University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 2
by Andrew Pitts

1. Let $\mathbf{C}$ be a category with binary products.

   (a) For morphisms $f \in \mathbf{C}(X,Y)$, $g_1 \in \mathbf{C}(Y,Z_1)$ and $g_2 \in \mathbf{C}(Y,Z_2)$, show that

   $$\langle g_1, g_2\rangle \circ f = \langle g_1 \circ f, g_2 \circ f\rangle \in \mathbf{C}(X, Z_1 \times Z_2) \tag{1}$$

   (b) For morphisms $f_1 \in \mathbf{C}(X_1,Y_1)$ and $f_2 \in \mathbf{C}(X_2,Y_2)$, define

   $$f_1 \times f_2 \triangleq \langle f_1 \circ \pi_1, f_2 \circ \pi_2\rangle \in \mathbf{C}(X_1 \times X_2, Y_1 \times Y_2) \tag{2}$$

   For any $g_1 \in \mathbf{C}(Z,X_1)$ and $g_2 \in \mathbf{C}(Z,X_2)$, show that

   $$(f_1 \times f_2) \circ \langle g_1, g_2\rangle = \langle f_1 \circ g_1, f_2 \circ g_2\rangle \in \mathbf{C}(Z, Y_1 \times Y_2) \tag{3}$$

   (c) Show that the operation $f_1, f_2 \mapsto f_1 \times f_2$ defined in part (1b) satisfies

   $$(h_1 \times h_2) \circ (k_1 \times k_2) = (h_1 \circ k_1) \times (h_2 \circ k_2) \tag{4}$$
   $$\mathrm{id}_X \times \mathrm{id}_Y = \mathrm{id}_{X \times Y} \tag{5}$$

2. Let $\mathbf{C}$ be a category with binary products $\_\times\_$ and a terminal object 1. Given objects $X, Y, Z \in \mathbf{C}$, construct isomorphisms

   $$\alpha_{X,Y,Z} : X \times (Y \times Z) \cong (X \times Y) \times Z \tag{6}$$
   $$\lambda_X : 1 \times X \cong X \tag{7}$$
   $$\rho_X : X \times 1 \cong X \tag{8}$$
   $$\tau_{X,Y} : X \times Y \cong Y \times X \tag{9}$$

3. A *pairing* for a monoid $(M, \cdot, e)$ consists of elements $p_1, p_2 \in M$ and a binary operation $\langle \_, \_\rangle : M \times M \to M$ satisfying for all $x, y, z \in M$

   $$p_1 \cdot \langle x, y\rangle = x \tag{10}$$
   $$p_2 \cdot \langle x, y\rangle = y \tag{11}$$
   $$\langle p_1, p_2\rangle = e \tag{12}$$
   $$\langle x, y\rangle \cdot z = \langle x \cdot z, y \cdot z\rangle \tag{13}$$

   Given such a pairing, show that the monoid, when regarded as a one-object category, has binary products.

4. A monoid $(M, \cdot_M, e_M)$ is said to be *abelian* if its multiplication is commutative: $(\forall x, y \in M)\ x \cdot_M y = y \cdot_M x$.

(a) If $(M, \cdot_M, e_M)$ is an abelian monoid, show that the functions $m \in \mathbf{Set}(M\times, M, M)$ and $u \in \mathbf{Set}(1, M)$ defined by

$$\begin{aligned} m(x,y) &= x \cdot_M y \qquad\qquad \text{(all } x, y \in M\text{)} \\ u(0) &= e_M \end{aligned}$$

determine morphisms in the catgory **Mon** of monoids, $m \in \mathbf{Mon}(M \times M, M)$ and $u \in \mathbf{Mon}(1, M)$ (where as usual we just write M for the monoid $(M, \cdot_M, e_M)$ and 1 for the terminal monoid $(1, \cdot_1, e_1)$ with 1 a one-element set, $\{0\}$ say, $0 \cdot_1 0 = 0$ and $e_1 = 0$).

