University of Cambridge
2023/24 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 1
by Andrew Pitts

1. (a) Show that the sets $2 = \{0, 1\}$ and $3 = \{0, 1, 2\}$ are not isomorphic in the category **Set** of sets and functions.

   (b) Let P be the pre-ordered set with underlying set $\{0, 1\}$ and pre-order: $0 \leq 0$, $1 \leq 1$. Let Q be the pre-ordered set with the same underlying set and pre-order: $0 \leq 0$, $0 \leq 1$, $1 \leq 1$. Show that P and Q are not isomorphic in the category **Preord** of pre-ordered sets and monotone functions.

   (c) Why are the sets $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (integers) and $\mathbb{Q}$ (rational numbers) isomorphic in **Set**? Regarding them as pre-ordered sets via the usual ordering on numbers, show that they are not isomorphic in **Preord**. [Hint: recall that $\mathbb{Q}$ has the property that for any two distinct elements there is a third distinct element lying between them in the ordering.]

2. Let **C** be a category and let $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$ be morphisms in **C**.

   (a) Prove that if f and g are both isomorphisms, with inverses f^{-1} and g^{-1} respectively, then $g \circ f$ is an isomorphism and its inverse is $f^{-1} \circ g^{-1}$.

   (b) Prove that if f and $g \circ f$ are both isomorphisms, then so is g.

   (c) If $g \circ f$ is an isomorphism, does that necessarily imply that either of f or g are isomorphisms?

3. Let **Mat** be a category whose objects are all the non-zero natural numbers $1, 2, 3, \ldots$ and whose morphisms $M \in \mathbf{Mat}(m, n)$ are $m \times n$ matrices with real number entries. If composition is given by matrix multiplication, what are the identity morphisms? Give an example of an isomorphism in **Mat** that is not an identity. Can two object m and n be isomorphic in **Mat** if $m \neq n$?

4. Let **C** be a category. A morphism $f : X \to Y$ in **C** is called a *monomorphism*, if for every object $Z \in \mathbf{C}$ and every pair of morphisms $g, h : Z \to X$ we have

   $$f \circ g = f \circ h \;\Rightarrow\; g = h$$

   It is called a *split monomorphism* if there is some morphism $g : Y \to X$ with $g \circ f = \mathrm{id}_X$, in which case we say that g is a *left inverse* for f.

   (a) Prove that every isomorphism is a split monomorphism and that every split monomorphism is a monomorphism.

   (b) Prove that if $f : X \to Y$ and $g : Y \to Z$ are monomorphisms, then $g \circ f : X \to Z$ is a monomorphism.

   (c) Prove that if $f : X \to Y$ and $g : Y \to Z$ are morphisms in **C**, and $g \circ f$ is a monomorphism, then f is a monomorphism.

(d) Characterize the monomorphisms in the category **Set** of sets and functions. Is every monomorphism in **Set** a split monomorphism?

(e) By considering the category **Set**, show that a split monomorphism can have more than one left inverse.

(f) Regarding a pre-ordered set $(P, \leq)$ as a category, which of its morphisms are monomorphisms and which are split monomorphisms?

5. The dual of *monomorphism* is called *epimorphism*: a morphism $f : X \to Y$ in $\mathbf{C}$ is an epimorphism iff $f \in \mathbf{C}^{\mathrm{op}}(Y, X)$ is a monomorphism in $\mathbf{C}^{\mathrm{op}}$.

   (a) Show that $f \in \mathbf{Set}(X, Y)$ is an epimorphism iff f is a surjective function.

   (b) Regarding a pre-ordered set $(P, \leq)$ as a category, which of its morphisms are epimorphisms?

   (c) Give an example of a category containing a morphism that is both an epimorphism and a monomorphism, but not an isomorphism. [Hint: consider your answers to (4f) and (5b).]

6. Let $\mathbf{C}$ be the category the following category:

   - $\mathbf{C}$-objects are triples (X, x_0, x_s) where $X \in \mathbf{Set}$, $x_0 \in X$ and $x_s \in \mathbf{Set}(X, X)$;
   - $\mathbf{C}$-morphisms $f \in \mathbf{C}((X, x_0, x_s), (Y, y_0, y_s))$ are functions $f \in \mathbf{Set}(X, Y)$ satisfying $f\, x_0 = y_0$ and $f \circ x_s = y_s \circ f$;
   - composition and identities are as for the category **Set**.

   (a) Show that $\mathbf{C}$ has a terminal object.

   (b) Show that $\mathbf{C}$ has an initial object whose underlying set is the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of natural numbers.

