Where we're at

We want:

- a mapping of PCF types τ to domains $[\![\tau]\!]$; ✓
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$;
- denotation of open terms will be continuous functions.

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- denotation of open terms will be continuous functions.

Such that:

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$.

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$.

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \mathrm{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$.

Denotational Semantics for PCF

Terms

To every typing judgement

$$\Gamma \vdash t : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash t : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains. In other words,

$$[\![-]\!] : \mathrm{PCF}_{\Gamma,\tau} \to [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\begin{aligned} \text{succ} : \quad \mathbb{N} &\to \mathbb{N} \\ n &\mapsto n + 1 \end{aligned}$$

$$\begin{aligned} \text{pred} : \quad \mathbb{N} &\to \mathbb{N} \\ 0 &\mapsto \text{undefined} \\ n + 1 &\mapsto n \end{aligned}$$

$$\begin{aligned} \text{zero?} : \quad \mathbb{N} &\to \mathbb{B} \\ 0 &\mapsto \text{true} \\ n + 1 &\mapsto \text{false} \end{aligned}$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$\begin{aligned} \text{succ}_\perp : \quad \mathbb{N}_\perp &\to \mathbb{N}_\perp \\ n &\mapsto n+1 \\ \perp &\mapsto \perp \end{aligned}$$

$$\begin{aligned} \text{pred}_\perp : \quad \mathbb{N}_\perp &\to \mathbb{N}_\perp \\ 0 &\mapsto \perp \\ n+1 &\mapsto n \\ \perp &\mapsto \perp \end{aligned}$$

$$\begin{aligned} \text{zero?}_\perp : \quad \mathbb{N}_\perp &\to \mathbb{B}_\perp \\ 0 &\mapsto \text{true} \\ n+1 &\mapsto \text{false} \\ \perp &\mapsto \perp \end{aligned}$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$
\begin{aligned}
[\![\texttt{0}]\!](\rho) &\stackrel{\text{def}}{=} 0 && \in \mathbb{N}_\bot \\
[\![\texttt{true}]\!](\rho) &\stackrel{\text{def}}{=} \text{true} && \in \mathbb{B}_\bot \\
[\![\texttt{false}]\!](\rho) &\stackrel{\text{def}}{=} \text{false} && \in \mathbb{B}_\bot
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\end{aligned}
$$

$$
\begin{aligned}
[\![\mathtt{succ}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{succ}_\bot([\![t]\!](\rho)) && \in \mathbb{N}_\bot \\
[\![\mathtt{pred}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{pred}_\bot([\![t]\!](\rho)) && \in \mathbb{N}_\bot \\
[\![\mathtt{zero?}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{zero?}_\bot([\![t]\!](\rho)) && \in \mathbb{B}_\bot
\end{aligned}
$$

$$
[\![\mathtt{succ}(t)]\!] = \text{succ}_\bot \circ [\![t]\!]
$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$
\begin{aligned}
[\![\texttt{0}]\!](\rho) &\overset{\text{def}}{=} 0 && \in \mathbb{N}_\bot \\
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\\
[\![\texttt{succ}(t)]\!](\rho) &\overset{\text{def}}{=} \text{succ}_\bot([\![t]\!](\rho)) && \in \mathbb{N}_\bot \\
[\![\texttt{pred}(t)]\!](\rho) &\overset{\text{def}}{=} \text{pred}_\bot([\![t]\!](\rho)) && \in \mathbb{N}_\bot \\
[\![\texttt{zero?}(t)]\!](\rho) &\overset{\text{def}}{=} \text{zero?}_\bot([\![t]\!](\rho)) && \in \mathbb{B}_\bot \\
\\
[\![\texttt{if}\ b\ \texttt{then}\ t\ \texttt{else}\ t']\!] &\overset{\text{def}}{=} \text{if}([\![b]\!](\rho), [\![t]\!](\rho), [\![t']\!](\rho)) && \in [\![\tau]\!]
\end{aligned}
$$

$$
[\![\texttt{if}\ b\ \texttt{then}\ t\ \texttt{else}\ t']\!] = \text{if} \circ \langle [\![b]\!], \langle [\![t]\!], [\![t']\!] \rangle \rangle
$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}$$

$$[\![\Gamma]\!] = \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$$

$$[\![\Gamma(x)]\!] = [\![\tau]\!]$$

$$[\![x]\!]\,(\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!]$$

$$[\![x]\!]\,(\rho) = \pi_x(\rho)$$

Denotation of the λ-calculus operations

$$\llbracket x \rrbracket(\rho) \stackrel{\text{def}}{=} \rho(x) \qquad \in \llbracket \Gamma(x) \rrbracket$$

$$\llbracket t_1\ t_2 \rrbracket(\rho) \stackrel{\text{def}}{=} (\llbracket t_1 \rrbracket(\rho))\ (\llbracket t_2 \rrbracket(\rho))$$

$$\llbracket t_1\ t_2 \rrbracket = \text{eval} \circ \langle \llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket \rangle$$

