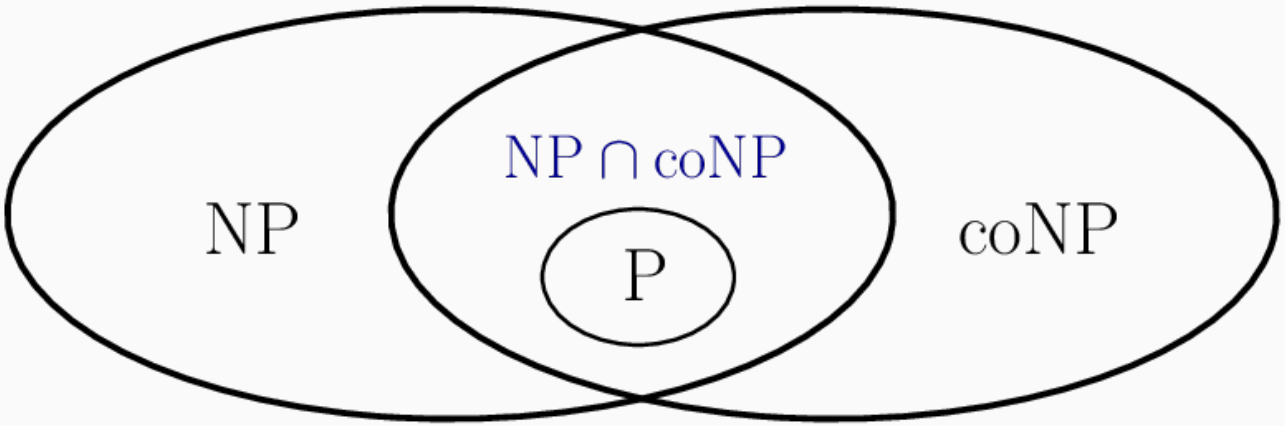
Complexity Theory

Lecture 9: Cryptography

Tom Gur

http://www.cl.cam.ac.uk/teaching/2324/Complexity

Recap



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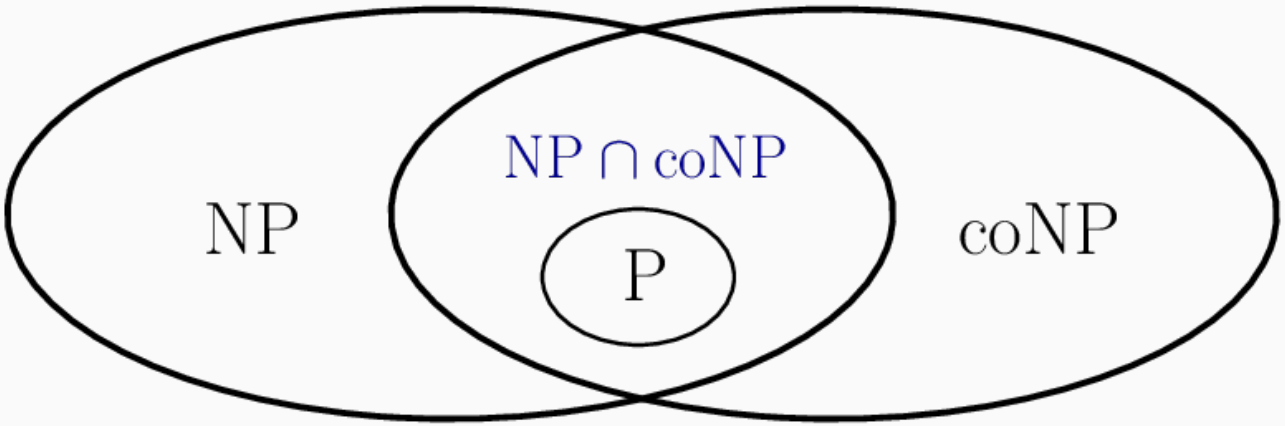

