Introduction to Probability

Lecture 11: Estimators (Part II)

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Easter 2024

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Outline

Recap

Estimating Population Size (First Version)

Mean Squared Error

Estimating Population Size (Second Version)

Recap: Unbiased Estimators and Bias

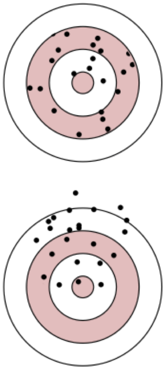
Definition

An estimator T is called an unbiased estimator for a parameter θ if

$$\mathbf{E}[\,T\,] = \theta,$$

irrespective of the value θ. The bias is defined as

$$\mathbf{E}[\,T\,] - \theta = \mathbf{E}[\,T - \theta\,].$$



Source: Edwin Leuven (Point Estimation)

- How can we measure the accuracy of an estimator? $\rightsquigarrow$ bias and mean-squared error
- If there are several unbiased estimators, which one to choose? $\rightsquigarrow$ mean-squared error (or variance)

An Unbiased Estimator may not always exist

Example 6

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

Answer

An Unbiased Estimator may not always exist (cntd. - non-examinable)

Example 6 (cntd.)

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

Answer

- Suppose there exists an unbiased estimator with $\mathbf{E}[\,T(X)\,] = 1/p$.
- Then

$$
\begin{aligned}
1 &= p \cdot \mathbf{E}[\,T(X)\,] \\
&= p \cdot \sum_{k=0}^{n} \mathbf{P}[\,X = k\,] \cdot T(k) \\
&= p \cdot \sum_{k=0}^{n} \binom{n}{k} p^k \cdot (1-p)^{n-k} \cdot T(k)
\end{aligned}
$$

- Last term is a polynomial of degree $n+1$ with constant term zero

$\Rightarrow p \cdot \mathbf{E}[\,T(X)\,] - 1$ is a (non-zero) polynomial of degree $\leq n+1$

$\Rightarrow$ this polynomial has at most $n+1$ roots

$\Rightarrow \mathbf{E}[\,T(X)\,]$ can be equal to $1/p$ for at most $n+1$ values of p, and thus cannot be an unbiased.

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Mean Squared Error

Estimating Population Size (Second Version)

Estimating Population Size (First Version)

- Suppose we have a sample of a few serial numbers (IDs) of some product
- We assume IDs are running from 1 to an unknown parameter N (so $N = \theta$)
- Each of the IDs is drawn without replacement from the discrete uniform distribution over $\{1, 2, \ldots, N\}$
- This is also known as Tank Estimation Problem or (Discrete) Taxi Problem

$$7, 3, 10, 46, 14$$

Warning

- As before, we denote the samples $X_1, X_2, \ldots, X_n$
- Since sampling is without replacement:
  - they are not independent! (but identically distributed)
  - their number must satisfy $n \leq N$

First Estimator Based on Sample Mean

Example 1

Construct an unbiased estimator using the sample mean.

Answer

Example: Odd Behaviour of T_1

- Suppose $n = 5$
- Let the sample be

$$7, 3, 10, 46, 14$$

- The estimator returns:

$$T_1 = 2 \cdot \overline{X}_n - 1 = 2 \cdot \frac{80}{5} - 1 = 31 \quad ☹$$

This estimator will often unnecessarily underestimate the true value N.

Challenging exercise: Find a lower bound on $\mathbf{P}\left[\, T_1 < \max(X_1, X_2, \ldots, X_n) \,\right]$

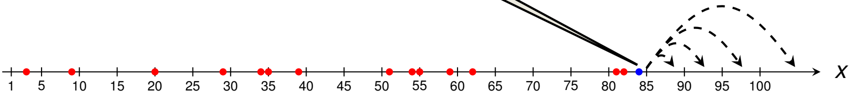
- Achieving unbiasedness alone is not a good strategy
- Improvement: find an estimator which always returns a value at least $\max(X_1, X_2, \ldots, X_n)$

Intuition: Constructing an Estimator based on Maximum

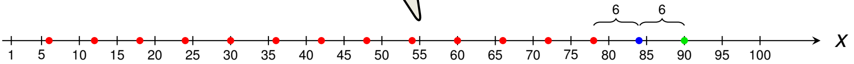
- Suppose $n = 15$
- Our samples are:

$$9, 82, 39, 35, 20, 51, 54, 62, 81, 29, 84, 59, 3, 34, 55$$

How much should we add to the maximum?



Rearrange the other 14 points equi-spaced between 0 and 84.



