Introduction to Probability

Lecture 2: Random variables, probability mass function, expectation

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

Outline

Random variable

Probability mass function

Cumulative distribution function

Expectation

What is a random variable?

Random variable

A random variable X is a function from the sample space to the real numbers.

- We can interpret X as a quantity whose value depends on the outcome of an experiment (some probabilistic process).
  - Roll two dice, X: sum of dice
  - Toss 3 coins, X: number of heads
  - Give a student a test, X: score
  - Stock market index
- Or can think of X as a variable in a programming language that takes on values, has a type, and has a domain over which it is applicable.
- Many different types of RV: indicator, binary, choice, Bernoulli, etc.
- Random variable can be discrete or continuous:
  - X has finitely many possible values: discrete.
  - X has every integer as a possible value: discrete.
  - X amount of time it takes to finish a race: continuous (possible value: $\{t : 0 \leq t < \infty\} = [0, \infty)$).

Examples of random variables

Example

We toss 3 fair coins. Let a random variable X be the total number of heads on the 3 coins. What are the probabilities of X taking on the following values: $X = 0, X = 1, X = 2, X = 3, X \geq 4$?

Answer

1. $\mathbf{P}[X = 0] = \frac{1}{8}$ where set of outcomes is $\{(T, T, T)\}$
2. $\mathbf{P}[X = 1] = \frac{3}{8}$ where set of outcomes is $\{(H, T, T), (T, H, T), (T, T, H)\}$
3. $\mathbf{P}[X = 2] = \frac{3}{8}$ where set of outcomes is $\{(H, H, T), (T, H, H), (H, T, H)\}$
4. $\mathbf{P}[X = 3] = \frac{1}{8}$ where set of outcomes is $\{(H, H, H)\}$
5. $\mathbf{P}[X \geq 4] = 0$ where set of outcomes is $\{\}$

Random variables are NOT events

random variables $\neq$ events

Tossing 3 fair coins example			
$X = x$	$\mathbf{P}[X = x]$	Set of outcomes	Possible event E
$X = 0$	$\frac{1}{8}$	$\{(T,T,T)\}$	Toss 0 heads
$X = 1$	$\frac{3}{8}$	$\{(H,T,T),(T,H,T),(T,T,H)\}$	Toss exactly 1 head
$X = 2$	$\frac{3}{8}$	$\{(H,H,T),(T,H,H),(H,T,H)\}$	Event where $X = 2$
			Toss exactly 2 heads
$X = 3$	$\frac{1}{8}$	$\{(H,H,H)\}$	Toss 0 tails
$X \geq 4$	0	$\{\}$	Toss 4 or more heads

We can define events by condition of the value of a random variable (RV takes on values that satisfy a numerical test).

Another example

Example

Tossing a coin has the probability p that it comes up heads. Toss a coin 5 times. Let X: the number of heads in 5 tosses. What is the range of X (i.e., what are the values that X can take on with non-zero probability)? What is $\mathbf{P}[\,X = k\,]$ where k is in the range of X?

Answer

- Notice that each coin toss is an independent trail.
- Recall $\mathbf{P}[\,2 \text{ heads}\,] = \binom{5}{2}p^2(1-p)^3$, $\mathbf{P}[\,3 \text{ heads}\,] = \binom{5}{3}p^3(1-p)^2$.
- Range of $X : \{0, 1, 2, 3, 4, 5\}$
- $\mathbf{P}[\,X = k\,] = \binom{5}{k}p^k(1-p)^{5-k}$

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Probability mass function definition (PMF)

Discrete random variable

A random variable X is discrete if its range has countably many values

$$X = x \text{ where } x \in \{x_1, x_2, x_3, \ldots\}$$

Probability mass function

The probability mass function (PMF) of a discrete random variable X is a function $p(a)$ of X that maps possible outcomes of a random variable to the corresponding probabilities:

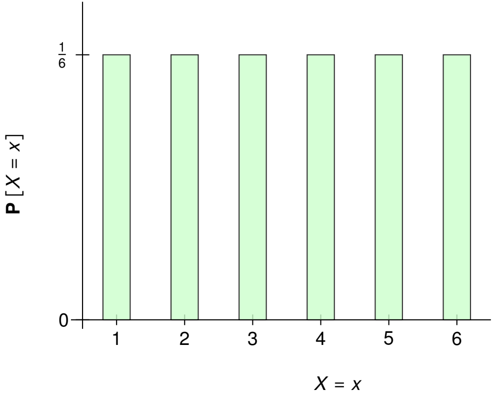
$$p(a) = \mathbf{P}\left[\, X = a \,\right] = p_X(a)$$

Recall that probabilities must sum to 1: $\sum_{i=1}^{\infty} p(a_i) = 1$.