Show futher that m and u make the monoid M into a "monoid object in the category **Mon**", in the sense that the following diagrams in **Mon** commute:

$$\begin{array}{ccccc} (M \times M) \times M & \xrightarrow{m\times \mathrm{id}} & M \times M & \xrightarrow{m} & M \\ {\scriptstyle \langle \pi_1\circ\pi_1, \langle \pi_2\circ\pi_1, \pi_2\rangle\rangle}\downarrow\cong & & & & \cong\downarrow{\scriptstyle \mathrm{id}} \\ M \times (M \times M) & \xrightarrow[\mathrm{id}\times m]{} & M \times M & \xrightarrow[m]{} & M \end{array} \qquad \text{(associativity)} \tag{14}$$

$$\begin{array}{ccccc} 1 \times M & \xrightarrow{u\times \mathrm{id}} & M \times M & \xrightarrow{m} & M \\ {\scriptstyle \pi_2}\downarrow\cong & & & & \cong\downarrow{\scriptstyle \mathrm{id}} \\ M & & \xrightarrow[\mathrm{id}]{} & & M \end{array} \qquad \text{(left unit)} \tag{15}$$

$$\begin{array}{ccccc} M \times 1 & \xrightarrow{\mathrm{id}\times u} & M \times M & \xrightarrow{m} & M \\ {\scriptstyle \pi_1}\downarrow\cong & & & & \cong\downarrow{\scriptstyle \mathrm{id}} \\ M & & \xrightarrow[\mathrm{id}]{} & & M \end{array} \qquad \text{(right unit)} \tag{16}$$

(b) Show that every monoid object in the category **Mon** (in the above sense) arises as in (4a). [Hint: if necessary, search the internet for "Eckmann-Hilton argument".]

5. Let **AbMon** be the category whose objects are abelian monoids (question 4) and whose morphisms, identity morphisms and composition are as in **Mon**.

   (a) Show that the product in **Mon** of two abelian monoids gives their product in **AbMon**.

   (b) Given $M, N \in \mathbf{AbMon}$ define morphisms $i \in \mathbf{AbMon}(M, M \times N)$ and $j \in \mathbf{AbMon}(N, M \times N)$ that make $M \times N$ into a *coproduct* in **AbMon**.

6. The category $\mathbf{Set}^{\omega}$ of 'sets evolving through discrete time' is defined as follows:

   - Objects are triples $(X, (\_)^+, |\_|)$, where $X \in \mathbf{Set}$, $(\_)^+ \in \mathbf{Set}(X, X)$ and $|\_| \in \mathbf{Set}(X, \mathbb{N})$ satisfy for all $x \in X$
   $$|x^+| = |x| + 1 \tag{17}$$
   [Think of $|x|$ as the instant of time at which x exists and $x \mapsto x^+$ as saying how an element evolves from one instant to the next.]
   - Morphisms $f : (X, (\_)^+, |\_|) \to (Y, (\_)^+, |\_|)$ are functions $f \in \mathbf{Set}(X, Y)$ satisfying for all $x \in X$
   $$(f\,x)^+ = f(x^+) \tag{18}$$
   $$|f\,x| = |x| \tag{19}$$

- Composition and identities are as in the category **Set**.

Show that $\mathbf{Set}^{\omega}$ has a terminal object and binary products.

7. Show that the category **PreOrd** of pre-ordered sets and monotone functions is a cartesian closed category.

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Question 1

(a) For $i = 1, 2$ we have $\pi_i \circ (\langle g_1, g_2\rangle \circ f) = (\pi_i \circ \langle g_1, g_2\rangle) \circ f = g_i \circ f = \pi_i \circ \langle g_1 \circ f, g_2 \circ f\rangle$ and hence by the uniqueness part of the universal property for the product $Z_1 \times Z_2$, it is the case that $\langle g_1, g_2\rangle \circ f = \langle g_1 \circ f, g_2 \circ f\rangle$.

