1. A pullback square in a category $C$ is a commutative diagram of the form

$$
\begin{array}{ccc}
B & \xrightarrow{u} & A \\
\downarrow{q} & & \downarrow{p} \\
Y & \xrightarrow{f} & X
\end{array}
$$

with the following universal property:

for all $C$-objects $W$ and $C$-morphisms

$$
Y \xrightarrow{\ell} W \xrightarrow{k} A
$$

satisfying $f \circ \ell = p \circ k$, there is a unique $C$-morphism

$$
\ell : W \rightarrow B
$$

satisfying $q \circ \ell = h$ and

$u \circ \ell = k$

(a) Let $C$ be a category and $f : Y \rightarrow X$ a morphism in $C$. Show that $f$ is a monomorphism (see Exercise Sheet 1, question 4) if and only if

$$
\begin{array}{ccc}
Y & \xrightarrow{id_Y} & Y \\
\downarrow{id_Y} & & \downarrow{f} \\
Y & \xrightarrow{f} & X
\end{array}
$$

is a pullback square in $C$.

(b) If (1) is a pullback square and $p$ is a monomorphism, show that $q$ is a monomorphism.

(c) If (1) is a pullback square and $p$ is an isomorphism, show that $q$ is an isomorphism.

(d) Given an example of a pullback square (1) in the category $\text{Set}$ of sets and functions, for which $q$ is an isomorphism, but $p$ is not a monomorphism. (Recall that in $\text{Set}$, monomorphisms and isomorphisms are given by the functions that are respectively injective and bijective.)

2. (a) Given morphisms $X' \xrightarrow{f} X$ and $Y \xrightarrow{g} Y'$ in a cartesian closed category $C$, show how to define a morphism $Y^X \rightarrow (Y')^{X'}$ in $C$.

(b) Given types $A'$, $A$, $B$ and $B'$ in simply typed lambda calculus (STLC), give a term $t$ satisfying

$$
\diamondsuit \vdash t : (A' \rightarrow A) \rightarrow (B \rightarrow B') \rightarrow (A \rightarrow B) \rightarrow (A' \rightarrow B')
$$

If the semantics in a cartesian closed category of $A'$, $A$, $B$ and $B'$ are the objects $X'$, $X$, $Y$ and $Y'$ respectively, what is the semantics of $t$?
3. Let \( C = \text{Set}^{\text{op}} \) be the opposite category of the category \( \text{Set} \) of sets and functions.

   (a) State, without proof, what is the product in \( C \) of two objects \( X \) and \( Y \).

   (b) Show by example that there are objects \( X \) and \( Y \) in \( C \) for which there is no exponential and hence that \( C \) is not a cartesian closed category.

4. [In this question I use the notation \( X \) inl \( X, Y \) \( \rightarrow \) \( X + Y \) inr \( X, Y \) \( \rightarrow \) \( Y \) for the coproduct (Lecture 4) of two object \( X \) and \( Y \) in a category, since it will be clearer to make explicit the objects \( X \) and \( Y \) in the notation for the associated coproduct injections, inl \( X, Y \) and inr \( X, Y \).]

A category \( C \) is \textit{distributive} if it has all binary products and binary coproducts, and for all objects \( X, Y, Z \in C \), (using the defining property of the coproduct \( X \times Y \xrightarrow{\text{inl}_{X,Y}} (X \times Y) + (X \times Z) \leftarrow \xrightarrow{\text{inr}_{X,Y,Z}} X \times Z \)), the unique morphism \( \delta_{X,Y,Z} : (X \times Y) + (X \times Z) \rightarrow X \times (Y + Z) \) that makes the following diagram commute

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\text{id} \times \text{inl}_{Y,Z}} & (X \times Y) + (X \times Z) \\
\downarrow \text{inl}_{X,Y,Z} & & \downarrow \delta_{X,Y,Z} \\
(X \times Y) + (X \times Z) & \xrightarrow{\delta_{X,Y,Z}} & X \times (Y + Z) \\
\downarrow \text{inr}_{X,Y,Z} & & \downarrow \text{id} \times \text{inr}_{Y,Z} \\
X \times Z & & \\
\end{array}
\]

is an isomorphism.

