Introduction to Probability

Lecture 4: More discrete distributions – Poisson, Geometric, Negative Binomial, Hypergeometric

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Outline

Poisson discrete random variable

Geometric discrete random variable

Negative binomial discrete random variable

Hypergeometric discrete random variable

Preliminaries:

The natural exponent e

e is a mathematical constant AKA the Euler number. e is very important for exponential functions. Here are some important identities:

$$e \approx 2.71828$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$e^{-\lambda} = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n$$

$$e^r = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$$

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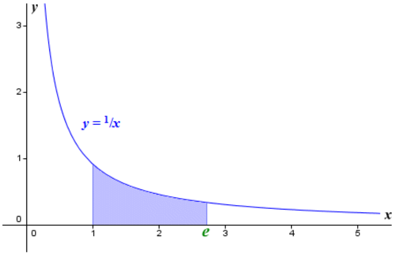

Binomial RV example: large n, small p

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   - $X \sim Bin(n = 60, p = \frac{\lambda}{n})$, so $\mathbf{P}[X{=}k] = \binom{n}{k}p^k(1-p)^{n-k} = \binom{60}{k}\left(\frac{8}{60}\right)^k\left(1-\frac{8}{60}\right)^{n-k}$

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2. Break an hour into **milliseconds**.
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3. Break an hour into **infinitely small units**.
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Computing Binomial in the limit

$$\mathbf{P}[X = k] = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \underset{expand}{=}$$

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Therefore, in our store footfall example: the probability of k people entering the store in the next 1 hour is:

$$\mathbf{P}[X{=}k] = \lim_{n\to\infty} \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!}e^{-\lambda}$$

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Poisson

Poisson discrete random variable

A Poisson RV X approximates Binomial where n is large, p is small, and $\lambda = np$ is "moderate". Thus we no longer need to know n and p, we only need to provide **rate** λ. X is the number of successes over the duration of the experiment.

$$\mathbf{X} \sim \mathbf{Pois}(\lambda)$$

Range: $\{0, 1, 2, \ldots\}$

PMF: $\mathbf{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}$

Expectation: $\mathbf{E}[X] = \lambda$

Variance: $\mathbf{V}[X] = \lambda$

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Examples: # earthquakes in a given year, # goals scored during a 90 minute football game, # misprints per page in a book, # emails per day.

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Key idea: Divide time into a **large number** of small increments. Assume that during each increment, there is some **small probability** of the event happening (independent of other increments).

Earthquake example

Example

Suppose there are an average of 2.79 major earthquakes in the world each year. What is the probability of getting 3 major earthquakes next year?

Answer

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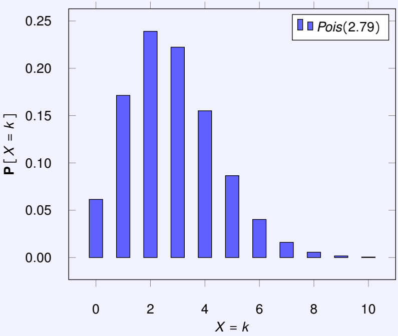

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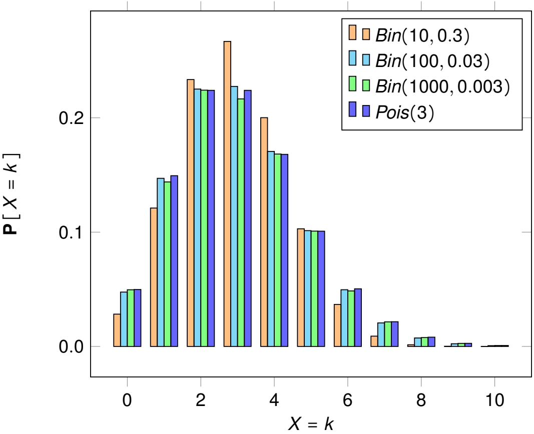

Poisson expectation

PMF: $= k \in \{0, 1, 2, \ldots \infty\}; \mathbf{P}[\, X = k \,] = \frac{\lambda^k}{k!} e^{-\lambda}$

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$$= \lambda e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} =$$

Poisson expectation

$$\text{PMF:} \ = k \in \{0, 1, 2, \ldots \infty\}; \mathbf{P}[\, X = k \,] = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\mathbf{E}[\, X \,] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} =$$

$$= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \ (\text{let } i = k - 1)$$

$$= \lambda e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Poisson variance

$$\mathbf{E}\left[X^2 \right] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} =$$

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$$\mathbf{V}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

Poisson variance

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$$\mathbf{V}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

$$\mathbf{E}\left[X^k\right] = \lambda \mathbf{E}\left[(X+1)^{k-1}\right]$$

Bernoulli, Poisson, and random processes

- A Poisson process is a model for a series of discrete events where the average time between events is known, but the exact timing of events is random.
  - The arrival of an event is independent of the event before (waiting time between events is memoryless).
  - The average rate (events per time period) is constant.
  - Two events cannot occur at the same time: each sub-interval of a Poisson process is a Bernoulli trial that is either a success or a failure.
- Example: your website goes down on average twice per 60 days; calling a help centre; movements of stock price...