(b)
$$\begin{aligned}(f_1 \times f_2) \circ \langle g_1, g_2\rangle &\triangleq \langle f_1 \circ \pi_1, f_2 \circ \pi_2\rangle \circ \langle g_1, g_2\rangle \\ &= \langle (f_1 \circ \pi_1) \circ \langle g_1, g_2\rangle, (f_2 \circ \pi_2) \circ \langle g_1, g_2\rangle\rangle \quad \text{(by part (a))} \\ &= \langle f_1 \circ (\pi_1 \circ \langle g_1, g_2\rangle), f_2 \circ (\pi_2 \circ \langle g_1, g_2\rangle)\rangle \\ &= \langle f_1 \circ g_1, f_2 \circ g_2\rangle\end{aligned}$$

(c)
$$\begin{aligned}(h_1 \times h_2) \circ (k_1 \times k_2) &\triangleq (h_1 \times h_2) \circ \langle k_1 \circ \pi_1, k_2 \circ \pi_2\rangle \\ &= \langle h_1 \circ (k_1 \circ \pi_1), h_2 \circ (k_2 \circ \pi_2)\rangle \quad \text{(by part (b))} \\ &= \langle (h_1 \circ k_1) \circ \pi_1, (h_2 \circ k_2) \circ \pi_2\rangle \\ &\triangleq (h_1 \circ k_1) \times (h_2 \circ k_2)\end{aligned}$$

For the second identity, note that $\mathrm{id}_X \times \mathrm{id}_Y \triangleq \langle \mathrm{id}_X \circ \pi_1, \mathrm{id}_Y \circ \pi_2\rangle = \langle \pi_1, \pi_2\rangle$. Since $\pi_i \circ \mathrm{id}_{X\times Y} = \pi_i = \pi_i \circ \langle \pi_1, \pi_2\rangle$, by the uniqueness part of the universal property for the product $X \times Y$, we have $\mathrm{id}_{X\times Y} = \langle \pi_1, \pi_2\rangle$. Therefore $\mathrm{id}_X \times \mathrm{id}_Y = \langle \pi_1, \pi_2\rangle = \mathrm{id}_{X\times Y}$.

Question 2 Define

$$\begin{aligned}\alpha_{X,Y,Z} &\triangleq \langle \mathrm{id}_X \times \pi_1, \pi_2 \circ \pi_2\rangle & \alpha^{-1}_{X,Y,Z} &\triangleq \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z\rangle \\ \lambda_X &\triangleq \pi_2 & \lambda^{-1}_X &\triangleq \langle \langle\rangle_X, \mathrm{id}_X\rangle \\ \rho_X &\triangleq \pi_1 & \rho^{-1}_X &\triangleq \langle \mathrm{id}_X, \langle\rangle_X\rangle \\ \tau_{X,Y} &\triangleq \langle \pi_2, \pi_1\rangle & \tau^{-1}_{X,Y} &\triangleq \langle \pi_2, \pi_1\rangle\end{aligned}$$

Then we have:

$$\begin{aligned}\alpha_{X,Y,Z} \circ \alpha^{-1}_{X,Y,Z} &\triangleq \langle \mathrm{id}_X \times \pi_1, \pi_2 \circ \pi_2\rangle \circ \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z\rangle \\ &= \langle (\mathrm{id}_X \times \pi_1) \circ \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z\rangle, \pi_2 \circ \pi_2 \circ \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z\rangle\rangle && \text{by (1)} \\ &= \langle \langle \pi_1 \circ \pi_1, \pi_1 \circ (\pi_2 \times \mathrm{id}_Z)\rangle, \pi_2 \circ \pi_2 \circ \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z\rangle\rangle && \text{by (3)} \\ &= \langle \langle \pi_1 \circ \pi_1, \pi_2 \circ \pi_1\rangle, \pi_2\rangle && \text{by (2)} \\ &= \langle \langle \pi_1, \pi_2\rangle \circ \pi_1, \pi_2\rangle && \text{by (1)}\end{aligned}$$

and since $\langle \pi_1, \pi_2 \rangle = \mathrm{id}$ (see the proof of question 1c), we get $\alpha_{X,Y,Z} \circ \alpha^{-1}_{X,Y,Z} = \langle \mathrm{id}_{X\times Y} \circ \pi_1, \pi_2 \rangle = \mathrm{id}_{(X\times Y)\times Z}$. Similar tedious calculations using the properties from question 1 give

$$
\begin{aligned}
\alpha^{-1} \circ \alpha &= \mathrm{id} \\
\lambda \circ \lambda^{-1} &= \mathrm{id} \\
\lambda^{-1} \circ \lambda &= \mathrm{id} \\
\rho \circ \rho^{-1} &= \mathrm{id} \\
\rho^{-1} \circ \rho &= \mathrm{id} \\
\tau \circ \tau^{-1} &= \mathrm{id} \\
\tau^{-1} \circ \tau &= \mathrm{id}
\end{aligned}
$$