eval : (D→E)×1) → E

$$\frac{\Gamma, x:\tau \vdash t : \sigma}{\Gamma \vdash \mathsf{fun}\, x:\tau.\, t : \tau \to \sigma}$$

$$
\begin{aligned}
[\![x]\!]\,(\rho) &\stackrel{\text{def}}{=} \rho(x) && \in [\![\Gamma(x)]\!] \\
[\![t_1\ t_2]\!]\,(\rho) &\stackrel{\text{def}}{=} ([\![t_1]\!]\,(\rho))\ ([\![t_2]\!]\,(\rho)) \\
[\![\mathsf{fun}\, x{:}\,\tau.\, t]\!]\,(\rho) &\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!]\,.\ [\![t]\!]\,(\rho, d)
\end{aligned}
$$

$$[\![\mathsf{fun}\, x{:}\,\tau.\, t]\!] = \mathrm{cur}([\![t]\!])$$

$$\mathrm{cur} : (D \times D' \to E) \to D \to D' \to E \qquad D : [\![\Gamma]\!] \quad D' : [\![\tau]\!] \quad E : [\![\sigma]\!]$$

$$([\![\Gamma]\!] \times [\![\tau]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!] \to [\![\sigma]\!]$$

$$[\![\Gamma, x:\tau]\!] \to [\![\sigma]\!] \to [\![\Gamma]\!] \to [\![\tau \to \sigma]\!]$$

Denotation of fixed points

$$[\![\texttt{fix}\ f]\!]\,(\rho) \ \stackrel{\text{def}}{=}\ \mathrm{fix}([\![f]\!]\,(\rho))$$

For any PCF term t such that $\Gamma \vdash t : \tau$, the object $[\![t]\!]$ is well-defined and a continuous function $[\![t]\!] : [\![\Gamma]\!] \to \tau$.

Denotation of PCF terms

For any PCF term t such that $\Gamma \vdash t : \tau$, the object $[\![t]\!]$
is well-defined and a continuous function $[\![t]\!] : [\![\Gamma]\!] \to \tau$.

If $t \in \mathrm{PCF}_\tau$: $\quad [\![t]\!] \quad \in \quad [\![\cdot]\!] \to [\![\tau]\!] \quad = \quad \mathbb{1} \to [\![\tau]\!] \quad \cong \quad [\![\tau]\!]$

$\cdot \vdash t : \tau$

Denotational Semantics for PCF

Compositionality

Suppose $t, u \in \mathrm{PCF}_{\Gamma,\tau}$, such that

$$\llbracket t \rrbracket = \llbracket u \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket \tau \rrbracket$$

Suppose moreover that $\mathcal{C}[-]$ is a PCF context such that $\Gamma' \vdash_{\Gamma,\tau} \mathcal{C} : \tau'$. Then

$$\llbracket \mathcal{C}[t] \rrbracket = \llbracket \mathcal{C}[u] \rrbracket : \llbracket \Gamma' \rrbracket \to \llbracket \tau' \rrbracket \, .$$

A denotation for evaluation contexts

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

A denotation for evaluation contexts

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

$$[\![-]\!]\,(d) = d$$
$$[\![\mathcal{C}\ t]\!]\,(d)(\rho) = ([\![\mathcal{C}]\!]\,(d)(\rho))([\![t]\!]\,(\rho))$$
$$\vdots$$

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\begin{aligned} [\![-]\!]\,(d) &= d \\ [\![\mathcal{C}\ t]\!]\,(d)(\rho) &= ([\![\mathcal{C}]\!]\,(d)(\rho))([\![t]\!]\,(\rho)) \\ &\ \vdots \end{aligned}$$

$[\![t]\!]$

$[\![\rho[\![t]\!]\!]$

$\mathcal{C} = -$

$[\![-[t]]\!]\rho = [\![t]\!] = [\![-]\!]([\![t]\!])$

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$ and $\Delta \vdash t : \sigma$, then

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}]\!]\,([\![t]\!])$$

Assume $[\![t]\!] = [\![u]\!] \in [\![\Delta]\!] \to [\![\sigma]\!]$ $t, u \in PCF_{\Delta,\sigma}$

And $\Gamma \vdash_{\Delta,\sigma} C : \tau$

$$[\![C[t]]\!] = [\![C]\!]([\![t]\!])$$
$$= [\![C]\!]([\![u]\!])$$
$$= [\![C[u]]\!]$$

