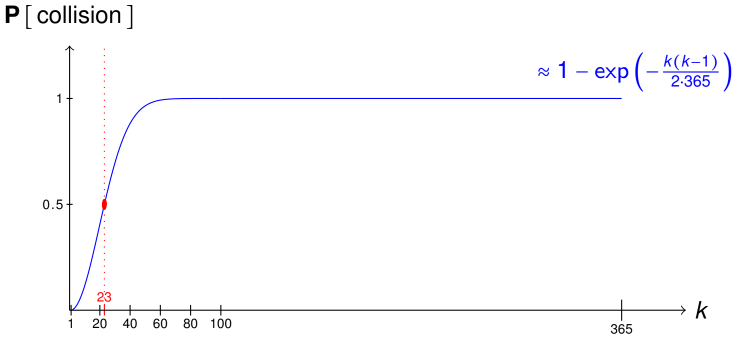

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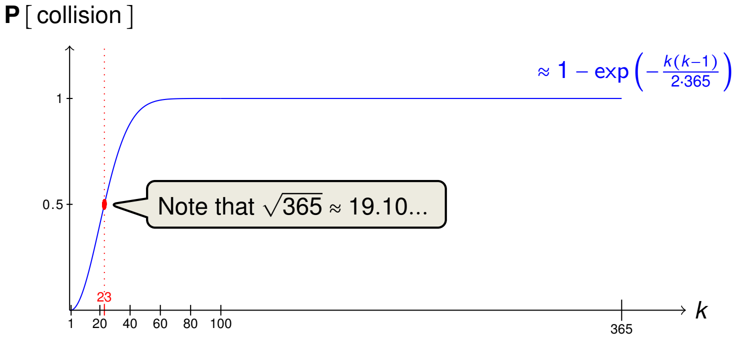

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For $i = 1, 2, \ldots$

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Expected Running Time (Knuth, Ramanujan)

$$\sqrt{\frac{\pi N}{2}} - \frac{1}{3} + O\left(\frac{1}{\sqrt{N}}\right).$$

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- Running Time: The expected time until the algorithm stops is:
  = the expected number of samples until a collision...

Same as the birthday problem, but now with $|S| = N$ days... ☺

Expected Running Time (Knuth, Ramanujan)

$$\sqrt{\frac{\pi N}{2}} - \frac{1}{3} + O\left(\frac{1}{\sqrt{N}}\right).$$

Exercise: Prove a bound of $\leq 2 \cdot \sqrt{N}$

Estimation via Collision: Getting the Estimator Unbiased

Example 6

One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[T] = |S|$ for any finite, non-empty set S.

Answer

Estimation via Collision: Getting the Estimator Unbiased

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One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[\,T\,] = |S|$ for any finite, non-empty set S.

Answer

- We outline a construction by induction.

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Example 6

One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[\,T\,] = |S|$ for any finite, non-empty set S.

Answer

- We outline a construction by induction.
- Case $|S| = 1$: Algo always stops after $i = 2$ rounds and returns $T(2)$.

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Answer

- We outline a construction by induction.
- Case $|S| = 1$: Algo always stops after $i = 2$ rounds and returns $T(2)$. We want

$$1 = \mathbf{E}[\,T\,] = T(2)$$

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Answer

- We outline a construction by induction.
- Case $|S| = 1$: Algo always stops after $i = 2$ rounds and returns $T(2)$. We want

$$1 = \mathbf{E}[\,T\,] = T(2) \qquad \Rightarrow \qquad T(2) = 1.$$

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- Case $|S| = 2$: Algo stops after 2 or 3 rounds (w.p. 1/2 each).

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$$2 = \mathbf{E}[T] = \frac{1}{2} \cdot T(2) + \frac{1}{2} \cdot T(3)$$

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- Case $|S| = 3$: gives $3 = \mathbf{E}[T] = \frac{1}{3} \cdot T(2) + \frac{4}{9} \cdot T(3) + \frac{2}{9} \cdot T(4)$

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- Case $|S| = 3$: gives $3 = \mathbf{E}[T] = \frac{1}{3} \cdot T(2) + \frac{4}{9} \cdot T(3) + \frac{2}{9} \cdot T(4)$ $\Rightarrow T(4) = 6$, similarly, $T(5) = 10$ etc.

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- can continue to define $T(i)$ inductively in this way (note T is unique)

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- can continue to define $T(i)$ inductively in this way (note T is unique) (proof that $T(i) = \binom{i}{2}$ is harder)