$\max(X_1, \ldots, X_n) + \frac{\max(X_1,\ldots,X_n)}{n-1}$

This suggests $84 + 6 = 90$ as the estimate!

Deriving the Estimator Based on Maximum

Example 2

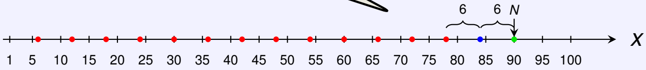
Construct an unbiased estimator using $\max(X_1, \ldots, X_n)$

Answer

- Calculate expectation of the maximum (for details see Dekking et al.)

$$\mathbf{E}\left[\,\max(X_1, \ldots, X_n)\,\right] = \ldots = \frac{n}{n+1} \cdot N + \frac{n}{n+1} = \frac{n}{n+1} \cdot (N+1).$$

Equi-spaced configuration would suggest $\max(X_1, \ldots, X_n) \approx \frac{n-1}{n} \cdot N$

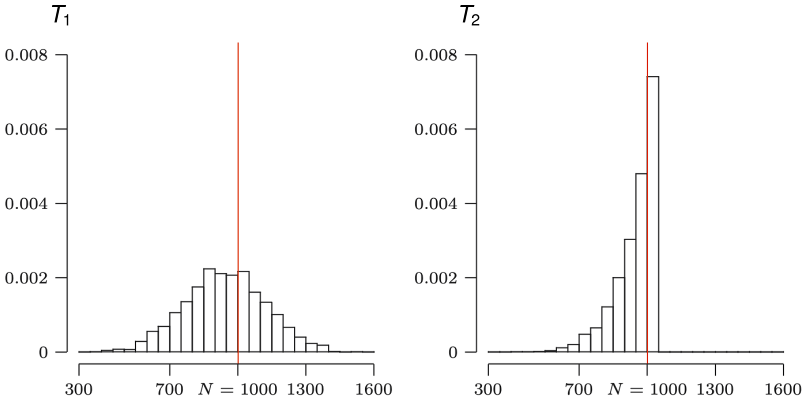


- Hence we obtain an unbiased estimator by

$$T_2 := \frac{n+1}{n} \cdot \max(X_1, \ldots, X_n) - 1.$$

- For our samples before, we get $t_2 = \frac{16}{15} \cdot 84 - 1 = 88.6$.

Empirical Analysis of the two Estimators



Source: Modern Introduction to Statistics

Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

Can we find a quantity that captures the superiority of T_2 over T_1?

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Mean Squared Error

Mean Squared Error Definition

Let T be an estimator for a parameter θ. The mean squared error of T is

$$\mathbf{MSE}[T] = \mathbf{E}\left[(T-\theta)^2\right].$$

- According to this, estimator T_1 better than T_2 if $\mathbf{MSE}[T_1] < \mathbf{MSE}[T_2]$.

Bias-Variance Decomposition

The mean squared error can be decomposed into:

$$\mathbf{MSE}[T] = \underbrace{(\mathbf{E}[T]-\theta)^2}_{=\,\mathrm{Bias}^2} + \underbrace{\mathbf{V}[T]}_{=\,\mathrm{Variance}}$$

- If T_1 and T_2 are both unbiased, T_1 is better than T_2 iff $\mathbf{V}[T_1] < \mathbf{V}[T_2]$.

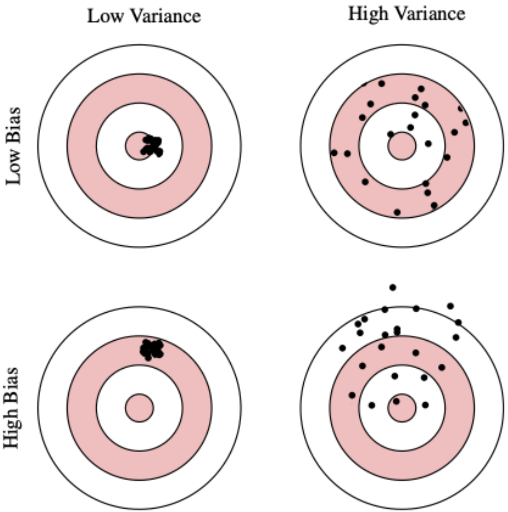
Bias-Variance Decomposition: Derivation

Example 3

We need to prove: $\mathbf{MSE}[T] = (\mathbf{E}[T] - \theta)^2 + \mathbf{V}[T]$.

Answer

Bias-Variance Decomposition: Illustration



Source: Edwin Leuven (Point Estimation)

Example 4

It holds that $\mathbf{MSE}\left[\, T_1 \,\right] = \Theta\left(\frac{N^2}{n}\right)$, where $T_1 = 2 \cdot \overline{X}_n - 1$.

