# Denotational Semantics
# Exercise Sheet

## § Topics 1–4

**Exercise 1 [2010 Paper 7 Question 8 (*b*)].**

For a partially ordered set $(P, \sqsubseteq)$, let $(\mathrm{Ch}(P), \sqsubseteq_{\mathrm{ptw}})$ be the partially ordered set of chains in $P$ ordered pointwise. That is,

$$\mathrm{Ch}(P) = \{x = \{x_n\}_{n \in \mathbb{N}} \mid \text{for all } i \leq j \text{ in } \mathbb{N},\ x_i \sqsubseteq x_j \text{ in } P\}$$

and

$$x \sqsubseteq_{\mathrm{ptw}} y \iff x_n \sqsubseteq y_n \text{ for all } n \in \mathbb{N}$$

Show that if $P$ is a domain then so is $\mathrm{Ch}(P)$.

**Exercise 2 [2014 Paper 7 Question 6 (*a*)].**

For partially ordered sets $(P, \sqsubseteq_P)$ and $(Q, \sqsubseteq_Q)$, define the set

$$(P \Rightarrow Q) = \{f \mid f \text{ is a monotone function from } (P, \sqsubseteq_P) \text{ to } (Q, \sqsubseteq_Q)\}$$

and, for all $f, g \in (P \Rightarrow Q)$, let

$$f \sqsubseteq_{(P \Rightarrow Q)} g \iff \forall\, p \in P.\ f(p) \sqsubseteq_Q g(p)$$

(*i*) Prove that $((P \Rightarrow Q), \sqsubseteq_{(P \Rightarrow Q)})$ is a partially ordered set.

(*ii*) Prove that if $(Q, \sqsubseteq_Q)$ is a domain then so is $((P \Rightarrow Q), \sqsubseteq_{(P \Rightarrow Q)})$.

**Exercise 3.**

Derive the result of Exercise 1 from that of Exercise 2(*ii*).

**Exercise 4 [2016 Paper 7 Question 7 (*a.ii*)].**

Let $\mathcal{P}(\mathbb{N}^2)$ be the domain of all subsets of pairs of natural numbers ordered by inclusion. Show that the function $f : \mathcal{P}(\mathbb{N}^2) \to \mathcal{P}(\mathbb{N}^2)$ given by

$$f(S) = \{(1,1)\} \cup \{(x+1, x \cdot y) \in \mathbb{N}^2 \mid (x,y) \in S\} \qquad (S \subseteq \mathbb{N}^2)$$

is continuous.

**Exercise 5 [2008 Paper 8 Question 14 (*c, d, e*)].**

(*i*) Let $\mathbb{O}$ be the domain with two elements $\bot \sqsubseteq \top$. For a domain $E$ and $e \in E$, define the function $g_e : E \to \mathbb{O}$ by

$$g_e(x) = \begin{cases} \bot & \text{if } x \sqsubseteq e \\ \top & \text{if } x \not\sqsubseteq e \end{cases}$$

Show that $g_e$ is continuous.

(*ii*) As an example of the definition in part (*i*), let $E = \mathbb{B}_\perp \times \mathbb{B}_\perp$, where $\mathbb{B} = \{true, false\}$, and consider $g_{(false,false)} : E \to \mathbb{O}$. Show that $g_{(false,false)}(x, y) = \top$ iff $x = true$ or $y = true$.

(*iii*) Let $f : D \to E$ be a function between domains $D$ and $E$. Show that $f$ is continuous iff $\forall\, e \in E.\ g_e \circ f$ is continuous.

**Exercise 6 [2011 Paper 7 Question 5 (*a*)].**

Let $\Omega$ be the domain of "vertical natural numbers" pictured in Figure 1 of the lecture notes.

(*i*) Is every monotone function from $\Omega$ to $\Omega$ continuous?

(*ii*) Does every monotone function from $\Omega$ to $\Omega$ have a least prefixed point?

**Exercise 7 [2019 Paper 9 Question 7 (*a*)].**

Suppose that $(D, \sqsubseteq)$ is a poset which is chain-complete but does not have a least element, and that $f : D \to D$ is a continuous function.

(*i*) Give an example of such $(D, \sqsubseteq)$ and $f$ for which $f$ has no fixed point.