   (a) Using the usual product and coproduct constructs in the category \( \text{Set} \) of sets and functions, show that it is a distributive category.

   (b) Give, with justification, an example of a category with binary products and coproducts that is not distributive.

   (c) If \( C \) is a distributive category and 0 is an initial object in \( C \), prove that for all \( X \in C \), the unique morphism \( 0 \rightarrow X \times 0 \) is an isomorphism.

5. A category \( C \) is called \textit{locally finite} if for all \( X, Y \in \text{obj} \ C \), the set of morphisms \( C(X, Y) \) is finite. \( C \) is said to be \textit{finite} if it is both locally finite and \( \text{obj} \ C \) is finite.

   (a) Prove that any finite category with binary products is a pre-order, that is, there is at most one morphism between any pair of objects. [Hint: if \( f, g : X \rightarrow Y \) were distinct, use them to construct too large a number of morphisms from \( X \) to the product \( Y^n \) of \( Y \) with itself \( n \ (>0) \) times, for some suitable some number \( n \).]

   (b) Is every locally finite category with binary products a pre-order? (Either prove it, or give a counterexample.)
Question 1

(a) Suppose (2) is a pullback square. To see that \( f \) is a monomorphism, suppose \( h, k : W \to Y \) satisfy \( f \circ h = f \circ k \); then by the universal property of (2), there is some (unique) \( \ell : W \to Y \) satisfying \( \text{id}_Y \circ \ell = h \) and \( \text{id}_Y \circ \ell = k \), so that \( h = k \). Thus \( f \) is a monomorphism.

Conversely, suppose \( f \) is a monomorphism. Then the pullback property holds for (2). For if \( Y \leftarrow W \to Y \) satisfy \( f \circ h = f \circ k \), then since \( f \) is a monomorphism, we have \( h = k \). Therefore, there is some \( \ell : W \to Y \) with \( \text{id}_Y \circ \ell = h \) and \( \text{id}_Y \circ \ell = k \), namely \( \ell = h = k \); and clearly \( \ell \) is unique with this property.

(b) Suppose \( p \) is a monomorphism in the pullback square (1) and that \( \ell_1, \ell_2 : W \to B \) satisfy \( q \circ \ell_1 = q \circ \ell_2 \). We have to show \( \ell_1 = \ell_2 \). But note that \( p \circ (u \circ \ell_1) = f \circ q \circ \ell_1 = f \circ q \circ \ell_2 = p \circ (u \circ \ell_2) \); and since \( p \) is a monomorphism, this implies that \( u \circ \ell_1 = u \circ \ell_2 \). Therefore, by the uniqueness part of the universal property of the pullback square (1), we do indeed have \( \ell_1 = \ell_2 \).

(c) Suppose \( p \) is a monomorphism in the pullback square (1). Applying the universal property with \( W = Y, h = \text{id}_Y \) and \( k = p^{-1} \circ f \) (for which we do indeed have \( f \circ h = f = p \circ (p^{-1} \circ f) = p \circ k \)), there is a unique \( \ell : Y \to B \) with \( q \circ \ell = \text{id}_Y \) and \( u \circ \ell = p^{-1} \circ f \). To see that \( q \) is an isomorphism it thus suffices to prove that \( \ell \circ q = \text{id}_B \). But \( q \circ (\ell \circ q) = \text{id}_Y \circ q = q \circ \text{id}_B \) and \( u \circ (\ell \circ q) = p^{-1} \circ f \circ q = p^{-1} \circ p \circ u = u = u \circ \text{id}_B \); so by the uniqueness part of the universal property of pullbacks, we do indeed have \( \ell \circ q = \text{id}_B \).

