# Introduction to Probability

Lecture 1: Conditional probabilities and Bayes' theorem

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# Outline

Logistics, motivation, background

Conditional probability

Bayes' Theorem

Independence

# Lecturers

Mateja Jamnik

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# Course logistics

Rough syllabus:

- Introduction to probability: 1 lecture
- Discrete and continuous random variables: 6 lectures
- Moments and limit theorems: 3 lectures
- Applications/statistics: 2 lectures

Recommended reading:

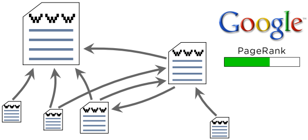
- **Ross, S.M. (2014). A First course in probability. Pearson (9th ed.).**
- **Dekking, F.M., et. al. (2005) A modern introduction to probability and statistics. Springer.**
- Bertsekas, D.P. & Tsitsiklis, J.N. (2008). Introduction to probability. Athena Scientific.
- Grimmett, G. & Welsh, D. (2014). Probability: an Introduction. Oxford University Press (2nd ed.).

# Why probability?

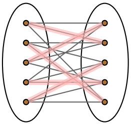
- Gives us mathematical tools to deal with uncertain events.
- It is used everywhere, especially in applications of machine learning.
- Machine learning: use probability to compute predictions about and from data.
- Probability is not statistics:
  - Both about random processes.
  - Probability: logically self-contained, few rules for computing, one correct answer.
  - Statistics: messier, more art, get experimental data and try to draw probabilistic conclusions, no single correct answer.

# Applications of probability

## Ranking Websites




## Matching



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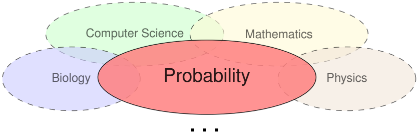
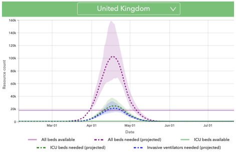
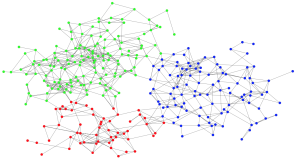
## Finance

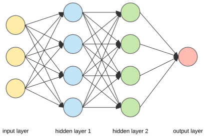

## Medicine



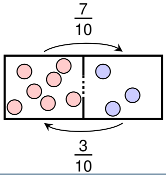

## Data Mining



## Deep Learning




## Particle Processes



# Prerequisite background

- Set theory
- Counting: product rule, sum rule, inclusion-exclusion
- Combinatorics: permutations
- Probability space: sample space, event space
- Axioms
- Union bound

- Look for revision material of above on the course website:

  `https://www.cl.cam.ac.uk/teaching/2324/IntroProb/`

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# Definition

**Conditional probability**

Consider an experiment with sample space $S$, and two events $E$ and $F$. Then, the (conditional) probability of event $E$ given $F$ has occurred (denoted $\mathbf{P}[E|F]$) with $\mathbf{P}[F] > 0$ is defined by

$$\mathbf{P}[E|F] = \frac{\mathbf{P}[E \cap F]}{\mathbf{P}[F]} = \frac{\mathbf{P}[EF]}{\mathbf{P}[F]}$$

Sample space: all possible outcomes consistent with $F$ (i.e., $S \cap F = F$)
Event space: all outcomes in $E$ consistent with $F$ (i.e., $E \cap F$)
Note: we assume that all outcomes are equally likely

$$\mathbf{P}[E|F] = \frac{\#\text{ outcomes in } E \cap F}{\#\text{ outcomes in } F} = \frac{\frac{\#\text{ outcomes in } E\cap F}{\#\text{ outcomes in } S}}{\frac{\#\text{ outcomes in } F}{\#\text{ outcomes in } S}} = \frac{\mathbf{P}[E \cap F]}{\mathbf{P}[F]}$$

# Example

**Example**

Two dice are rolled yielding value $D_1$ and $D_2$. Let $E$ be event that $D_1 + D_2 = 4$.