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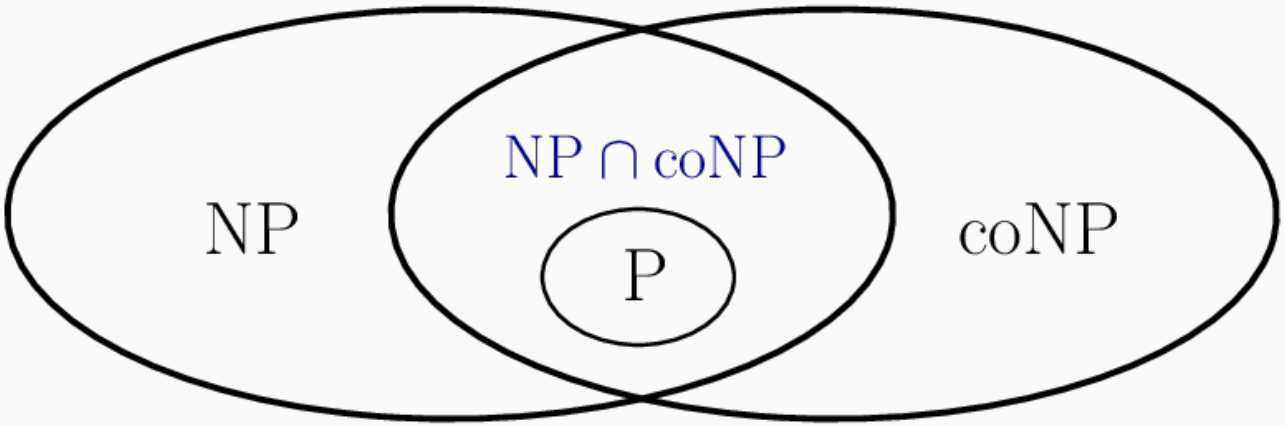

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Cliffhanger: Why can't we break RSA using unary encodings?

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Factor $\in$ BQP

Graph Isomorphism

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is there a *bijection*

$$\iota : V_1 \to V_2$$

such that for every $u, v \in V_1$,

$$(u, v) \in E_1 \quad \text{if, and only if,} \quad (\iota(u), \iota(v)) \in E_2.$$

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- not known (or *expected*) to be NP-complete
- shown to be in *quasi-polynomial time*, i.e. in

$$\mathrm{TIME}(n^{(\log n)^k})$$

for a constant k.

