$$\{a, b\} = \{c\}$$

$$\iff \{a, b\} \subseteq \{c\} \quad \wedge \quad \{c\} \subseteq \{a, b\}$$

$$\iff (a = c) \wedge (b = c) \quad \wedge \quad (c = a \vee c = b)$$

$$\iff (a = c) \wedge (b = c)$$

Proposition 104 *For all finite sets* $\mathcal{U}$,

$$\#\mathcal{P}(\mathcal{U}) = 2^{\#\mathcal{U}} \ .$$

$\mathcal{U} = \{u_1, \ldots u_n\}$

$\#\mathcal{U} = n$

PROOF IDEA:

$$\mathcal{P}(\mathcal{U}) = \{ S \mid S \subseteq \mathcal{U} \}$$

$S \subseteq \mathcal{U} \longmapsto$ array(S) of length n

$$\text{array}(S)[i] = \begin{cases} 0 & i \notin S \\ 1 & i \in S \end{cases}$$

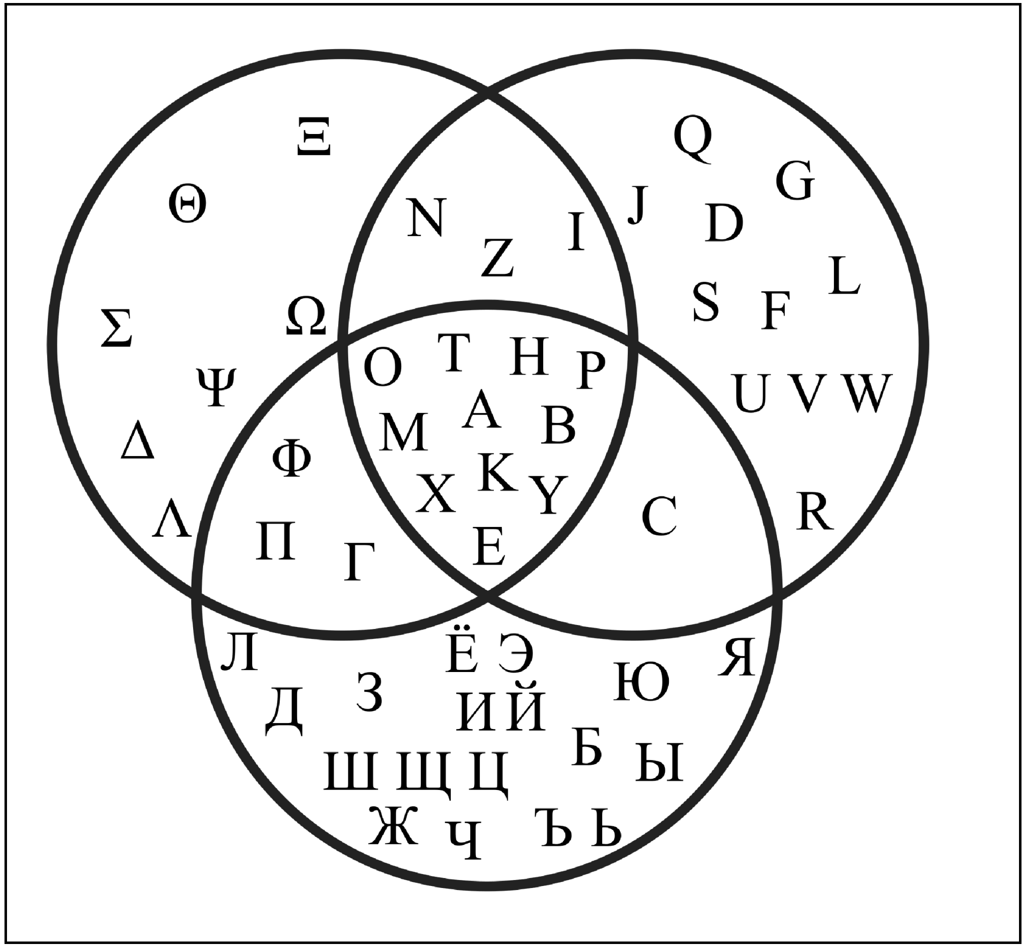
$\text{set}(A) = \{ u_i \mid A[i] = 1 \} \longleftarrow\!\!\!| \ A$