(d) In the category of sets, monomorphisms are injective functions (see Exercise Sheet 1, question 4(d)); and isomorphisms are bijections (see Lecture 2). Take \( A = \{0, 1\}, B = \emptyset, X = \emptyset \) and \( Y = \emptyset \). The functions \( f, p, q, u \) are uniquely determined and \( f \circ q = p \circ u \). Note that \( p : \{0, 1\} \to \{0\} \) is not injective and hence not a monomorphism; and \( q : \emptyset \to \emptyset \) is trivially a bijection, hence an isomorphism. The square is a pullback, since given and \( W, h, k \) as in the universal property, since \( h \) is a function from \( W \) to \( \emptyset \), it must be the case that \( W = \emptyset \), from which the unique existence of \( \ell : W \to B \) satisfying \( q \circ \ell = h \) and \( u \circ \ell = k \) follows trivially.

Question 2

(a) One can define \( g^f : Y^X \to (Y')^X \) by

\[
g^f \triangleq \text{cur} \left( Y^X \times Y' \xrightarrow{\text{id}_X \times f} Y^X \times X \xrightarrow{\text{app}} Y \xrightarrow{g} Y' \right)
\]

(b) \( t \triangleq \lambda f : A' \to A. \lambda g : B \to B'. \lambda h : A \to B. \lambda x' : A'. g(h(f x')) \)

\[
\llbracket t \rrbracket : (A' \Rightarrow A) \to (B \Rightarrow B') \Rightarrow (A \Rightarrow B) \Rightarrow (A' \Rightarrow B') =
\]

\[
\text{cur}(\text{cur}(\text{cur}(\text{app}(\pi_2 \circ \pi_1 \circ \pi_1), \text{app}(\pi_2 \circ \pi_1), \text{app}(\pi_2 \circ \pi_1 \circ \pi_1 \circ \pi_2))))
\]
**Question 3**

(a) The product of $X$ and $Y$ in $C$ is their coproduct in $\text{Set}$, which is the disjoint union

$$X \sqcup Y = \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}$$

together with the functions $\text{inl} \in \text{Set}(X, X \sqcup Y)$ and $\text{inr} \in \text{Set}(Y, X \sqcup Y)$ that respectively map $x \in X$ to $(0, x) \in X \sqcup Y$ and $y \in Y$ to $(1, y) \in X \sqcup Y$.

(b) Consider the one-element set $1 = \{0\}$ as an object of $C$. If the exponential $1^1$ existed in $C$, there would be a bijection $C(1 \times 1, 1) \cong C(1, 1^1)$. But from part (a)

$$C(1 \times 1, 1) \cong \text{Set}(1, 1 \sqcup 1)$$

is a two-element set, whereas

$$C(1, 1^1) \cong \text{Set}(1^1, 1)$$

has exactly one element no matter what set $1^1$ is. Thus for any set $X$, the sets $C(1 \times 1, 1)$ and $C(1, X)$ cannot be in bijection and therefore the exponential $1^1$ of 1 and 1 in $C$ cannot exist.

**Question 4**

(a) Product in $\text{Set}$ is given by Cartesian product ($\times$) of sets and coproduct by disjoint union ($\sqcup$) of sets. Thus given $X, Y, Z \in \text{Set}$, $\delta_{X,Y,Z}$ is the function $(X \times Y) \sqcup (X \times Z) \rightarrow X \times (Y \sqcup Z)$ satisfying for all $x \in X$, $y \in Y$ and $z \in Z$

$$\delta_{X,Y,Z}(0, (x, y)) = (x, (0, y))$$
$$\delta_{X,Y,Z}(1, (x, z)) = (x, (1, z))$$

and clearly this has a two-sided inverse $\delta^{-1}_{X,Y,Z} : X \times (Y \sqcup Z) \rightarrow (X \times Y) \sqcup (X \times Z)$ given by:

$$\delta^{-1}_{X,Y,Z}(x, (0, y)) = (0, (x, y))$$
$$\delta^{-1}_{X,Y,Z}(x, (1, z)) = (1, (x, z))$$

Alternative proof: use the fact that $\text{Set}$ is a cartesian closed category and then appeal to Exercise Sheet 3 question 6.