Cryptography

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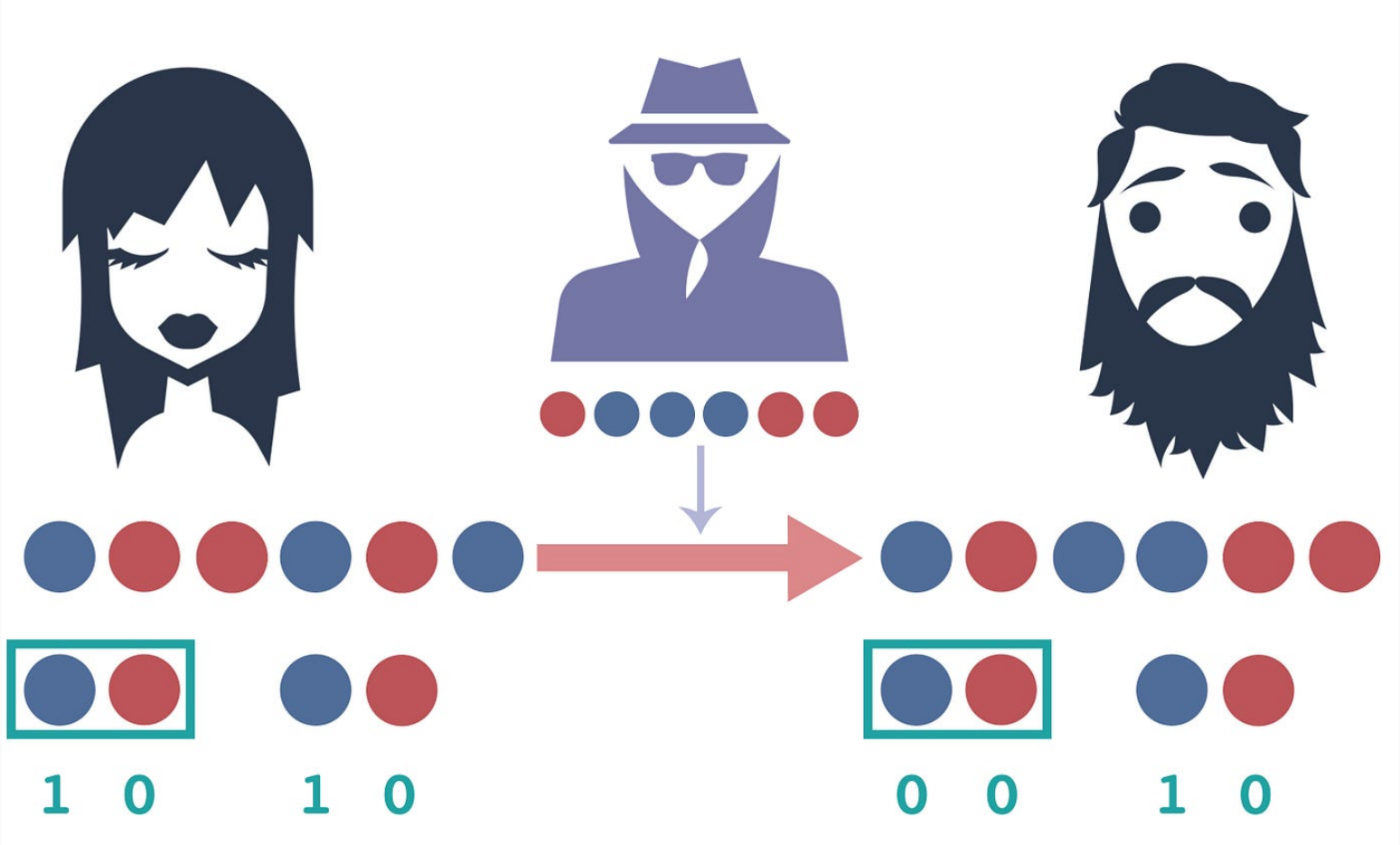