(*ii*) If $d \in D$ satisfies $d \sqsubseteq f(d)$, prove that there is a least element $e \in D$ satisfying $d \sqsubseteq e = f(e)$.

**Exercise 8 [2017 Paper 7 Question 7 (*b.ii*)].**

Give a concrete explicit description of the fixed point $\mathit{fix}(f) \subseteq \mathbb{N}^2$ of the continuous function $f$ in Exercise 4. Justify your answer.

**Exercise 9 [2007 Paper 8 Question 15 (*e*)].**

Suppose that $D$ is a domain and $f : D \times D \to D$ is a continuous function satisfying the property $\forall\, d, e \in D.\ f(d, e) = f(e, d)$. Let $g : D \times D \to D \times D$ be defined by

$$g(d_1, d_2) = (\, f(d_1, f(d_1, d_2))\,,\ f(f(d_1, d_2), d_2)\,)$$

Let $(u_1, u_2) = \mathit{fix}(g)$. Show that $u_1 = u_2$ using Scott induction.

**Exercise 10 [2011 Paper 7 Question 5 (*b*)].**

Let $D$ and $E$ be domains and let $f : D \to D$ and $g : E \to E$ be continuous functions.

(*i*) Define $f \times g : D \times E \to D \times E$ to be the continuous function given by $(f \times g)(d, e) = (f(d), g(e))$ and let $\pi_1 : D \times E \to D$ and $\pi_2 : D \times E \to E$ respectively denote the first and second projection functions. Show that $\mathit{fix}(f \times g) \sqsubseteq (\mathit{fix}(f), \mathit{fix}(g))$ and that $\mathit{fix}(f) \sqsubseteq \pi_1(\mathit{fix}(f \times g))$ and $\mathit{fix}(g) \sqsubseteq \pi_2(\mathit{fix}(f \times g))$.

(*ii*) It follows from part (*i*) that $\mathit{fix}(f \times g) = (\mathit{fix}(f), \mathit{fix}(g))$. Use this and Scott's Fixed Point Induction Principle to show that, for all strict continuous functions $h : D \to E$,

$$h \circ f = g \circ h \implies h(\mathit{fix}(f)) = \mathit{fix}(g)$$

## § Topics 5–8

**Exercise 1 [2009 Paper 9 Question 6 (*b.i*)].**
Define a closed PCF term $H : (nat \to nat \to nat) \to nat \to nat \to nat$ such that $[\![\mathbf{fix}(H)]\!] \in (\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp))$ satisfies

$$[\![\mathbf{fix}(H)]\!](i)(j) = \max(i, j)$$

for all $i, j \in \mathbb{N}$.

**Exercise 2 [2017 Paper 9 Question 5 (*c.i*)].**
For every pair of closed PCF expressions $M, N$ of type $nat$, let $F_{M,N}$ be the closed PCF expression of type $(nat \to nat) \to (nat \to nat)$ given by

$$\begin{array}{l} \mathbf{fn}\ f : nat \to nat.\ \mathbf{fn}\ n : nat. \\ \quad \mathbf{if}\ \mathbf{zero}(n)\ \mathbf{then}\ M \\ \quad \mathbf{else}\ \mathbf{if}\ \mathbf{zero}(\mathbf{pred}(n))\ \mathbf{then}\ N \\ \qquad \mathbf{else}\ \mathbf{succ}(f(\mathbf{pred}(n))) \end{array}$$

Give an explicit description of $[\![\mathbf{fix}(F_{M,N})]\!] \in (\mathbb{N}_\perp \to \mathbb{N}_\perp)$ in terms of $[\![M]\!], [\![N]\!] \in \mathbb{N}_\perp$. Justify your answer.

**Exercise 3 [2019 Paper 9 Question 7 (*b*)].**

(*i*) Define the notion of *contextual equivalence* for the language PCF.

(*ii*) State the *compositionality*, *soundness*, and *adequacy* properties of the denotational semantics of PCF. Explain why they imply that any two closed PCF terms of the same type with equal denotations are contextually equivalent.

(*iii*) Give an example of two contextually equivalent PCF terms that have unequal denotations.

