# Introduction to Probability

Lecture 6: Marginals and Joint Distributions

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# Motivation

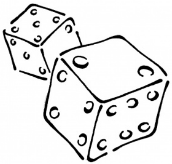
Experiments often involve several random variables, and some of them may influence each other.

To this end, we will introduce:

- Joint/Marginal distribution of two (or more) variables
- Independence of two (or more) variables
- Covariance of two variables

For simplicity, we will mainly focus on discrete random variables.

# Warm-Up Exercise



Example

Let $X_1, X_2 \in \{1, 2, \dots, 6\}$ be two independent rolls of an unbiased die. Let $S := X_1 + X_2$ and $M := \max\{X_1, X_2\}$. List the elements of the event $\{S = 7, M \leq 5\}$ and deduce the probability.

Answer

The elements are $\{(2,5),(3,4),(4,3),(5,2)\}$. Since each of these elements has a probability of $1/6 \cdot 1/6 = 1/36$, the sought probability is $4/36 = 1/9$.

# Joint Probability

**Joint Probability Mass Function**

The joint probability mass function of two discrete random variables $X$ and $Y$ is the function $p : \mathbb{R}^2 \to [0, 1]$, defined by:

$$p_{X,Y}(a, b) = \mathbf{P}[X = a, Y = b] \qquad \text{for } -\infty < a, b < \infty.$$

**Joint Distribution Function**

The joint distribution function of two (discrete or continuous) random variables $X$ and $Y$ is the function $F : \mathbb{R}^2 \to [0, 1]$, defined by:

$$F_{X,Y}(a, b) = \mathbf{P}[X \leq a, Y \leq b] \qquad \text{for } -\infty < a, b < \infty.$$

**Marginal Distribution**

Given a joint distribution $F_{X,Y}$ of two random variables $X$, $Y$, one obtains the marginal distribution of $X$ for any $a$ as follows:

$$F_X(a) = \mathbf{P}[X \leq a] = \lim_{b \to \infty} F_{X,Y}(a, b).$$

Joint Distribution contains (much) more information than the two marginals!

# Discrete Example 1

**Example**

Let $X_1, X_2 \in \{1, 2, \ldots, 6\}$ be independent rolls of an unbiased die. Let $S := X_1 + X_2$ and $M := \max\{X_1, X_2\}$. Compute the joint probability mass function $p$ of $S$ and $M$ and the marginal distributions of $S$ and $M$.

Answer

| | $b$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | $p_S(a)$ |
| 2 | 1/36 | 0 | 0 | 0 | 0 | 0 | 1/36 |
| 3 | 0 | 2/36 | 0 | 0 | 0 | 0 | 2/36 |
| 4 | 0 | 1/36 | 2/36 | 0 | 0 | 0 | 3/36 |
| 5 | 0 | 0 | 2/36 | 2/36 | 0 | 0 | 4/36 |
| 6 | 0 | 0 | 1/36 | 2/36 | 2/36 | 0 | 5/36 |
| 7 | 0 | 0 | 0 | 2/36 | 2/36 | 2/36 | 6/36 |
| 8 | 0 | 0 | 0 | 1/36 | 2/36 | 2/36 | 5/36 |
| 9 | 0 | 0 | 0 | 0 | 2/36 | 2/36 | 4/36 |
| 10 | 0 | 0 | 0 | 0 | 1/36 | 2/36 | 3/36 |
| 11 | 0 | 0 | 0 | 0 | 0 | 2/36 | 2/36 |
| 12 | 0 | 0 | 0 | 0 | 0 | 1/36 | 1/36 |
| $p_M(b)$ | 1/36 | 3/36 | 5/36 | 7/36 | 9/36 | 11/36 | 1 |

$(X_1, X_2) = (4, 3)$ or $(X_1, X_2) = (3, 4)$

$p_{S,M}(7, 4) = \frac{1}{36} + \frac{1}{36}$

$\{S = 8\} = \{S = 8, M = 4\} \cup \{S = 8, M = 5\} \cup \{S = 8, M = 6\}$

$p_S(8) = \frac{1}{36} + \frac{2}{36} + \frac{2}{36}$

# Discrete Example 2

Example

Suppose an urn contains balls numbered $1, 2, \ldots, N$. We draw $1 \leq n \leq N$ balls uniformly and without replacement from the urn. Let $X_i \in \{1, 2, \ldots, N\}$ be the number of the ball drawn in the $i$-th step. What is the marginal distribution of $X_i$?