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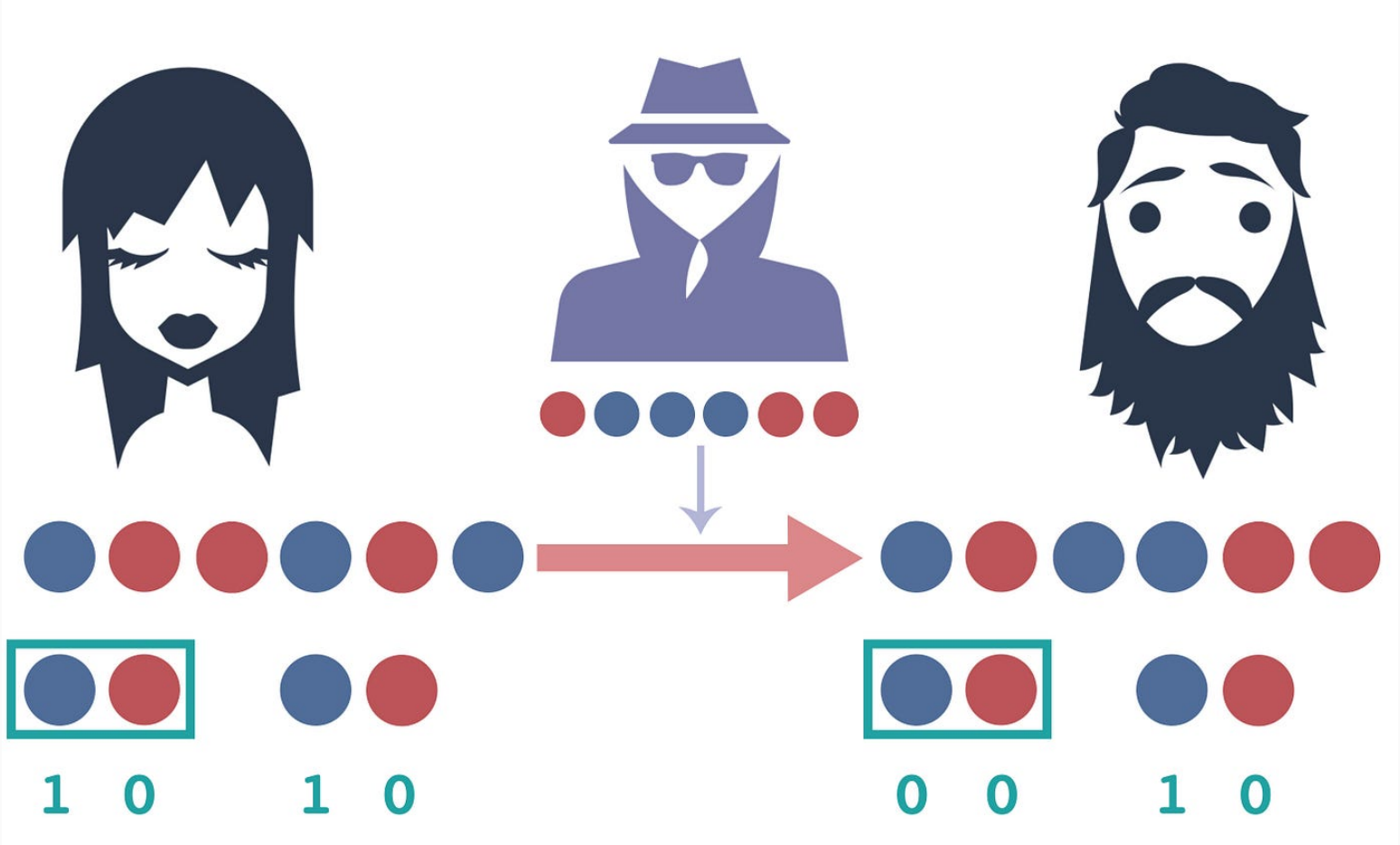

Alice wishes to communicate with Bob without Eve eavesdropping.

Private Key

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e – the encryption key

d – the decryption key

and two functions D and E such that:

for any x,

$$D(E(x, e), d) = x.$$

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For instance, taking $d = e$ and both D and E as *exclusive or*, we have the *one time pad*:

$$(x \oplus e) \oplus e = x$$

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Declassified files reveal how pre-WW2 Brits smashed Russian crypto

Moscow's agents used one-time pads, er, two times – ой!

John Leyden Thu 19 Jul 2018 | 18:35 UTC

Efforts by British boffins to thwart Russian cryptographic cyphers in the 1920s and 1930s have been declassified, providing fascinating insights into an obscure part of the history of code breaking.

America's National Security Agency this week released papers from John Tiltman, one of Britain's top cryptanalysts during the Second World War, describing his work in breaking Russian codes [PDF], in response to a Freedom of Information Act request.

The Russians started using one-time pads in 1928 – however, they made the grave cryptographic error of allowing these pads to be used twice, the release of Tiltman's papers has revealed for the first time.

By reusing one-time pads, Russian agents accidentally leaked enough information for eavesdroppers in Blighty to figure out the encrypted missives' plaintext. Two separate messages encrypted reusing the same key from a pad could be compared to ascertain the differences between their unencrypted forms, and from there eggheads could, using stats and knowledge of the language, work out the original words.