Assume

$$\Gamma \vdash u : \sigma$$
$$\Gamma, x{:}\sigma \vdash t : \tau$$

$$\Gamma \vdash t[u/x] : \tau$$

Then for all $\rho \in [\![\Gamma]\!]$

$$[\![t[u/x]]\!]\,(\rho) = [\![t]\!]\,(\rho[x \mapsto [\![u]\!]\,(\rho)]).$$

In particular when $\Gamma = \cdot$, $[\![t]\!] : [\![\sigma]\!] \to [\![\tau]\!]$ and

$$[\![t[u/x]]\!] = [\![t]\!]\,([\![u]\!])$$

$$[\![t]\!] : [\![x : \sigma]\!] \to [\![\tau]\!]$$
$$u \qquad [\![\sigma]\!]$$

Denotational Semantics for PCF

Soundness

Soundness

For all PCF types τ and all closed terms $t, v \in \mathrm{PCF}_\tau$ with v a value, if $t \Downarrow_\tau v$ is derivable, then

$$[\![t]\!] = [\![v]\!] \in [\![\tau]\!]$$

By rule induction on $t \Downarrow_\tau v$

Succ:

$$\frac{t \Downarrow_{nat} v}{succ(t) \Downarrow_{nat} succ(v)}$$

IH: $[\![t]\!] = [\![v]\!] \in [\![nat]\!] = \mathbb{N}_\bot$

$[\![succ(t)]\!] = succ_\bot([\![t]\!]) = succ_\bot([\![v]\!]) = [\![succ(v)]\!]$

Fun: $$\frac{t \Downarrow_{\sigma\to\tau} \text{fun}\, x:\sigma.t' \qquad t'[u/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v}$$

IH: $[\![t]\!] = [\![\text{fun}\, x:\sigma.t']\!] = \lambda d.\, [\![t']\!](\cdot[x\mapsto d])$

$[\![t']\!](d)$

$$[\![t'[u/x]]\!] = [\![v]\!]$$

$$\begin{aligned}
[\![t\ u]\!] &= [\![t]\!]([\![u]\!]) \\
&= (\lambda d.\, [\![t']\!](d))([\![u]\!]) \\
&= [\![t']\!]([\![u]\!]) \\
&= [\![t'[u/x]]\!] = [\![v]\!]
\end{aligned}$$

Fix:

$$\frac{t\,(\mathit{fix}\,t) \Downarrow_c v}{\mathit{fix}\,t \Downarrow_c v}$$

I.H.: $[\![t\,(\mathit{fix}\,t)]\!] = [\![v]\!]$

$$
\begin{aligned}
[\![\mathit{fix}\,t]\!] &= \mathit{fix}\,([\![t]\!]) \\
&= [\![t]\!]\,(\mathit{fix}\,[\![t]\!]) \\
&= [\![t\,(\mathit{fix}\,t)]\!] \\
&= [\![v]\!]
\end{aligned}
$$

Relating Denotational and Operational Semantics

For any **closed** PCF term t and value v of **ground** type $\gamma \in \{\texttt{nat}, \texttt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_\gamma v$$

$\underline{n}$

$true = [\![\underline{true}]\!]$

$[\![t]\!] = n \Rightarrow t \Downarrow_{nat} \underline{n}$

$n \in \mathbb{N}$

Reminder: Adequacy

For any **closed** PCF term t and value v of **ground** type $\gamma \in \{\texttt{nat}, \texttt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_\gamma v$$

Adequacy does **not** hold at function types or for open terms

Reminder: Adequacy

For any **closed** PCF term t and value v of **ground** type $\gamma \in \{\texttt{nat}, \texttt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_\gamma v$$

Adequacy does **not** hold at function types or for open terms

$$[\![\texttt{fun}\, x{:}\,\tau.\,(\texttt{fun}\, y{:}\,\tau.\, y)\, x]\!] \quad = \quad [\![\texttt{fun}\, x{:}\,\tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\texttt{fun}\, x{:}\,\tau.\,(\texttt{fun}\, y{:}\,\tau.\, y)\, x \not\Downarrow_{\tau\to\tau} \texttt{fun}\, x{:}\,\tau.\, x$$

Relating Denotational and Operational Semantics

Formal approximation relation

How to prove adequacy

$R : [\![\tau]\!] \to \mathrm{PCF}_\tau \to \mathrm{Prop}$

Proof idea: introduce a relation R such that

1. if $t \in \mathrm{PCF}_{\mathtt{nat}}$, $n \in \mathbb{N}$, and $R(n, t)$, then $t \Downarrow_\gamma \underline{n}$ (same for booleans);
2. for any well-typed term t, $R([\![t]\!], t)$;

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2. for any well-typed term t, $R(\llbracket t \rrbracket, t)$;

Assume $t, v \in \mathrm{PCF}_{\mathtt{nat}}$, $\llbracket t \rrbracket = \llbracket v \rrbracket$, and v is a value.

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Assume $t, v \in \mathrm{PCF}_{\mathtt{nat}}$, $\llbracket t \rrbracket = \llbracket v \rrbracket$, and v is a value.

Thus $v = \underline{n}$ for some $n \in \mathbb{N}$, and $\llbracket v \rrbracket = n$.

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Thus $v = \underline{n}$ for some $n \in \mathbb{N}$, and $\llbracket v \rrbracket = n$.

$$
\begin{aligned}
\llbracket t \rrbracket = \left\llbracket \underline{n} \right\rrbracket &= n \\
&\Rightarrow R(n, t) \\
&\Rightarrow t \Downarrow \underline{n} = v
\end{aligned}
$$