Example for a single die

- Let X be a RV representing a single die roll.
- Range of X : $\{1, 2, 3, 4, 5, 6\}$, thus X is a discrete RV.
- PMF of X:

$$p(x) = \mathbf{P}[\, X = x \,] = \begin{cases} \frac{1}{6} & x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$

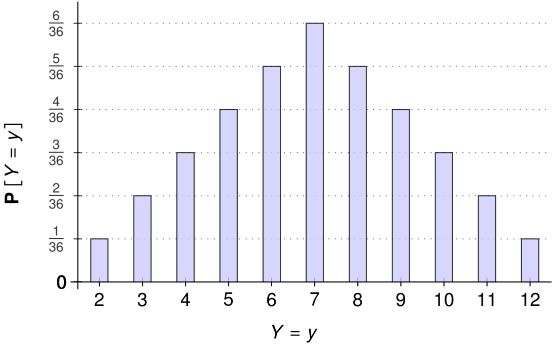


Example for two dice

- Let Y be a RV representing the sum of two independent dice rolls.
- Range of $Y : \{2, 3, \ldots, 11, 12\}$.
- PMF of Y:

$$p(y) = \mathbb{P}[\,Y = y\,] = \begin{cases} \frac{y-1}{36} & y \in \mathbb{Z}, 2 \le y \le 6 \\ \frac{13-y}{36} & y \in \mathbb{Z}, 7 \le y \le 12 \\ 0 & \text{otherwise} \end{cases}$$

- Check $\sum_{y=2}^{12} p(y) = 1$.



Properties of PMF

Let possible values of $X = \{a_1, a_2, a_3, \ldots\}$.

1. By Axiom 1: $0 \le p(a_i) \le 1$.

2. $p(a) = 0$ if a is not a possible value.

3. By Axiom 3: $\sum_{i=1}^{\infty} p(a_i) = 1$.

   $$\sum_{i=1}^{\infty} p(a_i) = \sum_{i=1}^{\infty} \mathbf{P}[\, X = a_i \,] = \mathbf{P}\left[\, \bigcup_{i=1}^{\infty} \{X = a_i\} \,\right] = \mathbf{P}[\, S \,] = 1$$

4. Notice that everything to do with discrete RVs is expressed in terms of (finite or infinite) sum.

5. For continuous RVs, these sums are replaced by integrals.

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Cumulative distribution function definition (CDF)

Another useful way to analyse probabilities.

Cumulative distribution function

The cumulative distribution function (CDF) of a random variable X is defined as

$$F(a) = F_X(a) = \mathbf{P}[X \leq a] \text{ where } -\infty < a < \infty$$

For a **discrete** random variable X, the CDF is

$$F(a) = \mathbf{P}[X \leq a] = \sum_{\text{all } x \leq a} p(x)$$

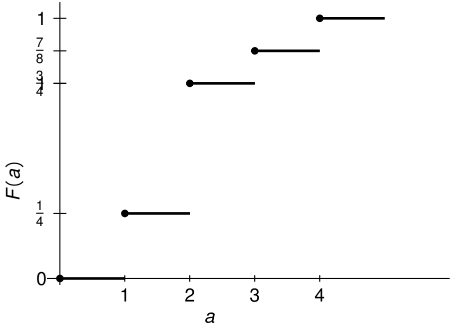
Note that for a discrete RV the CDF is a step function, i.e., the value of F is constant in the intervals (x_{i-1}, x_i) and then takes a step of size $p(x_i)$ at x_i.

Example

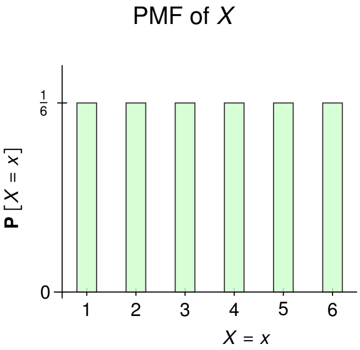
- Let the PMF for X be given by $p(1) = \frac{1}{4}, p(2) = \frac{1}{2}, p(3) = \frac{1}{8}, p(4) = \frac{1}{8}$.
- Then CDF is:

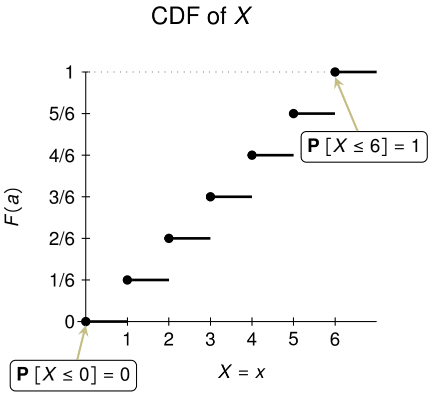
$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \leq a < 2 \\ \frac{3}{4} & 2 \leq a < 3 \\ \frac{7}{8} & 3 \leq a < 4 \\ 1 & 4 \leq a \end{cases}$$

