University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 5
by Andrew Pitts

1. For each set Σ, let $\mathrm{Pow}\,\Sigma$ be the set $\{S \mid S \subseteq \Sigma\}$ of all subsets of Σ. The operation of taking the union of two subsets and the empty subset together give the structure of a monoid on $\mathrm{Pow}\,\Sigma$. Writing $P\,\Sigma = (\mathrm{Pow}\,\Sigma, \_ \cup \_, \emptyset)$ for this monoid, show that $\Sigma \mapsto P\,\Sigma$ is the object part of a functor $P : \mathbf{Set} \to \mathbf{Mon}$.

2. Let $F : \mathbf{Set} \to \mathbf{Mon}$ be the free monoid functor, given by finite lists and mapping over finite lists (Lecture 10). Show that the operation that sends each finite list to the finite set of elements that it contains gives a natural transformation $\theta : F \to P$, where P is the functor from question 1.

3. Given small categories $\mathbf{C}, \mathbf{D}$, functors $F, G : \mathbf{C} \to \mathbf{D}$ and a natural transformation $\theta : F \to G$, show that θ is an isomorphism in the functor category $\mathbf{D}^{\mathbf{C}}$ if and only if for all $X \in \mathrm{obj}\,\mathbf{C}$, the morphism $\theta_X \in \mathbf{D}(F\,X, G\,X)$ is an isomorphism in the category $\mathbf{D}$.

4. In this question you will prove that there is no *natural* way to choose an element from an arbitrary non-empty set. Let $P^+ : \mathbf{Set} \to \mathbf{Set}$ be the functor assigning to each set X its set $P^+X \triangleq \{S \subseteq X \mid S \neq \emptyset\}$ of non-empty subsets; the action of P^+ on morphisms in $\mathbf{Set}$ sends each function $f : X \to Y$ to the function $P^+f : P^+X \to P^+Y$, where for all $S \in P^+X$

$$(P^+f)S \triangleq \{f\,x \mid x \in S\}$$

(This does indeed make P^+ into a functor.)

Suppose that for each set X we are given a function $ch_X : P^+X \to X$ with the property that $ch_X(S) \in S$ for all $S \in P^+X$. Show that these functions cannot be the components of a natural transformation $P^+ \to \mathrm{Id}_{\mathbf{Set}}$ from P^+ to the identity functor on the category $\mathbf{Set}$ of sets and functions. [Hint: consider the naturality condition for the function $\tau : \{0, 1\} \to \{0, 1\}$ with $\tau(0) = 1$ and $\tau(1) = 0$.]

5. Suppose we are given categories $\mathbf{C}, \mathbf{D}, \mathbf{E}$, functors $F, G, H : \mathbf{C} \to \mathbf{D}$ and $I, J, K : \mathbf{D} \to \mathbf{E}$, and natural transformations

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H \quad \text{and} \quad I \xrightarrow{\gamma} J \xrightarrow{\delta} K.$$

   (a) Using α and I, define a natural transformation $I\,\alpha : I \circ F \to I \circ G$.

   (b) Using F and γ, define a natural transformation $\gamma_F : I \circ F \to J \circ F$.

   (c) Using α and β, define a natural transformation $\beta \circ \alpha : F \to H$.

   (d) Using α and γ, define a natural transformation $\gamma * \alpha : I \circ F \to J \circ G$.

   (e) Show that the operations you defined in questions (5c) and (5d) satisfy: $(\delta * \beta) \circ (\gamma * \alpha) = (\delta \circ \gamma) * (\beta \circ \alpha)$. (This is called the *Interchange Law* for vertical and horizontal composition of natural transformations.)

6. Let $\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{G} \mathbf{C}$ and $(\theta_{X,Y} : \mathbf{D}(F\,X, Y) \cong \mathbf{C}((X, G\,Y) \mid X \in \mathrm{obj}\,\mathbf{C}, Y \in \mathrm{obj}\,\mathbf{D})$ be an adjunction between categories $\mathbf{C}$ and $\mathbf{D}$.

(a) Use θ to define natural transformations $\eta : \mathrm{Id}_{\mathbf{C}} \longrightarrow G \circ F$ and $\varepsilon : F \circ G \longrightarrow \mathrm{Id}_D$. (These are called respectively the *unit* and *counit* of the adjunction.)