Question 3 Regarding M as a category with a single object, $*$ say, it suffices to show that $* \xleftarrow{p_1} * \xrightarrow{p_2} *$ is a product in M, that is: for all $x, y \in M$, there is a unique $z \in M$ with $p_1 \cdot z = x$ and $p_2 \cdot z = y$. But $\langle x, y \rangle$ is such a z; and it is unique since if $p_1 \cdot z = x$ and $p_2 \cdot z = y$,then $z = e \cdot z = \langle p_1, p_2 \rangle \cdot z = \langle p_1 \cdot z, p_2 \cdot z \rangle = \langle x, y \rangle$.

Question 4

(a) Recall that the product monoid $(M, \cdot_M, e_M) \times (M, \cdot_M, e_M)$ is $(M \times M, \cdot, (e_M, e_M))$ where the binary operation $\_ \cdot \_ : (M \times M) \times (M \times M) \to (M \times M)$ is given by:

$$(x, y) \cdot (x', y') = (x \cdot_M x', y \cdot_M y')$$

Thus for all $x, x', y, y' \in M$ we have

$$
\begin{aligned}
m((x, y) \cdot (x', y')) &= m(x \cdot_M x', y \cdot_M y') \\
&\triangleq (x \cdot_M x') \cdot_M (y \cdot_M y') \\
&= x \cdot_M ((x' \cdot_M y) \cdot_M y') && \text{since } \cdot_M \text{ is associative} \\
&= x \cdot_M ((y \cdot_M x') \cdot_M y') && \text{since } \cdot_M \text{ is commutative} \\
&= (x \cdot_M y) \cdot_M (x' \cdot_M y') && \text{since } \cdot_M \text{ is associative} \\
&\triangleq m(x, y) \cdot_M m(x', y') \\
m(e_M, e_M) &\triangleq e_M \cdot_M e_M \\
&= e_M && \text{since } e_M \text{ is a unit for } \cdot_M
\end{aligned}
$$

so m is a monoid morphism; and u is one too because $u(0 \cdot 0) = u(0) = e_M = e_M \cdot_M e_M = u(0) \cdot_M u(0)$.

To see that m and u make M into a monoid object in **Mon**, just note that diagram (14) commutes because $(\forall x, y, z \in M)\ x \cdot_M (y \cdot_M z) = (x \cdot_M y) \cdot_M z$, (15) commutes because $(\forall x \in M)\ e_M \cdot_M x = x$ and (16) commutes because $(\forall x \in M)\ x \cdot_M e_M = x$.

(b) Suppose we are given monoid morphisms $m \in \mathbf{Mon}(M \times M, M)$ and $u \in \mathbf{Mon}(1, M)$ that make (14)–(16) commute. Since u is a monoid morphism we have $u(0) = e_M$ and therefore from the commutation of (15) and (16) we deduce that for all $x \in M$

$$m(e_M, x) = x = m(x, e_M) \tag{20}$$

Now by definition of the monoid multiplication operation for the product monoid $(M, \cdot_M, e_M) \times (M, \cdot_M, e_M)$ we have

$$(x, e_M) \cdot (e_M, y) = (x \cdot_M e_M, e_M \cdot_M y) = (x, y) = (e_M \cdot_M x, y \cdot_M e_M) = (e_M, y) \cdot (x, e_M)$$

Therefore since m is a monoid homomorphism, we have

$$m(x, e_M) \cdot_M m(e_M, y) = m((x, e_M) \cdot (e_M, y)) = m(x, y) = \\ m((e_M, y) \cdot (e_M, x)) = m(e_M, y) \cdot_M m(x, e_M)$$

and hence from (20) we get $x \cdot y = m(x, y) = y \cdot x$. Therefore $(M, \cdot_M, e_M)$ is abelian and the monoid object $((M, \cdot_M, e_M), m, u)$ in **Mon** coincides with the one from part (a).