**Exercise 4 [2017 Paper 9 Question 5 (*b.ii*)].**
Prove or disprove that

$$(\mathbf{fn}\ n : nat.\ n) \cong_{\mathrm{ctx}} (\ \mathbf{fn}\ n : nat.\ \mathbf{succ}(\mathbf{pred}(n))\ ) : nat \to nat$$

**Exercise 5 [2017 Paper 9 Question 5 (*c.ii*)].**
For $M, N$ and $F_{M,N}$ as in Exercise 2, prove or disprove that there are closed PCF expressions $M, N$ of type $nat$ such that

$$\mathbf{fix}(F_{M,N}) \cong_{\mathrm{ctx}} (\mathbf{fn}\ n : nat.\ \mathbf{pred}(n)) : nat \to nat$$

You may use any standard results provided that you state them clearly.

**Exercise 6 [2011 Paper 9 Question 3 (*b*)].**
Consider the following two statements for PCF terms $M_1$ and $M_2$ for which the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold for some type environment $\Gamma$ and type $\tau$.

(1) For all PCF contexts $\mathcal{C}[-]$ for which $\mathcal{C}[M_1] : bool$ and $\mathcal{C}[M_2] : bool$,

$$\mathcal{C}[M_1] \Downarrow_{bool} \iff \mathcal{C}[M_2] \Downarrow_{bool}$$

(2) For all PCF contexts $\mathcal{C}[-]$ for which $\mathcal{C}[M_1] : bool$ and $\mathcal{C}[M_2] : bool$,

$$\mathcal{C}[M_1] \Downarrow_{bool} \mathbf{true} \iff \mathcal{C}[M_2] \Downarrow_{bool} \mathbf{true}$$

(*i*) Show that (1) implies (2).

(*ii*) Show that (2) implies that $M_1$ and $M_2$ are contextually equivalent.

**Exercise 7 [2010 Paper 9 Question 5 (*c*)].**
Is the following statement true or false, and why?

For all closed PCF-terms $M_1$ and $M_2$ of type $nat \to nat$,
if $M_1 \cong_{\mathrm{ctx}} M_2 : nat \to nat$ then $[\![M_1]\!] = [\![M_2]\!]$ in $\mathbb{N}_\bot \to \mathbb{N}_\bot$.

**Exercise 8 [2015 Paper 9 Question 4 (*c*)].**
Let $M$ be the PCF+por term

$$\begin{array}{l}\mathbf{fn}\ f : (nat \to bool) \to bool.\\ \quad \mathbf{fn}\ P : nat \to bool.\\ \quad\quad \mathbf{por}(P(\mathbf{0}), f(\mathbf{fn}\ n : nat.\ P(\mathbf{succ}(n))))\end{array}$$

Give an explicit description of $[\![\mathbf{fix}(M)]\!] \in ((\mathbb{N}_\bot \to \mathbb{B}_\bot) \to \mathbb{B}_\bot)$.

**Exercise 9 [2021 Paper 9 Question 7].**
Say whether the following statements are true or false with justification. You may use standard results provided that you state them clearly.

(*i*) For all PCF types $\tau$ and terms $M \in \mathrm{PCF}_\tau$, if $[\![M]\!] = \bot_{[\![\tau]\!]}$ then $M \cong_{\mathrm{ctx}} \Omega_\tau : \tau$.

(*ii*) For all PCF types $\tau$ and terms $M \in \mathrm{PCF}_\tau$, if $[\![M]\!] = \bot_{[\![\tau]\!]}$ then $M \not\Downarrow_\tau$.

(*iii*) For all PCF types $\tau$ and terms $M \in \mathrm{PCF}_\tau$, if $M \cong_{\mathrm{ctx}} \Omega_\tau : \tau$ then $M \not\Downarrow_\tau$.

(*iv*) For all PCF types $\tau$ and terms $M \in \mathrm{PCF}_\tau$, if $M \not\Downarrow_\tau$ then $M \cong_{\mathrm{ctx}} \Omega_\tau : \tau$.

(*v*) For all PCF types $\tau$ and terms $M \in \mathrm{PCF}_\tau$, if $M \cong_{\mathrm{ctx}} \Omega_\tau : \tau$ then $[\![M]\!] = \bot_{[\![\tau]\!]}$.

**Exercise 10 [1998 Paper 7 Question 5].**
Suppose that $lam : (D \to D) \to D$ and $app : D \to (D \to D)$ are continuous functions for a domain $D$. Use this data to give a denotational semantics for the terms of the untyped $\lambda$-calculus by answering Question 5 from Paper 7 of the 1998 CS Tripos.