To count $\mathcal{P}(U)$ is to count the number of arrays of 0 & 1's of length n.
The number of which is 2^n.

$$\#\{S \mid S \subseteq U\} = \sum_{k=0}^{n} \#\{S \mid S \subseteq U \text{ of size } k\}$$

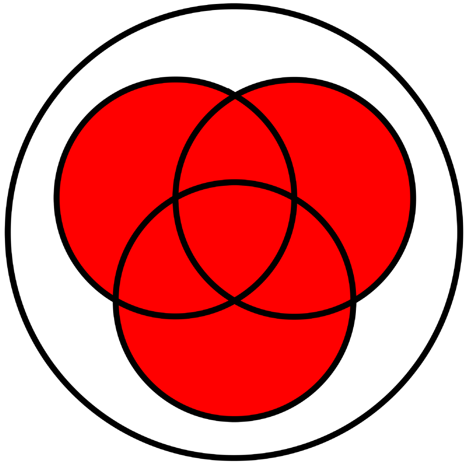
$$= \sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

Venn diagrams[a]

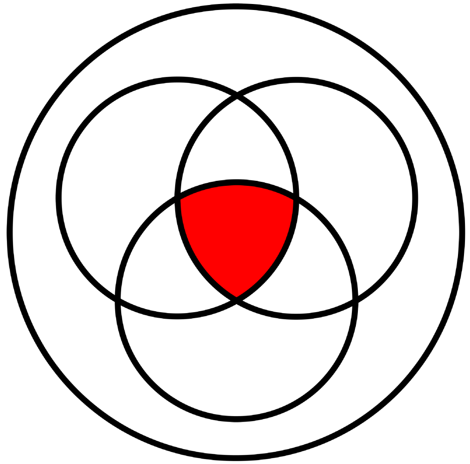


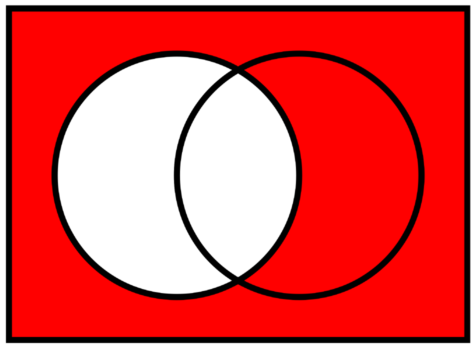
[a]From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

The powerset Boolean algebra

$$\left(\quad \mathcal{P}(U)\ ,\quad \emptyset\ ,\quad U\ ,\quad \cup\ ,\quad \cap\ ,\quad (\cdot)^c \quad \right)$$

$$\emptyset = \{ x \in U \mid \underline{false} \}$$

$$U = \{ x \in U \mid \underline{true} \}$$

For all $A, B \in \mathcal{P}(U)$,

$$A \cup B = \{ x \in U \mid x \in A \vee x \in B \} \in \mathcal{P}(U)$$

$$A \cap B = \{ x \in U \mid x \in A \wedge x \in B \} \in \mathcal{P}(U)$$

$$A^c = \{ x \in U \mid \neg(x \in A) \} \in \mathcal{P}(U)$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathcal{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A = A = \mathcal{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad , \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) \;=\; A \;=\; A \cap (A \cup B)$$

$$A \cup (A \cap B) = A$$

$\Longleftrightarrow$

$$A \cup (A \cap B) \subseteq A \quad \wedge \quad A \subseteq A \cup (A \cap B)$$

$\forall x \in A \cup (A \cap B)$

RTP: $x \in A$

By assumption:

$x \in A \vee x \in (A \cap B)$

$\Longleftrightarrow$ $x \in A \vee (x \in A \wedge x \in B)$

In either case $x \in A$ and we are done.

Lemma: $X \subseteq X \cup Y$

$\forall x \in X$:

RTP $x \in X \vee x \in Y$

But $x \in X$ and we are done.