(b) Prove that the natural transformations defined in part (6a) satisfy $\varepsilon_F \circ F\eta = \mathrm{id}_F$ and $G\varepsilon \circ \eta_G = \mathrm{id}_G$

$$
\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & F \circ G \circ F \arrow[d, "\varepsilon_F"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\eta_G"] \arrow[dr, "\mathrm{id}_G"'] & G \circ F \circ G \arrow[d, "G\varepsilon"] \\
& G
\end{tikzcd}
\tag{1}
$$

where we are using notation as in questions (5a) and (5b). (These are called the *triangular identities* for the unit and counit of the adjunction.)

7. Given functors $\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{G} \mathbf{C}$ and natural transformations $\eta : \mathrm{Id}_{\mathbf{C}} \longrightarrow G \circ F$ and $\varepsilon : F \circ G \longrightarrow \mathrm{Id}_D$ satisfying the triangular identities (1), show that F is left adjoint to G.

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Question 1 For each function $f \in \mathbf{Set}(\Sigma, \Sigma')$ there is a function $\mathrm{Pow}\, f \in \mathbf{Set}(\mathrm{Pow}\,\Sigma, \mathrm{Pow}\,\Sigma')$ defined by

$$(\mathrm{Pow}\, f)\, S \triangleq \{f\, x \mid x \in S\} \tag{2}$$

Note that this is a monoid homomorphism $(\mathrm{Pow}\,\Sigma, \cup, \emptyset) \to (\mathrm{Pow}\,\Sigma', \cup, \emptyset)$, because

- $\mathrm{Pow}\, f\, \emptyset = \{f\, x \mid x \in \emptyset\} = \emptyset$
- $\mathrm{Pow}\, f\, (S \cup S') = \{f\, x \mid x \in S \cup S'\} = \{f\, x \mid x \in S \;\vee\; x \in S'\} = \{f\, x \mid x \in S\} \cup \{f\, x \mid x \in S'\} = (\mathrm{Pow}\, f\, S) \cup (\mathrm{Pow}\, f\, S')$.

So we get $P\, f \in \mathbf{Mon}((\mathrm{Pow}\,\Sigma, \cup, \emptyset), (\mathrm{Pow}\,\Sigma', \cup, \emptyset))$. Furthermore

- $(\mathrm{Pow}\,\mathrm{id}_\Sigma)\, S = \{\mathrm{id}_\Sigma x \mid x \in S\} = S$ for all $S \in \mathrm{Pow}\,\Sigma$; and hence $P\,\mathrm{id}_\Sigma = \mathrm{id}_{P\,\Sigma}$.
- $\mathrm{Pow}\, g(\mathrm{Pow}\, f\, S) = \{g\, y \mid y \in \mathrm{Pow}\, f\, S\} = \{g(f\, x) \mid x \in S\} = \{(g \circ f)x \mid x \in S\} = \mathrm{Pow}(g \circ f)\, S$ for all $S \in \mathrm{Pow}\,\Sigma$; and hence $P(g \circ f) = (P\, g) \circ (P\, f)$.

So P is a functor $\mathbf{Set} \to \mathbf{Mon}$.

Question 2 For each set Σ, define $\theta_\Sigma \in \mathbf{Set}(\mathrm{List}\,\Sigma, \mathrm{Pow}\,\Sigma)$ by recursion on the length of lists:

$$\begin{aligned}\theta_\Sigma(\mathrm{nil}) &= \emptyset\\ \theta_\Sigma(a :: \ell) &= \{a\} \cup \theta_\Sigma(\ell)\end{aligned}$$

Then one can prove $(\forall \ell, \ell' \in \mathrm{List}\,\Sigma)\; \theta_\Sigma(\ell \mathbin{@} \ell') = \theta_\Sigma(\ell) \cup \theta_\Sigma(\ell')$ by induction on the length of ℓ. So we get that θ_Σ is in $\mathbf{Mon}(F\Sigma, P\Sigma)$. To show that these morphisms form a natural transformation $\theta : F \to P$, we have to show for each $f \in \mathbf{Set}(\Sigma, \Sigma')$ that $P\, f \circ \theta_\Sigma = \theta_{\Sigma'} \circ F\, f$; and by definition of F and P, this means proving $(\forall \ell \in \mathrm{List}\,\Sigma)\;\; \mathrm{Pow}\, f(\theta_\Sigma(\ell)) = \theta_{\Sigma'}(\mathrm{List}\, f\, \ell)$, which follows easily from the definitions of $\mathrm{Pow}\, f$, $\mathrm{List}\, f$ and θ_Σ, by induction on the length of ℓ.