1. What is $\mathbf{P}[\,E\,]$?
2. Let event $F$ be $D_1 = 2$. What is $\mathbf{P}[\,E|F\,]$?

Answer

# Rules revisited

**Chain rule**

Rearranging the definition of conditional probability gives us:

$$\mathbf{P}[\,EF\,] = \mathbf{P}[\,E|F\,]\,\mathbf{P}[\,F\,]$$

Generalisation of the Chain rule:

**Multiplication rule**

$$\mathbf{P}[\,E_1 E_2 \cdots E_n\,] = \mathbf{P}[\,E_1\,]\,\mathbf{P}[\,E_2|E_1\,]\,\mathbf{P}[\,E_3|E_2 E_1\,] \cdots \mathbf{P}[\,E_n|E_1 \cdots E_{n-1}\,]$$

# Example

**Example**

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. What is the probability that each pile has exactly 1 ace?

Answer

Define:

$E_1$= *ace*♥ is in any one pile

$E_2$= *ace*♥ and *ace*♠ are in different piles

$E_3$= *ace*♥, *ace*♠ and *ace*♣ are in different piles

$E_4$= all aces are in different piles

$$\mathbf{P}[\, E_1 E_2 E_3 E_4 \,] = \mathbf{P}[\, E_1 \,]\,\mathbf{P}[\, E_2 | E_1 \,]\,\mathbf{P}[\, E_3 | E_1 E_2 \,]\,\mathbf{P}[\, E_4 | E_1 E_2 E_3 \,]$$

We have $\mathbf{P}[\, E_1 \,] = 1$. For rest we consider complement of next ace being in the same pile and thus have:

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# Law of total probability

The law of total probability (a.k.a. Partition theorem)

For events $E$ and $F$ where $\mathbf{P}[\,F\,] > 0$, then for any event $E$

$$\mathbf{P}[\,E\,] = \mathbf{P}[\,EF\,] + \mathbf{P}\left[\,EF^c\,\right] = \mathbf{P}[\,E|F\,]\,\mathbf{P}[\,F\,] + \mathbf{P}\left[\,E|F^c\,\right]\mathbf{P}\left[\,F^c\,\right]$$

In general, for disjoint events $F_1, F_2, \ldots F_n$ s.t. $F_1 \cup F_2 \cup \cdots \cup F_n = S$,

$$\mathbf{P}[\,E\,] = \sum_{i=1}^{n} \mathbf{P}[\,E|F_i\,]\,\mathbf{P}[\,F_i\,]$$

Intuition:

Want to know probability of $E$. There are two scenarios, $F$ and $F^c$. If we know these and the probability of $E$ conditioned on each scenario, we can compute the probability of $E$.

# Lightbulb example

Example

There are 3 boxes each containing a different number of light bulbs. The first box has 10 bulbs of which 4 are dead, the second has 6 bulbs of which 1 is dead, and the third box has 8 bulbs of which 3 are dead. What is the probability of a dead bulb being selected when a bulb is chosen at random from one of the 3 boxes (each box has equal chance of being picked)?

Answer

Let event $E$ = "dead bulb is picked", and $F_1$ = "bulb is picked from first box", $F_2$ = "bulb is picked from second box" and $F_3$ = "bulb is picked from third box". We know:

$$\mathbf{P}[\, E|F_1 \,] = \frac{4}{10}, \mathbf{P}[\, E|F_2 \,] = \frac{1}{6}, \mathbf{P}[\, E|F_3 \,] = \frac{3}{8}$$

We need to compute $\mathbf{P}[\, E \,]$, and we know that $\mathbf{P}[\, F_i \,] = \frac{1}{3}$:

# Bayes' theorem

How many spam emails contain the word "Dear"?

$$\mathbf{P}[\,E|F\,] = \mathbf{P}[\,\text{"Dear"}|\text{spam}\,]$$

But how about what is the probability that an email containing "Dear" is spam?