Question 5

(a) If M and N are both abelian monoids, then the product operation of the monoid $M \times N$ satisfies for all $x, x' \in M$ and $y, y' \in N$

$$\begin{aligned} (x, y) \cdot (x', y') &\triangleq (x \cdot x', y \cdot y') \\ &= (x' \cdot x, y' \cdot y) && \text{since } M \text{ and } N \text{ are abelian} \\ &\triangleq (x', y') \cdot (x, y) \end{aligned}$$

so that $M \times N$ is also abelian. Therefore the universal property of $M \xleftarrow{\pi_1} M \times N \xrightarrow{\pi_2} N$ in **Mon** restricts to give the correct universal property for a product in **AbMon**.

(b) The functions

$$\begin{aligned} i(x) &\triangleq (x, e) \\ j(y) &\triangleq (e, y) \end{aligned}$$

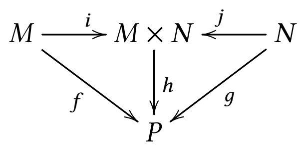
clearly give morphisms $M \xrightarrow{i} M \times N \xleftarrow{j} N$ in **AbMon**. We show that it is a coproduct diagram. Given any morphisms $M \xrightarrow{f} P \xleftarrow{g} N$ in **AbMon**, consider the function $h : M \times N \to P$ defined by

$$h(x, y) \triangleq (f\,x) \cdot (g\,y)$$

It is a morphism in $\mathbf{AbMon}(M \times N, P)$ because $h(e, e) = (f\,e) \cdot (g\,e) = e \cdot e = e$ and

$$\begin{aligned} h((x, y) \cdot (x', y')) &\triangleq f(x \cdot x') \cdot g(y \cdot y') \\ &= (f\,x \cdot f\,x') \cdot (g\,y \cdot g\,y') && \text{since } f \text{ and } g \text{ are morphisms} \\ &= f\,x \cdot (f\,x' \cdot g\,y) \cdot g\,y' && \text{associativity} \\ &= f\,x \cdot (g\,y \cdot f\,x') \cdot g\,y' && \text{since } P \text{ is abelian} \\ &= (f\,x \cdot g\,y) \cdot (f\,x' \cdot g\,y') && \text{associativity} \\ &\triangleq h(x, y) \cdot h(x', y') \end{aligned}$$

Furthermore, since $h(i\,x) = h(x, e) = f\,x \cdot g\,e = f\,x \cdot e = f\,x$ and $h(j\,y) = h(e, y) = f\,e \cdot g\,y = e \cdot g\,y = g\,y$, we have that



commutes. Finally, h is the unique such morphism, since if $h' \in \mathbf{AbMon}(M{\times}N, P)$ also satisfies $h' \circ i = f$ and $h' \circ j = g$, then

$$h'(x,y) = h'((x,e)\cdot(e,y)) = h'(x,e)\cdot h'(e,y) = h'(i\,x)\cdot h'(j\,y) = f\,x \cdot g\,y \triangleq h(x,y).$$

so that $h' = h$.

Question 6

- *Terminal object* is $(\mathbb{N}, (\_)^+, |\_|)$, where for all $n \in \mathbb{N}$, $n^+ \triangleq n+1$ and $|n| \triangleq n$, which trivially have the required property (17). For each object $(X, (\_)^+, |\_|) \in \mathbf{Set}^\omega$, the unique morphism $(X, (\_)^+, |\_|) \to (\mathbb{N}, (\_)^+, |\_|)$ is given by $|\_|$.

- *Binary product* of $(X, (\_)^+, |\_|)$ and $(Y, (\_)^+, |\_|)$ is $(X, (\_)^+, |\_|) \xleftarrow{\pi_1} (P, (\_)^+, |\_|) \xrightarrow{\pi_2} (Y, (\_)^+, |\_|)$, where

$$\begin{aligned}
P &\triangleq \{(x,y) \in X \times Y \mid |x| = |y|\}\\
(x,y)^+ &\triangleq (x^+, y^+)\\
|(x,y)| &\triangleq |x|(= |y|)\\
\pi_1(x,y) &\triangleq x\\
\pi_2(x,y) &\triangleq y
\end{aligned}$$

Given morphisms $(X, (\_)^+, |\_|) \xleftarrow{f} (Z, (\_)^+, |\_|) \xrightarrow{g} (Y, (\_)^+, |\_|)$, the unique morphism $\langle f, g\rangle : (Z, (\_)^+, |\_|) \to (P, (\_)^+, |\_|)$ with $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$ maps each $z \in Z$ to

$$\langle f, g\rangle\, z \triangleq (f\,z, g\,z)$$

(which does lie in P because $|f\,z| = |z| = |g\,z|$).

