Where we're at

- We know how to construct fixed points using domains and continuous functions (Kleene's fixed point theorem).

Where we're at

- We know how to construct fixed points using domains and continuous functions (Kleene's fixed point theorem).

- We know how to construct interesting domains and continuous functions.

Where we're at

- We know how to construct fixed points using domains and continuous functions (Kleene's fixed point theorem).
- We know how to construct interesting domains and continuous functions.
- Missing: how to reason on fixed points?

Scott Induction

Let D be a domain, $f : D \to D$ be a continuous function and $S \subseteq D$ be a subset of D. If the set S

(i) contains $\perp$,

(ii) is stable under f, *i.e.* $f(S) \subseteq S$,

(iii) is chain-closed, *i.e.* the lub of any chain of elements of S is also in S,

then $\mathrm{fix}(f) \in S$.

$\perp \in S$ i)

$f(\perp) \in S$ ii)

$f^2(\perp) \in S$ ii) ...

$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \dots \in S$

$\mathrm{fix}(f) = \bigsqcup_n f^n(\perp)$

Let D be a domain, $f\colon D \to D$ be a continuous function and $S \subseteq D$ be a subset of D. If the set S

(i) contains $\bot$,

(ii) is stable under f, *i.e.* $f(S) \subseteq S$,

(iii) is chain-closed, *i.e.* the lub of any chain of elements of S is also in S,

then $\mathrm{fix}(f) \in S$.

$S = \{x \in D \mid \Phi(x)\}$

$$\textsc{ScottInd}\ \frac{\Phi(\bot) \qquad \Phi(x) \Rightarrow \Phi(f(x)) \qquad (\forall i \in \mathbb{N}.\ \Phi(x_i)) \Rightarrow \Phi(\bigsqcup_{i\in\mathbb{N}} x_i)}{\Phi(\mathrm{fix}(f))}$$

All the following are chain-closed:

All the following are chain-closed:

$$\{(x,y) \in D \times D \mid x \sqsubseteq y\}\ , \qquad d\downarrow \stackrel{\text{def}}{=} \{x \in D \mid x \sqsubseteq d\} \quad \text{and} \quad \{(x,y) \in D \times D \mid x = y\}$$

$x_0 \sqsubseteq y_0$

$x_1 \sqsubseteq y_1$

$\vdots$

$\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n$

$\bigsqcup_n (x_n, y_n) = (\bigsqcup_n x_n, \bigsqcup_n y_n) \in \ldots$

All the following are chain-closed:

$$\{(x,y) \in D \times D \mid x \sqsubseteq y\}\ , \qquad d\downarrow \stackrel{\text{def}}{=} \{x \in D \mid x \sqsubseteq d\} \qquad \text{and} \qquad \{(x,y) \in D \times D \mid x = y\}$$

$f^{-1}S = \{x \in D \mid f(x) \in S\}$ if $S \subseteq E$ is chain-closed, and $f : D \to E$ is continuous

All the following are chain-closed:

$$\{(x,y) \in D{\times}D \mid x \sqsubseteq y\}\ , \qquad d\downarrow \stackrel{\text{def}}{=} \{x \in D \mid x \sqsubseteq d\} \qquad \text{and} \qquad \{(x,y) \in D{\times}D \mid x = y\}$$

$$f^{-1}S = \{x \in D \mid f(x) \in S\} \qquad \text{if } S \subseteq E \text{ is chain-closed, and } f : D \to E \text{ is continuous}$$

$$S \cup T \qquad \text{and} \qquad \bigcap_{i \in I} S_i \qquad \text{if } S, T \text{ and } S_i \text{ are}$$

$x_0 \leq x_n \ldots \bigcap_{i \in I} S_i$

$x_0 \leq x_1 \ldots \in S_i \; \forall i$

$\bigcup_n x_n \subset S_i \; \forall i$

$\bigcup_n x_n \in \bigcap_{i \in I} S_i$

$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \ldots \in S \cup T$

An infinite number of x_is in one of S or T, assume it's S

$x_{\varphi(0)} \sqsubseteq x_{\varphi(1)} \sqsubseteq \ldots \in S$ $\quad \varphi : \mathbb{N} \to \mathbb{N}$