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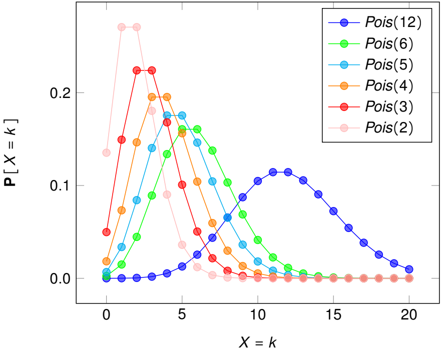

Outline

Poisson discrete random variable

Geometric discrete random variable

Negative binomial discrete random variable

Hypergeometric discrete random variable

Geometric

Geometric discrete random variable

X is a geometric RV if X is a number of independent Bernoulli trials until the first success, and p is the probability of success on each Bernoulli trial.

$$\mathbf{X} \sim \mathbf{Geo}(\mathbf{p})$$

$$\begin{aligned} \text{Range:} &\quad \{1, 2, \ldots\} \\ \text{PMF:} &\quad \mathbf{P}[X = n] = (1-p)^{n-1} p \\ \text{Expectation:} &\quad \mathbf{E}[X] = \frac{1}{p} \\ \text{Variance:} &\quad \mathbf{V}[X] = \frac{1-p}{p^2} \end{aligned}$$

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Examples: tossing a coin ($\mathbf{P}[\, head \,] = p$) until first heads appears, generating bits with $\mathbf{P}[\, bit = 1 \,] = p$ until first 1 is generated.

Geometric RV properties

PMF (E_i is the event that the i-th trial succeeds):

$$\mathbf{P}[\, X = n \,] = \mathbf{P}\left[\, E_1^c E_2^c \ldots E_{n-1}^c E_n \,\right] =$$

CDF ($\mathbf{P}[\, X > n \,]$ is the probability that at least the first n trials fail):

Geometric RV properties

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$$\mathbf{P}[X \leq n] = 1 - \mathbf{P}[X > n] =$$

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&= (1-p)^{n-1}p
\end{aligned}$$

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&= 1 - \mathbf{P}[\,E_1^c\,]\,\mathbf{P}[\,E_2^c\,]\dots\mathbf{P}[\,E_n^c\,] = \\
&= 1 - (1-p)^n
\end{aligned}$$

Die example

Example

You roll a fair 6-sided die until it comes up with # 6. What is the probability that it will take 3 rolls?

Answer

Die example

Example

You roll a fair 6-sided die until it comes up with # 6. What is the probability that it will take 3 rolls?

Answer

Let X be a RV for # of rolls. Probability for any # on die is $\frac{1}{6}$.

Define RVs: $X \sim Geo(\frac{1}{6})$, want $\mathbf{P}[\,X = 3\,]$.

Solve:

Outline

Negative binomial

Negative binomial discrete random variable

X is a negative binomial RV if X is the number of independent Bernoulli trials until r successes and p is the probability of success on each trial.

$$\mathbf{X \sim NegBin(r, p)}$$

$$\text{Range:} \quad \{r, r+1, \ldots\}$$

$$\text{PMF:} \quad \mathbf{P}[X = n] = \binom{n-1}{r-1}(1-p)^{n-r}p^r$$

$$\text{Expectation:} \quad \mathbf{E}[X] = \frac{r}{p}$$

$$\text{Variance:} \quad \mathbf{V}[X] = \frac{r(1-p)}{p^2}$$

Negative binomial

Negative binomial discrete random variable

X is a negative binomial RV if X is the number of independent Bernoulli trials until r successes and p is the probability of success on each trial.

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$$\text{Expectation:} \quad \mathbf{E}[X] = \frac{r}{p}$$

$$\text{Variance:} \quad \mathbf{V}[X] = \frac{r(1-p)}{p^2}$$

Examples: tossing a coin until r-th heads appears, generating bits until the first r 1's are generated.

Note: $Geo(p) = NegBin(1, p)$.

NegBin example

Example (not real life!)

A PhD student is expected to publish 2 papers to graduate. A conference accepts each paper randomly and independently with probability $p = 0.25$. On average, how many papers will the student need to submit to a conference in order to graduate?

Answer

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

- Need to show that $Z \sim NegBin(m + n, p)$.

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

- Need to show that $Z \sim NegBin(m + n, p)$.
- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\, heads \,] = p$.

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

- Need to show that $Z \sim NegBin(m + n, p)$.
- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\, heads \,] = p$.
- Let X be a RV for # of coin tosses until m heads are observed. Thus $X \sim NegBin(m, p)$.