$$\mathbf{P}[\,F|E\,] = \mathbf{P}[\,\text{spam}|\text{"Dear"}\,]$$

**Bayes' theorem**

For any events $E$ and $F$ where $\mathbf{P}[\,E\,] > 0$ and $\mathbf{P}[\,F\,] > 0$,

$$\mathbf{P}[\,F|E\,] = \frac{\mathbf{P}[\,E|F\,]\,\mathbf{P}[\,F\,]}{\mathbf{P}[\,E\,]}$$

and in expanded form,

$$\mathbf{P}[\,F|E\,] = \frac{\mathbf{P}[\,E|F\,]\,\mathbf{P}[\,F\,]}{\mathbf{P}[\,E|F\,]\,\mathbf{P}[\,F\,] + \mathbf{P}[\,E|F^c\,]\,\mathbf{P}[\,F^c\,]} = \frac{\mathbf{P}[\,E|F\,]\,\mathbf{P}[\,F\,]}{\sum_{i=1}^{n}\mathbf{P}[\,E|F_i\,]\,\mathbf{P}[\,F_i\,]}$$

using the Law of Total Probability. Note that all events $F_i$ must be mutually exclusive (non-overlapping) and exhaustive (their union is the complete sample space) .

# Example

**Example**

60% of all email in 2022 is spam. 20% of spam contains the word "Dear". 1% of non-spam contains the word "Dear". What is the probability that an email is spam given it contains the word "Dear"?

Answer

- Let event $E$ ="Dear", event $F$ = spam.

## Bayes' terminology

$$\underbrace{\mathbf{P}[\,F|E\,]}_{\text{posterior}} = \frac{\overbrace{\mathbf{P}[\,E|F\,]}^{\text{likelihood}} \cdot \overbrace{\mathbf{P}[\,F\,]}^{\text{prior}}}{\underbrace{\mathbf{P}[\,E\,]}_{\text{normalisation constant}}}$$

$F$: hypothesis, $E$: evidence

$\mathbf{P}[\,F\,]$: "prior probability" of hypothesis

$\mathbf{P}[\,E|F\,]$: probability of evidence given hypothesis (likelihood)

$\mathbf{P}[\,E\,]$: calculated by making sure that probabilities of all outcomes sum to 1 (they are "normalised")

# Confusion matrix (error matrix)

Used in classification tasks for predicting output error.

| | | True condition | |
|---|---|---|---|
| | Total population | Condition positive $F$ | Condition negative $F^c$ |
| Predicted condition | Predicted condition positive $E$ | True positive $\mathbf{P}[E \mid F]$ | False positive $\mathbf{P}[E \mid F^c]$ |
| | Predicted condition negative $E^c$ | False negative $\mathbf{P}[E^c \mid F]$ | True negative $\mathbf{P}[E^c \mid F^c]$ |

# Medical testing example

Example

- A test is 98% effective at detecting the disease COVID-19 ("true positive").
- The test has a "false positive" rate of 1%.
- 0.5% of the population has COVID-19.
- What is the likelihood you have COVID-19 if you test positive?

Answer

- Let $E$: test positive, $F$: actually have COVID-19.
- Need to find $\mathbf{P}[F \mid E]$.

# Bayesian intuition

- 33% chance of having COVID-19 after testing positive may seem surprising.
- But the space of facts is now conditioned on a positive test result (people who test positive and have COVID-19 **and** people who test positive and don't have COVID-19).

| | $F$ yes disease | $F^c$ no disease |
|---|---|---|
| $E$ test+ | True positive<br>$\mathbf{P}[E \mid F] = 0.98$ | False positive<br>$\mathbf{P}[E \mid F^c] = 0.01$ |
| $E^c$ test- | False negative<br>$\mathbf{P}[E^c \mid F] = 0.02$ | True negative<br>$\mathbf{P}[E^c \mid F^c] = 0.99$ |

- But what is a chance of having COVID-19 if you test and it comes back negative?

$$\mathbf{P}\left[F \mid E^c\right] = \frac{\mathbf{P}[E^c \mid F]\,\mathbf{P}[F]}{\mathbf{P}[E^c \mid F]\,\mathbf{P}[F] + \mathbf{P}[E^c \mid F^c]\,\mathbf{P}[F^c]} \approx 0.0001$$

- We update our beliefs with Bayes' theorem:
  I have 0.5% chance of having COVID-19. I take the test:
  - Test is positive: I now have 33% chance of having COVID-19.
  - Test is negative: I now have 0.01% chance of having COVID-19.
- So it makes sense to take the test.