$\underbrace{\bigsqcup_n x_{\varphi(n)}}_{\in S} = \bigsqcup_n x_n$

$$\dfrac{\dfrac{\text{chain}}{x_m \sqsubseteq x_{\varphi(m)} \sqsubseteq \bigsqcup_n x_{\varphi(n)}}\ \text{Bound}}{\dfrac{\forall m,\ x_m \sqsubseteq \bigsqcup_n x_{\varphi(n)}}{\bigsqcup_m x_m \sqsubseteq \bigsqcup_n x_{\varphi(n)}}\ \text{Least}}$$

$$\dfrac{\dfrac{\dfrac{\text{Bound}}{\forall n.\ x_{\varphi(n)} \sqsubseteq \bigsqcup_m x_m}}{\bigsqcup_n x_{\varphi(n)} \sqsubseteq \bigsqcup_m x_m}\ \text{Least} \qquad \bigsqcup_m x_m \sqsubseteq \bigsqcup_n x_{\varphi(n)}}{\bigsqcup_n x_{\varphi(n)} = \bigsqcup_n x_n}\ \text{ASym}$$

Building chain-closed sets

All the following are chain-closed:

$$\{(x,y) \in D{\times}D \mid x \sqsubseteq y\}\ , \qquad d\downarrow \stackrel{\text{def}}{=} \{x \in D \mid x \sqsubseteq d\} \qquad \text{and} \qquad \{(x,y) \in D{\times}D \mid x = y\}$$

$$f^{-1}S = \{x \in D \mid f(x) \in S\} \qquad \text{if } S \subseteq E \text{ is chain-closed, and } f : D \to E \text{ is continuous}$$

$$S \cup T \qquad \text{and} \qquad \bigcap_{i \in I} S_i \qquad \text{if } S, T \text{ and } S_i \text{ are}$$

$$\forall S \stackrel{\text{def}}{=} \{y \in E \mid \forall x \in D.\,(x,y) \in S\} \subseteq E \qquad \text{if } S \subseteq D \times E \text{ is}$$

$\Phi(x,y)$ chain-closed $\Rightarrow$ $\forall x\ \Phi(x,y)$ chain-closed

Assume $f(d) \sqsubseteq d$, *i.e.* d is a pre-fixed point of the continuous $f : D \to D$. By Scott induction on $d \downarrow$, $\mathrm{fix}(f) \sqsubseteq d$.

iii) chain-closed

i) $\bot \in d\downarrow \iff \bot \sqsubseteq d$ ✓

ii) $f(d\downarrow) \subseteq d\downarrow \iff (\forall x,\ x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d)$ but $x \sqsubseteq d \Rightarrow f(x) \sqsubseteq f(d)$ in d

by Scott ind $\mathrm{fix} f \in d\downarrow$

Example: downset

Assume $f(d) \sqsubseteq d$, *i.e.* d is a pre-fixed point of the continuous $f : D \to D$. By Scott induction on $d\downarrow$, $\mathrm{fix}(f) \sqsubseteq d$.

Proof!

Let $w_\infty \colon \mathrm{State}_\perp \to \mathrm{State}_\perp$ be the denotation of

$$\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)$$

Recall that $w_\infty = \mathrm{fix}(F)$ where

$[X \mapsto x, Y \mapsto y]$

$$F(w)(x, y) = \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

$$F(w)(\perp) = \perp$$

Let $w_\infty \colon \text{State}_\perp \to \text{State}_\perp$ be the denotation of

$$\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)$$

Recall that $w_\infty = \text{fix}(F)$ where

$$F(w)(x, y) = \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$
$$F(w)(\perp) = \perp$$

Claim:

$$\forall x.\, \forall y \geq 0.\, w_\infty(x, y) \Downarrow \implies \pi_Y\left(w_\infty(x, y)\right) \geq 0$$

Let $w_\infty \colon \text{State}_\perp \to \text{State}_\perp$ be the denotation of

$$\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)$$

Recall that $w_\infty = \text{fix}(F)$ where

$$F(w)(x, y) = \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

$$F(w)(\perp) = \perp$$

$w_\infty(x, y) \neq \perp$

Claim:

$$\phi(w_\infty) := \forall x.\, \forall y \geq 0.\, w_\infty(x, y) \Downarrow \implies \pi_Y\,(w_\infty(x, y)) \geq 0$$

Proof: by Scott induction!