Here is another proof, which uses the universal property of the free monoid $F\Sigma$ instead of recursion/induction on lists.

For each set Σ, let $s_\Sigma \in \mathbf{Set}(\Sigma, \mathrm{Pow}\,\Sigma)$ be the function mapping each $x \in \Sigma$ to $s_\Sigma(x) \triangleq \{x\} \in \mathrm{Pow}\,\Sigma$. Using the universal property of the free monoid $i_\Sigma : \Sigma \to \mathrm{List}\,\Sigma$, there is a unique monoid homomorphism $\widehat{s_\Sigma} \in \mathbf{Mon}(F\,\Sigma, P\,\Sigma)$ with $\widehat{s_\Sigma} \circ i_\Sigma = s_\Sigma$. We take θ_Σ to be $\widehat{s_\Sigma}$ and show that these functions together give a natural transformation $\theta : F \to P$.

So we have to show for each $f \in \mathbf{Set}(\Sigma, \Sigma')$ that $\theta_{\Sigma'} \circ F\, f = P\, f \circ \theta_\Sigma \in \mathbf{Mon}(F\Sigma, P\Sigma')$. By the uniqueness part of the universal property of the free monoid $i_\Sigma : \Sigma' \to \mathrm{List}\,\Sigma$, for this it suffices to show that the two monoid homomorphisms $\theta_{\Sigma'} \circ F\, f$ and $P\, f \circ \theta_\Sigma$, when composed with the function i_Σ, give equal functions in $\mathbf{Set}(\Sigma, \mathrm{Pow}\,\Sigma')$. But

$$\begin{aligned}(P\, f \circ \theta_\Sigma) \circ i_\Sigma \triangleq ((\mathrm{Pow}\, f) \circ \widehat{s_\Sigma}) \circ i_\Sigma &= (\mathrm{Pow}\, f) \circ (\widehat{s_\Sigma} \circ i_\Sigma)\\ &= (\mathrm{Pow}\, f) \circ s_\Sigma \qquad \text{by definition of } \widehat{s_\Sigma}\end{aligned}$$

whereas

$$
\begin{aligned}
(\theta_{\Sigma'} \circ F\,f) \circ i_\Sigma \triangleq (\widehat{s_{\Sigma'}} \circ F\,f) \circ i_\Sigma &= \widehat{s_{\Sigma'}} \circ (F\,f \circ i_\Sigma) \\
&= \widehat{s_{\Sigma'}} \circ (i_{\Sigma'} \circ f) && \text{since } i \text{ is a natural transformation} \\
&= (\widehat{s_{\Sigma'}} \circ i_{\Sigma'}) \circ f \\
&= s_{\Sigma'} \circ f && \text{by definition of } \widehat{s_{\Sigma'}}
\end{aligned}
$$

So it suffices to prove that $(\mathrm{Pow}\,f) \circ s_\Sigma = s_{\Sigma'} \circ f \in \mathbf{Set}(\Sigma, \mathrm{Pow}\,\Sigma')$. But for all $x \in \Sigma$, we have $((\mathrm{Pow}\,f) \circ s_\Sigma)\,x = \mathrm{Pow}\,f\,(s_\Sigma\,x) = \mathrm{Pow}\,f\,\{x\} = \{f\,y \mid y \in \{x\}\} = \{f\,x\} = s_{\Sigma'}(f\,x) = (s_{\Sigma'} \circ f)\,x$.

Question 3 If $\theta \in \mathbf{D}^{\mathbf{C}}(F,G)$ is an isomorphism, then there is a natural transformation $\theta^{-1} \in \mathbf{D}^{\mathbf{C}}(G,F)$ with $\theta^{-1} \circ \theta = \mathrm{id}_F$ and $\theta \circ \theta^{-1} = \mathrm{id}_G$. By definition of identity and composition for natural transformations, that means that for all $X \in \mathrm{obj}\,\mathbf{C}$ we have $(\theta^{-1})_X \circ \theta_X = \mathrm{id}_{F\,X}$ and $\theta_X \circ (\theta^{-1})_X = \mathrm{id}_{G\,X}$. Therefore each $\theta_X \in \mathbf{D}(F\,X, G\,X)$ is an isomorphism in $\mathbf{D}$ with inverse $(\theta^{-1})_X$.