(b) Since $\text{Set}$ has binary products and coproducts, by duality, $\text{Set}^{\text{op}}$ has binary co-products and products. It is not distributive because, for example, if we take $X = Y = Z = 1 = \{0\}$ then $(X \times Y) + (X \times Z)$ in $\text{Set}^{\text{op}}$ is given by a set $\{1 \cup 1\} \times \{1 \cup 1\}$ with four elements, whereas $X \times (Y + Z)$ in $\text{Set}^{\text{op}}$ is given by a set $\{1 \cup 1\} \times \{1 \times 1\}$ with only two elements; these cannot be isomorphic in $\text{Set}^{\text{op}}$ because isomorphism is a self-dual concept and we know that isomorphism in $\text{Set}$ is given by bijection.

(c) Write $i$ for the unique morphism $0 \rightarrow X \times 0$. We will show that $\pi_2 : X \times 0 \rightarrow 0$ is its two-sided inverse. It is certainly the case that $\pi_2 \circ i = \text{id} : 0 \rightarrow 0$, because 0 is initial. So it just remains to show that $i \circ \pi_2 = \text{id} : X \times 0 \rightarrow X \times 0$.

Consider the unique morphism $[\text{id}, i \circ \pi_2] : (X \times 0) + (X \times 0) \rightarrow X \times 0$ whose compositions with $\text{inl}_{X \times 0, X \times 0}$ and $\text{inr}_{X \times 0, X \times 0}$ are $\text{id}$ and $i \circ \pi_2$ respectively. If suffices to prove

$$\text{inl}_{X \times 0, X \times 0} = \text{inr}_{X \times 0, X \times 0}$$

(4)
since then \( \text{id} = [\text{id}, i \circ \pi_2] \circ \text{inl}_{X \times 0, X \times 0} = [\text{id}, i \circ \pi_2] \circ \text{inr}_{X \times 0, X \times 0} = i \circ \pi_2 \). To see (4), take \( Y = Z = 0 \) in (3) to deduce that

\[
\text{inl}_{X \times 0, X \times 0} = (\text{id} \times \text{inl}_{0,0}) \circ \delta^{-1}_{X,0,0} \\
\text{inr}_{X \times 0, X \times 0} = (\text{id} \times \text{inr}_{0,0}) \circ \delta^{-1}_{X,0,0}
\]

But since 0 is initial we have \( \text{inl}_{0,0} = \text{inr}_{0,0} : 0 \to 0 + 0 \) and therefore \( \text{inl}_{X \times 0, X \times 0} = \text{inr}_{X \times 0, X \times 0} \).

**Question 5**

(a) Given any \( Y \in \text{obj} \, C \), for each natural number \( n > 0 \), by iterating the binary product we can form the product of \( n \) copies of \( Y \); this is an object \( Y^n \) together with morphisms \( \pi_i^n : Y^n \to Y \) \( (i = 1, \ldots, n) \) that have the universal property that for each \( n \)-tuple of morphisms \( (f_i : Y \to Y \mid i = 1, \ldots, n) \), there is a unique morphism \( h : Y \to Y^n \) satisfying \( \forall j = 1, \ldots, n, \pi_j^n \circ h = f_j \). Therefore given any \( f, g : X \to Y \) in \( C \), for each \( n > 0 \) there are morphisms \( h_i^n : X \to Y^n \) \( (i = 1, \ldots, n) \) where for each \( j = 1, \ldots, n \)

\[
\pi_j^n \circ h_i^n = \begin{cases} f & \text{if } i = j \\ g & \text{if } i \neq j \end{cases}
\]

Since \( C \) is finite, we can pick \( n \) sufficiently large that for some \( i \neq j \) we have \( h_i^n = h_j^n \); and then \( f = \pi_j^n \circ h_i^n = \pi_i^n \circ h_j^n = g \). So \( C \) is a pre-order.

(b) Clearly the category whose objects are *finite* sets and whose morphisms are functions (with composition and identities as for \( \text{Set} \)) is locally finite (there are only finitely many different functions from one finite set to another), but is not a pre-order. It has binary products, given as in \( \text{Set} \) by Cartesian product of sets.