$\phi(w) := \forall x \in \mathbb{Z}\ \forall y \subseteq \mathbb{N}.\ w(x,y)\Downarrow \Rightarrow \pi_x(w(x,y)) \geq 0$

i) $\Phi(\bot)$. $\bot(x,y) = \bot$ vacuously True ✓

ii) $\phi(\alpha) \Rightarrow \phi(F(\alpha))$

$x \in \mathbb{Z}$ $y \in \mathbb{N}$, $F(w)(x,y)\Downarrow$

a) $x \leq 0$ $F(w)(x,y) = (x,y)$ ✓

a) $x > 0$ $F(w)(x,y) = (x \cdot 1, x \cdot y)$ $\pi_y(F(w)(x,y)) = x \cdot y > 0$ ✓

iii) ϕ is chain-closed $\forall$ is always chain-closed

fix $x \in \mathbb{Z}$, $y \in \mathbb{N}$

$w(x,y)\not\Downarrow \Rightarrow \pi_x(w(x,y)) \geq 0$ is chain-closed

$w_0 \sqsubseteq w_1 \sqsubseteq \ldots$

if all $w_i \not\Downarrow$ $\Rightarrow (\bigsqcup_n w_n)(x,y) \not\Downarrow$ ✓

b) $w_i(x,y) \Downarrow F(w_i)(x,y)\big) \geq 0$

$$\Big(\bigsqcup_n w_n\Big)(x,y) = w_i(x,y)$$

$$F\Big(\bigsqcup_n w_n\Big)(x,y) = \Big(\bigsqcup_n F(w_n)\Big)(x,y) = \bigcup_n \big(F(w_n)(x,y)\big) = F(w_i)(x,y)$$

✓

PCF

PCF

Terms and types

Syntax of PCF

Types: $\tau ::= \texttt{nat} \mid \texttt{bool} \mid \tau \to \tau$

Syntax of PCF

Types: $\tau ::= \mathtt{nat} \mid \mathtt{bool} \mid \tau \to \tau$

Terms:

$$
\begin{aligned}
t \quad ::= \quad & \mathtt{0} \mid \mathtt{succ}(t) \mid \mathtt{pred}(t) \mid \\
& \mathtt{true} \mid \mathtt{false} \mid \mathtt{zero?}(t) \mid \mathtt{if}\ t\ \mathtt{then}\ t\ \mathtt{else}\ t \\
& x \mid \mathtt{fun}\ x{:}\tau.\, t \mid t\ t \mid \mathtt{fix}(t)
\end{aligned}
$$

$f : (\sigma_1 \to \sigma_2 \to \sigma_3 \cdots \to \tau)$ $\quad$ $\mathrm{fix}(f)$

let rec $f\ (x_1 : \sigma_1) \ldots (x_n : \sigma_n) : \tau :=$

$f\ x_1 \ldots x_n$

Typing for PCF (I)

$\boxed{\Gamma \vdash t : \tau}$ The term t has type τ in context Γ

$$\textsc{Zero}\ \frac{}{\Gamma \vdash \mathtt{0} : \mathtt{nat}} \qquad \textsc{Succ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{succ}(t) : \mathtt{nat}} \qquad \textsc{Pred}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{pred}(t) : \mathtt{nat}}$$

$\boxed{\Gamma \vdash t : \tau}$ The term t has type τ in context Γ

$$\textsc{Zero}\ \frac{}{\Gamma \vdash \mathtt{0} : \mathtt{nat}} \qquad \textsc{Succ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{succ}(t) : \mathtt{nat}} \qquad \textsc{Pred}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{pred}(t) : \mathtt{nat}}$$

$$\textsc{True}\ \frac{}{\Gamma \vdash \mathtt{true} : \mathtt{bool}} \qquad \textsc{False}\ \frac{}{\Gamma \vdash \mathtt{false} : \mathtt{bool}} \qquad \textsc{IsZ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{zero?}(t) : \mathtt{bool}}$$

$$\textsc{If}\ \frac{\Gamma \vdash b : \mathtt{bool} \qquad \Gamma \vdash t : \tau \qquad \Gamma \vdash t' : \tau}{\Gamma \vdash \mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t' : \tau}$$

$$\textsc{Var}\ \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \textsc{Fun}\ \frac{\Gamma, x{:}\sigma \vdash t : \tau}{\Gamma \vdash \mathtt{fun}\, x{:}\sigma.\, t : \sigma \to \tau} \qquad \textsc{App}\ \frac{\Gamma \vdash f : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash f\, u : \tau}$$

$$\textsc{Fix}\ \frac{\Gamma \vdash f : \tau \to \tau}{\Gamma \vdash \mathtt{fix}(f) : \tau}$$

$$\textsc{Var}\ \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \textsc{Fun}\ \frac{\Gamma, x{:}\sigma \vdash t : \tau}{\Gamma \vdash \mathsf{fun}\, x{:}\sigma.\, t : \sigma \to \tau} \qquad \textsc{App}\ \frac{\Gamma \vdash f : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash f\, u : \tau}$$

$$\textsc{Fix}\ \frac{\Gamma \vdash f : \tau \to \tau}{\Gamma \vdash \mathtt{fix}(f) : \tau}$$

$$\mathrm{PCF}_{\Gamma,\tau} \stackrel{\text{def}}{=} \{t \mid \Gamma \vdash t : \tau\} \qquad \mathrm{PCF}_{\tau} \stackrel{\text{def}}{=} \mathrm{PCF}_{\cdot,\tau}$$