Conversely, if each $\theta_X \in \mathbf{D}(F\,X, G\,X)$ is an isomorphism in $\mathbf{D}$, then the inverse morphisms $(\theta_X)^{-1}$ are natural in X because for any $f \in \mathbf{C}(X,Y)$ we have

$$
\begin{aligned}
F\,f \circ (\theta_X)^{-1} &= (\theta_Y)^{-1} \circ \theta_Y \circ F\,f \circ (\theta_X)^{-1} && \text{because } (\theta_Y)^{-1} \circ \theta_Y = \mathrm{id}_{F\,Y} \\
&= (\theta_Y)^{-1} \circ G\,f \circ \theta_X \circ (\theta_X)^{-1} && \text{because } \theta_X \text{ is natural in } X \\
&= (\theta_Y)^{-1} \circ G\,f && \text{because } \theta_X \circ (\theta_X)^{-1} = \mathrm{id}_{G\,X}
\end{aligned}
$$

and so determine a natural transformation $\theta \in \mathbf{D}^{\mathbf{C}}(G,F)$ with $(\theta^{-1})_X \triangleq (\theta_X)^{-1}$ for each $X \in \mathrm{obj}\,\mathbf{C}$. This gives an inverse for θ. For $(\theta^{-1} \circ \theta)_X = (\theta^{-1})_X \circ \theta_X = (\theta_X)^{-1} \circ \theta_X = \mathrm{id}_{F\,X} = (\mathrm{id}_F)_X$, so that $\theta^{-1} \circ \theta = \mathrm{id}_F$; and similarly, $\theta \circ \theta^{-1} = \mathrm{id}_G$.

Question 4 If ch_X were natural in X, then taking $X = 2 = \{0,1\}$ and letting τ be as in the hint, there would be a commutative square in **Set**:

$$
\begin{tikzcd}
P^+2 \arrow[r, "ch_2"] \arrow[d, "P^+\tau"'] & 2 \arrow[d, "\tau"] \\
P^+2 \arrow[r, "ch_2"'] & 2
\end{tikzcd}
\tag{3}
$$

Consider $\{0,1\} \in P^+2$. We have

$$
P^+\tau\,\{0,1\} = \{\tau\,0, \tau\,1\} = \{1,0\} = \{0,1\} \tag{4}
$$

Since $ch_2(\{0,1\}) \in \{0,1\}$, either $ch_2(\{0,1\}) = 0$, or $ch_2(\{0,1\}) = 1$. In the first case we get

$$
\begin{aligned}
1 = \tau\,0 = \tau(ch_2\{0,1\}) &= ch_2(P^+\tau\,\{0,1\}) && \text{by (3)} \\
&= ch_2\{0,1\} && \text{by (4)} \\
&= 0 && \text{by assumption}
\end{aligned}
$$

which is a contradiction; and in the second case we get a similar contradiction. So (3) cannot commute and in particular ch_X cannot be natural in X.

Question 5

(a) Define $(I\,\alpha)_X \triangleq I(\alpha_X) : I(F\,X) \to I(G\,X)$. Since α_X is natural in $X \in \mathrm{obj}\,\mathbf{C}$, we have $G\,f \circ \alpha_X = \alpha_Y \circ F\,f$; and then since I is a functor, we get $I(G\,f) \circ I(\alpha_X) = I(\alpha_Y) \circ I(F\,f)$. So $(I\,\alpha)_X$ is natural in X.

(b) Define $(\gamma_F)_X \triangleq \gamma_{(F\,X)} : I(F\,X) \to J(F\,X)$. Since γ_Y is natural in $Y \in \mathrm{obj}\,\mathbf{D}$, $(\gamma_F)_X$ is natural in $X \in \mathrm{obj}\,\mathbf{C}$.

(c) Define $(\beta \circ \alpha)_X \triangleq \beta_X \circ \alpha_X : F\,X \to H\,X$. Since α_X and β_X are natural in $X \in \mathrm{obj}\,\mathbf{C}$, so is $(\beta \circ \alpha)_X$.