Answer

We first compute the joint distribution. For distinct $a_1, a_2, \ldots, a_n$,

$$p(a_1, a_2, \ldots, a_n) = \mathbf{P}[X_1 = a_1, X_2 = a_2, \ldots, X_n = a_n] = \frac{1}{N(N-1)\cdots(N-n+1)}.$$

Fix $i$ and consider the marginal distribution of $X_i$:

$$p_{X_i}(k) = \sum_{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n} p(a_1, \ldots, a_{i-1}, k, a_{i+1}, \ldots, a_n)$$

$$= (N-1)(N-2)\cdots(N-n+1) \cdot \frac{1}{N(N-1)\cdots(N-n+1)}$$

$$= \frac{1}{N}$$

The $X_i$'s are **not** independent, yet their marginals are identical!

Same argument applies to the hypergeometric distribution, with balls of two different colours.

# Joint Distributions of Continuous Variables

**Definition**

Random variables $X$ and $Y$ have a joint continuous distribution if for some function $f : \mathbb{R}^2 \to \mathbb{R}$ and for all numbers $a_1 \leq b_1$ and $a_2 \leq b_2$,

$$\mathbf{P}[a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y)\, dx\, dy.$$

The function $f$ has to satisfy $f(x, y) \geq 0$ for all $x$ and $y$, and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$. We call $f$ the joint probability density.

As in one-dimensional case we switch from $F$ to $f$ by differentiating (or integrating):

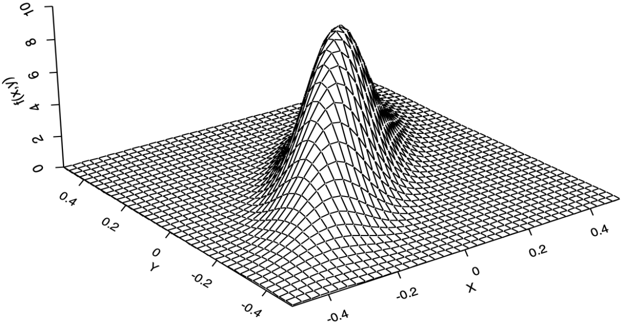
$$F(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y) dx\, dy \qquad \text{and} \qquad f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

# Example of a Joint Distribution of Continuous Random Variables

- Consider the density:

$$f(x, y) = \frac{30}{\pi} \cdot e^{-50x^2 - 50y^2 + 80xy},$$

  where $-\infty < x, y < \infty$.
- This is an example of a so-called bivariate normal probability density function.




Source: Modern Introduction to Statistics

# Dealing with Continuous Variables

Example (1/2)

Suppose that the joint probability density of $X$ and $Y$ is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & \text{for } 0 < x < \infty, 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Compute (i) $\mathbf{P}[X > 1, Y < 1]$ and (ii) $\mathbf{P}[X < Y]$.

Answer

(i) We first compute:

$$\begin{aligned} \mathbf{P}[X > 1, Y < 1] &= \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dxdy \\ &= \int_0^1 2e^{-2y}(-e^{-x})\Big|_1^\infty dy = \int_0^1 2e^{-2y}(0 - (-e^{-1}))dy \\ &= e^{-1}\int_0^1 2e^{-2y}dy = e^{-1}\cdot(1 - e^{-2}). \end{aligned}$$

# Dealing with Continuous Variables (cont.)

Example (2/2)

Suppose that the joint probability density of $X$ and $Y$ is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & \text{for } 0 < x < \infty, 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Compute (i) $\mathbf{P}[\,X > 1, Y < 1\,]$ and (ii) $\mathbf{P}[\,X < Y\,]$.

Answer

(ii) We have:

$$\begin{aligned} \mathbf{P}[\,X < Y\,] &= \int_0^\infty \int_0^y 2e^{-x}e^{-2y} dxdy \\ &= \int_0^\infty 2e^{-2y}(1 - e^{-y})dy \\ &= \int_0^\infty 2e^{-2y} dy - \int_0^\infty 2e^{-3y} dy \\ &= 1 - \frac{2}{3} = \frac{1}{3}. \end{aligned}$$