PCF

Operational Semantics

Values: $$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x{:}\,\tau.\, t$$

Values:

$$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x{:}\,\tau.\, t$$

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v}$$

PCF EVALUATION

Values:

$$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x{:}\,\tau.\, t$$

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v} \qquad \textsc{Succ}\ \frac{t \Downarrow_{\mathtt{nat}} v}{\mathtt{succ}(t) \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)} \qquad \textsc{Pred}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)}{\mathtt{pred}(t) \Downarrow_{\mathtt{nat}} v}$$

PCF evaluation

Values:

$$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x{:}\, \tau.\, t$$

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v} \qquad \textsc{Succ}\ \frac{t \Downarrow_{\mathtt{nat}} v}{\mathtt{succ}(t) \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)} \qquad \textsc{Pred}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)}{\mathtt{pred}(t) \Downarrow_{\mathtt{nat}} v}$$

$$\textsc{ZeroZ}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{0}}{\mathtt{zero?}(t) \Downarrow_{\mathtt{bool}} \mathtt{true}} \qquad \dots \qquad \textsc{IfT}\ \frac{b \Downarrow_{\mathtt{bool}} \mathtt{true} \qquad t_1 \Downarrow_\tau v}{\mathtt{if}\ b\ \mathtt{then}\ t_1\ \mathtt{else}\ t_2 \Downarrow_\tau v} \qquad \dots$$

Values:

$$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x{:}\,\tau.\, t$$

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v} \qquad \textsc{Succ}\ \frac{t \Downarrow_{\mathtt{nat}} v}{\mathtt{succ}(t) \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)} \qquad \textsc{Pred}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)}{\mathtt{pred}(t) \Downarrow_{\mathtt{nat}} v}$$

$$\textsc{ZeroZ}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{0}}{\mathtt{zero?}(t) \Downarrow_{\mathtt{bool}} \mathtt{true}} \qquad \ldots \qquad \textsc{IfT}\ \frac{b \Downarrow_{\mathtt{bool}} \mathtt{true} \qquad t_1 \Downarrow_\tau v}{\mathtt{if}\ b\ \mathtt{then}\ t_1\ \mathtt{else}\ t_2 \Downarrow_\tau v} \qquad \ldots$$

$$\textsc{Fun}\ \frac{t \Downarrow_{\sigma \to \tau} \mathtt{fun}\, x{:}\,\sigma.\, t' \qquad t'[u/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v} \qquad \textsc{Fix}\ \frac{t\ (\mathtt{fix}(t)) \Downarrow_\tau v}{\mathtt{fix}(t) \Downarrow_\tau v}$$

Values: $$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x{:}\,\tau.\, t$$

$$\text{VAL}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v} \qquad \text{SUCC}\ \frac{t \Downarrow_{\mathtt{nat}} v}{\mathtt{succ}(t) \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)} \qquad \text{PRED}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)}{\mathtt{pred}(t) \Downarrow_{\mathtt{nat}} v}$$

$$\text{ZEROZ}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{0}}{\mathtt{zero?}(t) \Downarrow_{\mathtt{bool}} \mathtt{true}} \qquad \dots \qquad \text{IFT}\ \frac{b \Downarrow_{\mathtt{bool}} \mathtt{true} \qquad t_1 \Downarrow_\tau v}{\mathtt{if}\ b\ \mathtt{then}\ t_1\ \mathtt{else}\ t_2 \Downarrow_\tau v} \qquad \dots$$

$$\text{FUN}\ \frac{t \Downarrow_{\sigma\to\tau} \mathtt{fun}\, x{:}\,\sigma.\, t' \qquad t'[u/x] \Downarrow_\tau v}{t\, u \Downarrow_\tau v} \qquad \text{FIX}\ \frac{t\, (\mathtt{fix}(t)) \Downarrow_\tau v}{\mathtt{fix}(t) \Downarrow_\tau v}$$

Alternatively: small-step $t \rightsquigarrow_\tau u$, we have $t \Downarrow_\tau v$ iff $t \rightsquigarrow_\tau^\star u$.