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# Independent events

**Independence**

Two events $E$ and $F$ are independent if and only if

$$\mathbf{P}[\,EF\,] = \mathbf{P}[\,E\,]\,\mathbf{P}[\,F\,]$$

Otherwise, they are called dependent events.

In general, $n$ events $E_1, E_2, \ldots, E_n$ are mutually independent if for every subset of these events with $r$ elements (where $r \leq n$) it holds that

$$\mathbf{P}[\,E_a E_b \cdots E_r\,] = \mathbf{P}[\,E_a\,]\,\mathbf{P}[\,E_b\,] \cdots \mathbf{P}[\,E_r\,]$$

Therefore for 3 events $E, F, G$ to be independent, we must have

$$\begin{aligned}
\mathbf{P}[\,EFG\,] &= \mathbf{P}[\,E\,]\,\mathbf{P}[\,F\,]\,\mathbf{P}[\,G\,] \\
\mathbf{P}[\,EF\,] &= \mathbf{P}[\,E\,]\,\mathbf{P}[\,F\,] \\
\mathbf{P}[\,EG\,] &= \mathbf{P}[\,E\,]\,\mathbf{P}[\,G\,] \\
\mathbf{P}[\,FG\,] &= \mathbf{P}[\,F\,]\,\mathbf{P}[\,G\,]
\end{aligned}$$

# Independence of complement

Notice an equivalent definition for independent events $E$ and $F$ ($\mathbf{P}[F] > 0$)

$$\mathbf{P}[E|F] = \mathbf{P}[E]$$

Proof:

**Independence of complement**

If events $E$ and $F$ are independent, then $E$ and $F^c$ are independent:

$$\mathbf{P}[EF^c] = \mathbf{P}[E]\mathbf{P}[F^c]$$

Proof:

## Example

**Example**

Each roll of a die is an independent trial. We have two rolls of $D_1$ and $D_2$. Let event $E : D_1 = 1$, $F : D_2 = 6$ and event $G : D_1 + D_2 = 7$ (thus $G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$).

1. Are $E$ and $F$ independent?
2. Are $E$ and $G$ independent?
3. Are $E, F, G$ independent?

Answer

# Conditional independence

**Conditional independence**

Two events $E$ and $F$ are called conditionally independent given a third event $G$ if

$$\mathbf{P}[\,EF|G\,] = \mathbf{P}[\,E|G\,]\,\mathbf{P}[\,F|G\,]$$

Or equivalently,

$$\mathbf{P}[\,E|FG\,] = \mathbf{P}[\,E|G\,]$$

Notice that:

- Dependent events can become conditionally independent.
- Independent events can become conditionally dependent.
- Knowing when conditioning breaks or creates independence is a big part of building complex probabilistic models.

# Example revisited

Example

Each roll of a die is an independent trial. We have two rolls of $D_1$ and $D_2$. Let event $E : D_1 = 1$, $F : D_2 = 6$ and event $G : D_1 + D_2 = 7$ (thus $G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$).

1. Are $E$ and $F$ independent?
2. Are $E$ and $F$ independent given $G$?

Answer

# Summary of conditional probability

Conditioning on event $G$:

| Name of rule | Original rule | Conditional rule |
|---|---|---|
| 1st axiom of probability | $0 \le \mathbf{P}[E] \le 1$ | $0 \le \mathbf{P}[E\mid G] \le 1$ |
| Complement | $\mathbf{P}[E] = 1 - \mathbf{P}[E^c]$ | $\mathbf{P}[E\mid G] = 1 - \mathbf{P}[E^c\mid G]$ |
| Chain rule | $\mathbf{P}[EF] = \mathbf{P}[E\mid F]\mathbf{P}[F]$ | $\mathbf{P}[EF\mid G] = \mathbf{P}[E\mid FG]\mathbf{P}[F\mid G]$ |
| Bayes' theorem | $\mathbf{P}[F\mid E] = \dfrac{\mathbf{P}[E\mid F]\mathbf{P}[F]}{\mathbf{P}[E]}$ | $\mathbf{P}[F\mid EG] = \dfrac{\mathbf{P}[E\mid FG]\mathbf{P}[F\mid G]}{\mathbf{P}[E\mid G]}$ |