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

- Need to show that $Z \sim NegBin(m + n, p)$.
- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\ heads\ ] = p$.
- Let X be a RV for # of coin tosses until m heads are observed. Thus $X \sim NegBin(m, p)$.
- Now, continue to toss a coin after m heads are observed, until n more heads are observed. Thus, for this part of the sequence, $Y \sim NegBin(n, p)$.

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

- Need to show that $Z \sim NegBin(m + n, p)$.
- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\,heads\,] = p$.
- Let X be a RV for # of coin tosses until m heads are observed. Thus $X \sim NegBin(m, p)$.
- Now, continue to toss a coin after m heads are observed, until n more heads are observed. Thus, for this part of the sequence, $Y \sim NegBin(n, p)$.
- Looking at it from the beginning we tossed independently the coin until we observed $m + n$ heads, thus $Z = X + Y$ and thus $Z \sim NegBin(m + n, p)$.

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

- Need to show that $Z \sim NegBin(m + n, p)$.
- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\,heads\,] = p$.
- Let X be a RV for # of coin tosses until m heads are observed. Thus $X \sim NegBin(m, p)$.
- Now, continue to toss a coin after m heads are observed, until n more heads are observed. Thus, for this part of the sequence, $Y \sim NegBin(n, p)$.
- Looking at it from the beginning we tossed independently the coin until we observed $m + n$ heads, thus $Z = X + Y$ and thus $Z \sim NegBin(m + n, p)$.

- Note: if $X_1, X_2, \ldots X_m$ are m independent $Geo(p)$ RVs, then the RV $X = X_1 + X_2 + \cdots + X_m$ has $NegBin(m, p)$ distribution.

Outline

Poisson discrete random variable

Geometric discrete random variable

Negative binomial discrete random variable

Hypergeometric discrete random variable

Hypergeometric

Hypergeometric discrete random variable

X is a hypergeometric RV that samples n objects, without replacement, with i successes (random draw for which the object drawn has a specified feature), from a finite population of size N that contains exactly m objects with that feature.

$$\mathbf{X \sim Hyp(N, n, m)}$$

Range: $\{0, 1, \ldots, n\}$

PMF: $\mathbf{P}[X = i] = \dfrac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$

Expectation: $\mathbf{E}[X] = n\dfrac{m}{N}$

Variance: $\mathbf{V}[X] = n\dfrac{m}{N}\left(1 - \dfrac{m}{N}\right)\left(1 - \dfrac{n-1}{N-1}\right)$

Example: an urn has N balls of which m are white and $N - m$ are black; we take a random sample **without replacement** of size n and measure X: # of white balls in the sample.

Example

A street has 40 houses of which 5 houses are inhabited by families with an income below the poverty line. In a survey, 7 houses are sampled at random from this street. What is the probability that: (a) none of the 5 families with income below poverty line are sampled? (b) 4 of them are sampled? (c) no more than 2 are sampled? (d) at least 3 are sampled?

Answer

Survey sampling

Example

A street has 40 houses of which 5 houses are inhabited by families with an income below the poverty line. In a survey, 7 houses are sampled at random from this street. What is the probability that: (a) none of the 5 families with income below poverty line are sampled? (b) 4 of them are sampled? (c) no more than 2 are sampled? (d) at least 3 are sampled?

Answer

Let X: # of families sampled which are below the poverty line.

$X \sim Hyp(N = 40, n = 7, m = 5)$.

Summary of discrete RV

	$Ber(p)$	$Bin(n,p)$	$Pois(\lambda)$	$Geo(p)$	$NegBin(r,p)$	$Hyp(N,n,m)$
PMF	$\mathbf{P}[X=1]=p$	$\mathbf{P}[X=k]=\binom{n}{k}p^k(1-p)^{n-k}$	$\mathbf{P}[X=k]=\frac{\lambda^k}{k!}e^{-\lambda}$	$\mathbf{P}[X=n]=(1-p)^{n-1}p$	$\mathbf{P}[X=n]=\binom{n-1}{r-1}(1-p)^{n-r}p^r$	$\mathbf{P}[X=i]=\frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$
$\mathbf{E}[X]$	p	np	λ	$\frac{1}{p}$	$\frac{r}{p}$	$n\frac{m}{N}$
$\mathbf{V}[X]$	$p(1-p)$	$np(1-p)$	λ	$\frac{1-p}{p^2}$	$\frac{r(1-p)}{p^2}$	$n\frac{m}{N}\left(1-\frac{m}{N}\right)\left(1-\frac{n-1}{N-1}\right)$
Descr.	1 experiment with prob p of success	n independent trials with prob p of success	# successes over experiment duration, $\lambda = np$ rate of success	# independent trials until first success	# independent trials until r successes	# successes of drawing item with a feature (without replacement) in a sample of size n from a population of size N with m items with the feature