Question 7 I do not give the proof that a one-element poset is terminal in **PreOrd**, or that the binary product of $(P, \leq)$ and $(Q, \leq)$ in **PreOrd** is given by the cartesian product of underlying sets together with the partial order

$$(p_1, q_1) \leq (p_2, q_2) \triangleq p_1 \leq p_2 \wedge q_1 \leq q_2 \qquad \text{for all } p_1, p_2 \in P \text{ and } q_1, q_2 \in Q.$$

Let us show that the exponential of $(P, \leq)$ and $(Q, \leq)$ is given by:

$$\begin{aligned}
P \to Q &\triangleq \{f \in Q^P \mid (\forall p, p')\ p \leq p' \in P \implies f\,p \leq f\,p' \in Q\} && (21)\\
f \leq f' \in P \to Q &\triangleq (\forall p \in P)\ f\,p \leq f'\,p && (22)\\
\mathrm{app}(f, p) &\triangleq f\,p && (23)
\end{aligned}$$

Two things need checking (that we don't do here):

- (22) does define a partial order on the set (21), and

- (23) does give a monotone function.

So we have a morphism $\mathrm{app} : (P \to Q, \leq) \times (P, \leq) \longrightarrow (Q, \leq)$ in **PreOrd** and we need to see that it has the universal property of the exponential of $(P, \leq)$ and $(Q, \leq)$.

Given $f : (R, \leq) \times (P, \leq) \longrightarrow (Q, \leq)$ in **PreOrd**, since $f \in \mathbf{Set}(R \times P, Q)$ we have the function $\mathrm{cur}\, f \in \mathbf{Set}(R, Q^P)$, where as usual, $\mathrm{cur}\, f\, r\, p = f(r, p)$ for all $r \in R$ and $p \in P$. Note that

$$
\begin{aligned}
p \leq p' \in P &\Rightarrow (r, p) \leq (r, p') \in R \times P \\
&\Rightarrow \mathrm{cur}\ f\, r\, p = f(r, p) \leq f(r, p') = \mathrm{cur}\, f\, r\, p' \qquad \text{since } f \text{ is monotone}
\end{aligned}
$$

so that for each $r \in R$, we have $\mathrm{cur}\, f\, r \in P \to Q$. In other words, $\mathrm{cur}\, f \in \mathbf{Set}(R, P \to Q)$. Furthermore $\mathrm{cur}\, f$ is a monotone function, because

$$
\begin{aligned}
r \leq r' \in R &\Rightarrow (\forall p \in P)\ (r, p) \leq (r', p) \in R \times P \\
&\Rightarrow (\forall p \in P)\ \ \mathrm{cur}\, f\, r\, p = f(r, p) \leq f(r', p) = \mathrm{cur}\, f\, r' p \qquad \text{since } f \text{ is monotone}
\end{aligned}
$$

Note that $\mathrm{app} \circ (\mathrm{cur}\, f \times \mathrm{id}_P) = f \in \mathbf{PreOrd}((R, \leq) \times (P, \leq), (Q, \leq))$, because for all $(r, p) \in R \times P$

$$
(\mathrm{app} \circ (\mathrm{cur}\, f \times \mathrm{id}_P))(r, p) = \mathrm{app}((\mathrm{cur}\, f \times \mathrm{id}_P)(r, p)) = \mathrm{app}(\mathrm{cur}\, f\, r, p) = \mathrm{cur}\, f\, r\, p = f(r, p)
$$

Finally, $\mathrm{cur}\, f$ is the only element $g \in \mathbf{PreOrd}((R, \leq), (P \to Q, \leq))$ satisfying $\mathrm{app} \circ (g \times \mathrm{id}_P) = f$, since the latter equation implies that $g\, r\, p = (\mathrm{app} \circ (g \times \mathrm{id}_P))(r, p) = f(r, p) = \mathrm{cur}\, f\, r\, p$ for all $(r, p) \in R \times P$. Hence for any $r \in R$, $g\, r$ and $\mathrm{cur}\, f\, r$ are equal functions from P to Q; and therefore $g = \mathrm{cur}\, f$.