$\boxtimes$

- The complement operation $(\cdot)^{\mathrm{c}}$ satisfies complementation laws.

$$A \cup A^{\mathrm{c}} = \mathcal{U} \ , \quad A \cap A^{\mathrm{c}} = \emptyset$$

Proposition 105 *Let* $\mathcal{U}$ *be a set and let* $A, B \in \mathcal{P}(\mathcal{U})$.

1. $\forall X \in \mathcal{P}(\mathcal{U}).\ A \cup B \subseteq X \iff (A \subseteq X \wedge B \subseteq X)$.

2. $\forall X \in \mathcal{P}(\mathcal{U}).\ X \subseteq A \cap B \iff (X \subseteq A \wedge X \subseteq B)$.

PROOF:

$A \subseteq A \cup B \wedge B \subseteq A \cup B$.

$A \cup B \subseteq X \Rightarrow (A \subseteq X \wedge B \subseteq X) \quad \forall X \subseteq U$

$(A \subseteq X \wedge B \subseteq X) \Rightarrow A \cup B \subseteq X \quad \forall X \subseteq U$

$A \cup B$ is contained in every set that contains both A and B; in other words, $A \cup B$ is the smallest set is contained on every set containing A and B.

Corollary 106 *Let* U *be a set and let* $A, B, C \in \mathcal{P}(U)$.

1. $C = A \cup B$

To show C is the union of A and B equivalently show:

iff

① $[A \subseteq C \wedge B \subseteq C]$

$\wedge$

② $[\forall X \in \mathcal{P}(U).\ (A \subseteq X \wedge B \subseteq X) \implies C \subseteq X]$

2. $C = A \cap B$

To show C is the intersection of A and B equivalently show:

iff

① $[C \subseteq A \wedge C \subseteq B]$

$\wedge$

② $[\forall X \in \mathcal{P}(U).\ (X \subseteq A \wedge X \subseteq B) \implies X \subseteq C]$

NB: $\gcd(a,b) \mid a \;\wedge\; \gcd(a,b) \mid b$

$$\forall d.\ (d \mid a \wedge d \mid b) \Rightarrow d \mid \gcd(a,b)$$

Sets and logic

$\mathcal{P}(\mathsf{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
U	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{c}$	$\neg(\cdot)$

big $\cap$ $\qquad$ $\forall$

big $\cup$ $\qquad$ $\exists$

Pairing axiom

For every a and b, there is a set with a and b as its only elements.

$$\{ a , b \}$$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \lor x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{ a \}$, and referred to as a *singleton*.

Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

Proposition 107 *For all* a, b, c, x, y,

1. $\{a\} = \{x, y\} \implies x = y = a$
2. $\{c, x\} = \{c, y\} \implies x = y$

PROOF:

Assume $\{c, x\} = \{c, y\}$.

Since $\{c, x\} \subseteq \{c, y\}$ we have ① $(x = c \vee x = y)$

Since $\{c, y\} \subseteq \{c, x\}$ we have ② $(y = c \vee y = x)$

<u>RTP</u> $x = y$

From ① and ②, $x = y$ always.

$\boxtimes$

Given a and b, define

$$\langle a,b\rangle =_{\text{def}} \{\{a\},\{a,b\}\}$$

Consider

$$\langle a,b\rangle = \langle x,y\rangle$$

Then $\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$

So $\{a\}=\{x\}$ or $\{a\}=\{x,y\}$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\boxed{a=x} \qquad\qquad \boxed{a=x}=y$$

Therefore $\{\{a\},\{a,b\}\} = \{\{a\},\{a,y\}\}$

Thus

$$\{a, b\} = \{a, y\}$$

and

$$\boxed{b = y}$$

We have shown:

$$\boxed{\langle a, b\rangle = \langle x, y\rangle \Rightarrow (a = x) \wedge (b = y)}$$

defining property of order pairing

Ordered pairing

Notation:

$$(a, b) \quad \text{or} \quad \langle a, b \rangle$$

Fundamental property:

$$(a, b) = (x, y) \implies a = x \wedge b = y$$