- Graphical depiction of function:



Example for a single die





Properties of CDF

1. $0 \leq F(x) \leq 1$ for all x
2. $\lim_{x \to -\infty} F(x) = 0$
3. $\lim_{x \to \infty} F(x) = 1$
4. $F(x)$ is a non-decreasing function of x (if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$)

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Expectation - expected value

Expectation

The expectation of a discrete random variable X is defined as

$$\mathbf{E}[X] = \sum_{x:\mathbf{P}[x]>0} x\mathbf{P}[x]$$

- Expectation is the average value of the random variable over many repetitions of the experiment it represents.
- It is the sum over all values of $X = x$ that have non-zero probability.
- AKA: mean, expected value, weighted average, centre of mass, first moment.

Example of a die roll

What is the expected value of a 6-sided die roll (i.e., what is the average value of a die roll)?

1. Define random variables:

$$X = \text{RV for value of roll}$$

$$\mathbf{P}[\,X = x\,] = \begin{cases} \frac{1}{6} & x \in \{1, \ldots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

2. Solve:

$$\mathbf{E}[\,X\,] = 1(\frac{1}{6}) + 2(\frac{1}{6}) + 3(\frac{1}{6}) + 4(\frac{1}{6}) + 5(\frac{1}{6}) + 6(\frac{1}{6}) = \frac{7}{2}$$

Example of school classes

Example

A school has 3 classes with 5, 10 and 150 students. What is the average class size?

Answer

Interpretation 1: Randomly choose a class with equal probability. Thus, X = size of chosen class

$$\mathbf{E}[X] = 5(\frac{1}{3}) + 10(\frac{1}{3}) + 150(\frac{1}{3}) = \frac{165}{3} = 55$$

Interpretation 2: Randomly choose a student with equal probability. Thus, Y = size of chosen class

$$\mathbf{E}[Y] = 5(\frac{5}{165}) + 10(\frac{10}{165}) + 150(\frac{150}{165}) = \frac{22635}{165} = 137$$

This is a general phenomenon: it occurs because the more students are in a class, the more likely it is that a randomly chosen student would be in that class. As a result, bigger classes are given more weight than smaller classes.

Example of Roulette Version 1

Example

A roulette wheel has 36 places numbered from 1 to 36. In addition, 18 of them are coloured red and the other 18 are coloured black. A ball is thrown to take one of 36 places. A gambler can bet:

- on the colour of the place that the ball takes. A correct, either red or black, place wins them a 1 to 1 ratio payout;
- on the number of the place that the ball takes. A correct number wins them a 35 to 1 ratio payout.

What is the expected value if a gambler bets on

1. the colour of the place in the roulette;
2. the number of the place in the roulette that the ball will fall.

Are the two different betting games fair?

Answer

Example of Roulette Version 1 Cont.

Example

What is the expected value if a gambler bets on

1. the colour of the place in the roulette;
2. the number of the place in the roulette that the ball will fall.

Are the two different betting games fair?

Answer

1. Let E_X : bet on colour.
   - If loose, then $X = -1$. Thus $\mathbf{P}[\, loose_X \,] = \frac{1}{2}$.
   - If win, then $X = 1$. Thus $\mathbf{P}[\, win_X \,] = \frac{1}{2}$.
   - Thus, $\mathbf{E}[\, X \,] = (-1)(\frac{1}{2}) + (1)(\frac{1}{2}) = 0$, This game is "fair".
2. Let E_Y : bet on number.
   - If loose, then $Y = -1$. Thus $\mathbf{P}[\, loose_Y \,] = \frac{35}{36}$.
   - If win, then $Y = 35$. Thus $\mathbf{P}[\, win_Y \,] = \frac{1}{36}$.
   - Thus, $\mathbf{E}[\, Y \,] = (-1)(\frac{35}{36}) + (35)(\frac{1}{36}) = 0$, This game is "fair" too.

Example of Roulette Version 2

Example

Change the game to add two green places, 0 and 00. Now there are a total of 38 places. Payouts are the same as before. What are the expected values now?

Answer

1. Let E_X : bet on red colour.
   Thus, $\mathbf{E}[\,X\,] = (-1)(\frac{20}{38}) + (1)(\frac{18}{38}) = -\frac{1}{19}$.

2. Let E_Y : bet on number 10.
   Thus, $\mathbf{E}[\,Y\,] = (-1)(\frac{37}{38}) + (35)(\frac{1}{38}) = -\frac{1}{19}$.

So, no, these games are not fair, as the gambler would loose £$\frac{1}{19}$ = 5.3 pence per game.