(d) Define $(\gamma * \alpha)_X \triangleq \gamma_{G\,X} \circ I(\alpha_X) : I(F\,X) \to J(G\,X)$. This is natural in X, because for any $f \in \mathbf{C}(X, Y)$

$$
\begin{aligned}
J(G\,f) \circ (\gamma * \alpha)_X &\triangleq J(G\,f) \circ \gamma_{G\,X} \circ I(\alpha_X) \\
&= J(G\,f) \circ J(\alpha_X) \circ \gamma_{F\,X} && \text{by natuality for } \gamma \\
&= J(G\,f \circ \alpha_X) \circ \gamma_{F\,X} && \text{by functoriality for } J \\
&= J(\alpha_Y \circ F\,f) \circ \gamma_{F\,X} && \text{by natuality for } \alpha \\
&= J(\alpha_Y) \circ J(F\,f) \circ \gamma_{F\,X} && \text{by functoriality for } J \\
&= J(\alpha_Y) \circ \gamma_{F\,Y} \circ I(F\,f) && \text{by natuality for } \gamma \\
&= \gamma_{G\,Y} \circ I(\alpha_Y) \circ I(F\,f) && \text{by natuality for } \gamma \\
&\triangleq (\gamma * \alpha)_Y \circ I(F\,f)
\end{aligned}
$$

(e)
$$
\begin{aligned}
((\delta * \beta) \circ (\gamma * \alpha))_X &\triangleq (\delta * \beta)_X \circ (\gamma * \alpha)_X \\
&\triangleq \delta_{H\,X} \circ J(\beta_X) \circ \gamma_{G\,X} \circ I(\alpha_X) \\
&= \delta_{H\,X} \circ \gamma_{H\,X} \circ I(\beta_X) \circ I(\alpha_X) && \text{by naturality for } \gamma \\
&\triangleq (\delta \circ \gamma)_{H\,X} \circ I(\beta_X) \circ I(\alpha_X) \\
&= (\delta \circ \gamma)_{H\,X} \circ I(\beta_X \circ \alpha_X) && \text{by functoriality for } I \\
&\triangleq (\delta \circ \gamma)_{H\,X} \circ I((\beta \circ \alpha)_X) \\
&\triangleq ((\delta \circ \gamma) * (\beta \circ \alpha))_X
\end{aligned}
$$

Question 6

(a) We use the notation $\overline{g} \triangleq \theta_{X,Y}(g)$ and $\overline{f} \triangleq \theta^{-1}_{X,Y}(f)$ from Lecture 13.

Define $\eta_X \triangleq \overline{\mathrm{id}_{F\,X}} \in \mathbf{C}(X, G(F\,X))$. This is natural in $X \in \mathrm{obj}\,\mathbf{C}$, because using naturality for θ (twice) we have

$$
G(F\,f) \circ \eta_X \triangleq G(F\,f) \circ \overline{\mathrm{id}_{F\,X}} = \overline{F\,f \circ \mathrm{id}_{F\,X}} = \overline{\mathrm{id}_{F\,Y} \circ F\,f} = \overline{\mathrm{id}_{F\,Y}} \circ f \triangleq \eta_Y \circ f
$$

Dually, define $\varepsilon_Y \triangleq \overline{\mathrm{id}_{G\,Y}} \in \mathbf{D}(F(G\,Y), Y)$ and prove it is natural in $Y \in \mathrm{obj}\,\mathbf{D}$ by a similar calculation.

(b) $$\begin{aligned}
(\varepsilon_F \circ F\eta)_X &\triangleq (\varepsilon_F)_X \circ (F\eta)_X \\
&\triangleq \varepsilon_{FX} \circ F(\eta_X) \\
&\triangleq \overline{\mathrm{id}_{G(FX)}} \circ F(\eta_X) \\
&= \overline{\mathrm{id}_{G(FX)} \circ \eta_X} && \text{by naturality of } \theta \\
&= \overline{\eta_X} \\
&\triangleq \overline{\overline{id_{FX}}} \\
&= \mathrm{id}_{FX} && \text{since } \theta \text{ is an isomorphism} \\
&\triangleq (\mathrm{id}_F)_X
\end{aligned}$$

The proof that $(G\varepsilon \circ \eta_G)_Y = (\mathrm{id}_G)_Y$ is dual to the one above.

Question 7 This is a standard result; see for example Proposition 10.1 on page 254 of Awodey's *Category Theory* book.