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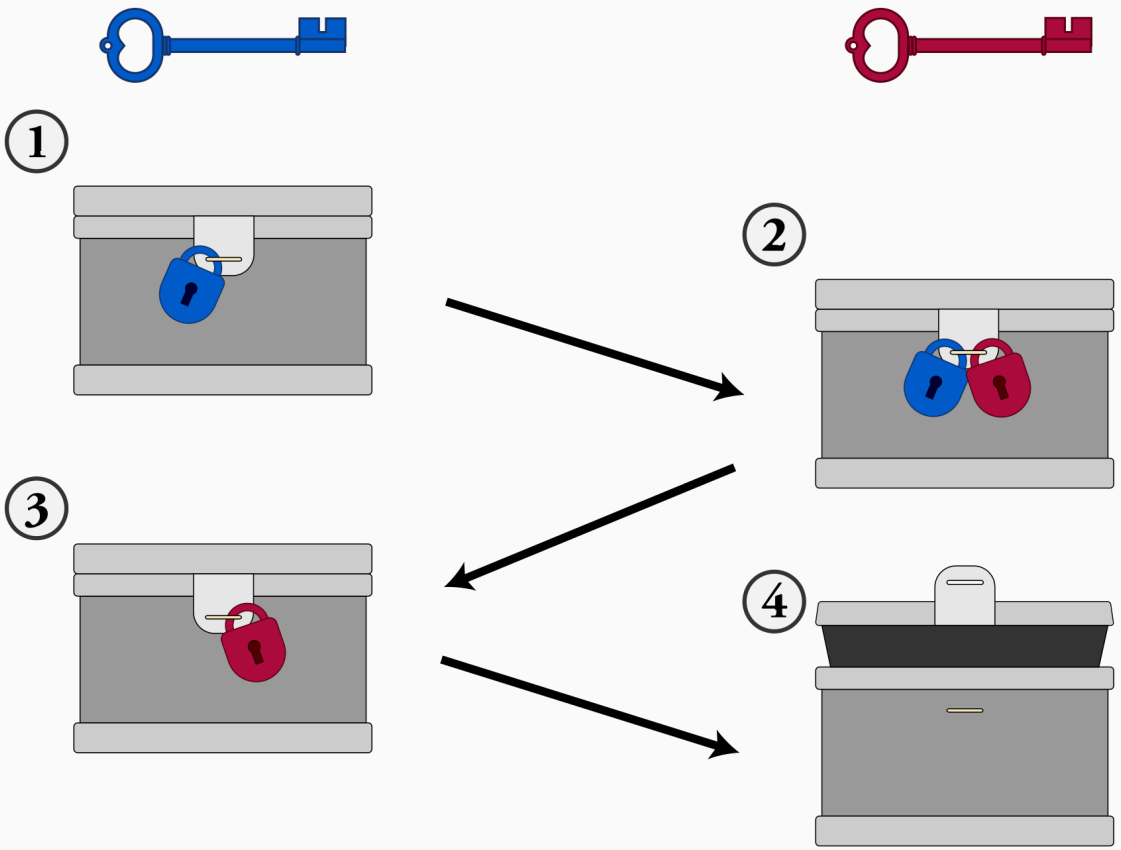

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Thus, public key cryptography is not *provably secure* in the way that the one time pad is. It relies on the assumption that $\mathrm{P} \neq \mathrm{NP}$.

Abstracting chests and locks with computational hardness

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It is strongly believed that the RSA function:

$$f(x, e, p, q) = (x^e \bmod pq, pq, e)$$

is a one-way function.

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UP is the class of languages accepted by unambiguous machines in polynomial time.

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Equivalently, UP is the class of languages of the form

$$\{x \mid \exists y R(x, y)\}$$

Where R is polynomial time computable, polynomially balanced, *and* for each x, there is *at most one* y such that $R(x, y)$.

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One-way functions exist *if, and only if,* $\mathsf{P} \neq \mathsf{UP}$.

Bonus: randomisation and BPP