Answer

Analysis of the MSE for T_2 (non-examinable)

Example 5

It holds that $\mathbf{MSE}[\,T_2\,] = \Theta\left(\frac{N^2}{n^2}\right)$, where $T_2 = \frac{n+1}{n} \cdot \max(X_1, \ldots, X_n) - 1$.

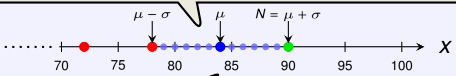
Answer

- T_2 is unbiased $\Rightarrow$ need $\mathbf{V}[\,T_2\,]$ which reduces to $\mathbf{V}[\,\max(X_1, \ldots, X_n)\,]$
- One can prove:

For details see Dekking et al.

$$\mathbf{V}[\,\max(X_1, \ldots, X_n)\,] = \cdots = \frac{n(N+1)(N-n)}{(n+2)(n+1)^2} = \Theta\left(\frac{N^2}{n^2}\right)$$

Equi-spaced (idealised) configuration suggests a standard deviation of $\sigma \approx \frac{N}{n}$



Maximum could have equally likely taken any value between 79 and 90

- $\mathbf{MSE}[\,T_2\,]$ is much lower than $\mathbf{MSE}[\,T_1\,] = \Theta\left(\frac{N^2}{n}\right)$, i.e., $\frac{\mathbf{MSE}[\,T_1\,]}{\mathbf{MSE}[\,T_2\,]} = \frac{n+2}{3}$
- $\Rightarrow$ confirms simulations suggesting that T_2 is better than T_1!
- can be shown T_2 is the best unbiased estimator, i.e., it minimises MSE.

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A New Estimation Problem

Previous Model

- Population/ID space $S = \{1, 2, \ldots, N\}$
- We take uniform samples from S without replacement
- **Goal:** Find estimator for N

Similar idea applies to situations where elements are not labelled before we see them first time (Mark & Recapture Method)

New Model

- Population/ID space of size $|S| = N$
- We take uniform samples from S with replacement
- **Goal:** Find estimator for N

- Suppose $n = 6$, $N = 11$, $S = \{3, 4, 7, 8, 10, 15.83356, 20, 21, 56, 81, 10000\}$
- Let the sample be

$$10, 81, 20, 3, 81, 10000$$

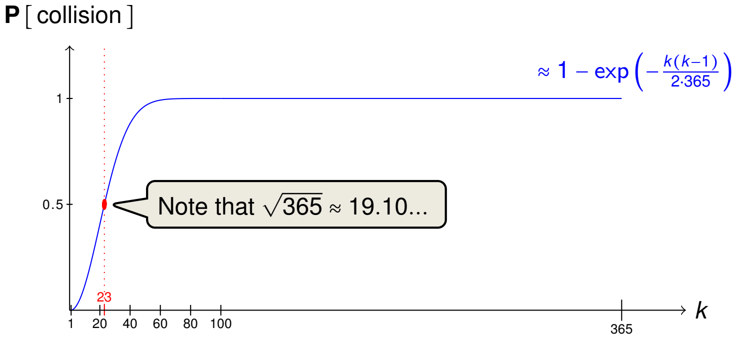
Let us call this a **collision**

As we do not know S, our only clue are elements that **were sampled twice.**

Birthday Problem

Birthday Problem: Given a set of k people

- What is the probability of having two with the same birthday (i.e., having at least one collision)?
- What is the expected number of people one needs to ask until the first collision occurs?



Estimation via Collision: The Algorithm

Recall: As we do not know S, our only information are **collisions**.

FIND-FIRST-COLLISION(S)

$C = \varnothing$
For $i = 1, 2, \ldots$
  Take next i.i.d. sample X_i from S
  If $X_i \notin C$ **then** $C \leftarrow C \cup \{X_i\}$
  else return $T(i)$
End For

$T(i)$ will be the value of the estimator if algo returns after i rounds. (We want T unbiased)

- Running Time: The expected time until the algorithm stops is:
  $=$ the expected number of samples until a collision...

Same as the birthday problem, but now with $|S| = N$ days... ☺

Expected Running Time (Knuth, Ramanujan)

$$\sqrt{\frac{\pi N}{2}} - \frac{1}{3} + O\left(\frac{1}{\sqrt{N}}\right).$$

Exercise: Prove a bound of $\leq 2 \cdot \sqrt{N}$

Estimation via Collision: Getting the Estimator Unbiased

Example 6

One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[\,T\,] = |S|$ for any finite, non-empty set S.

Answer