7. In a category $\mathbf{C}$ with a terminal object 1, a morphism $p : 1 \to X$ is called a *point* (or *global element*) of the object X. $\mathbf{C}$ is said to be *well-pointed* if for all objects $X, Y \in \mathbf{C}$, two morphisms $f, g : X \to Y$ are equal if their compositions with all points of X are equal:

$$(\forall p \in \mathbf{C}(1, X),\ f \circ p = g \circ p) \ \Rightarrow\ f = g \tag{1}$$

   (a) Show that **Set** is well-pointed.

   (b) Is the opposite category $\mathbf{Set}^{\mathrm{op}}$ well-pointed? [Hint: observe that the left-hand side of the implication in (1) is vacuously true in the case that $\mathbf{C}(1, X)$ is empty.]

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Question 1

(a) In Lecture 2 we saw that a morphism in **Set** is an isomorphism iff it is a bijection; but there is no bijection $3 \cong 2$, since any function $f : 3 \to 2$ cannot be injective.

(b) Any function $f : Q \to P$ that is monotonic satisfies $f\,0 \leq f\,1$ in P and hence $f\,0 = f\,1$ (because of the definition of $<$ for P). So f is not a bijection. But any isomorphism in **Preord** is in particular an isomorphism in **Set** of the underlying sets (why?) and hence a bijection.

(c) Recall that the set $\mathbb{Q}$ of rational numbers is countably infinite and so is in bijection with $\mathbb{Z}$. Thus $\mathbb{Z}$ and $\mathbb{Q}$ are isomorphic in **Set**. However, as a pre-ordered set the rationals are dense: writing $x < y$ to mean $x \leq y \wedge x \neq y$, we have $(\forall x, y \in \mathbb{Q})\ x < y \Rightarrow (\exists z \in \mathbb{Q})\ x < z \wedge z < y$; whereas $(\mathbb{Z}, \leq)$ is not a dense pre-ordered set. It is not hard to see that the density property of pre-ordered sets is preserved under isomorphism. So $(\mathbb{Z}, \leq)$ cannot be isomorphic to $(\mathbb{Q}, \leq)$ in **Preord**.

Question 2

(a)

$$\begin{aligned}
(g \circ f) \circ (f^{-1} \circ g^{-1}) &= (g \circ (f \circ f^{-1})) \circ g^{-1} && \text{(associativity)}\\
&= (g \circ \mathrm{id}_Y) \circ g^{-1} && \text{(definition of } f^{-1})\\
&= g \circ g^{-1} && \text{(unity)}\\
&= \mathrm{id}_Z && \text{(definition of } g^{-1})
\end{aligned}$$

and a similar proof shows that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}_X$. So $g \circ f$ is an isomorphism with inverse $f^{-1} \circ g^{-1}$.

(b) If f and $g \circ f$ have inverses $f^{-1} \in \mathbf{C}(Y, X)$ and $(g \circ f)^{-1} \in \mathbf{C}(Z, X)$, then consider $h \triangleq f \circ (g \circ f)^{-1} \in \mathbf{C}(Z, Y)$. We have

$$g \circ h = g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} = \mathrm{id}_Z$$

and

$$h \circ g = (f \circ (g \circ f)^{-1}) \circ g = f \circ (g \circ f)^{-1} \circ g \circ f \circ f^{-1} = f \circ f^{-1} = \mathrm{id}_Y$$

so that g is an isomorphism with inverse h.

(c) No. In the category **Set** take $X = \{0\} = Z$, $Y = \{0, 1\}$, $f \in \mathbf{Set}(X, Y)$ to be the function $f\,0 = 0$ and $g \in \mathbf{Set}(Y, Z)$ to be the function with constant value 0. Then neither f nor g are isomorphisms (since they are not bijections), but $g \circ f = \mathrm{id}_X$ is one.

Question 3 The identity morphism $\mathrm{id}_n \in \mathbf{Mat}(n,n)$ is the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is 1 if $i = j$ and is 0 otherwise.

The morphism $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbf{Mat}(2,2)$ is a non-identity isomorphism since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_2$.

Two objects m and n are isomorphic in **Mat** only if $m = n$. For if $M \in \mathbf{Mat}(m,n)$ is an isomorphism, then $M = \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_m \end{pmatrix}$ consists of m rows that are linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$: for if $\lambda_1 \vec{v}_1 + \cdots + \lambda_m \vec{v}_m = \vec{0} \in \mathbb{R}^n$, that is, $\begin{pmatrix} \lambda_1 & \cdots & \lambda_m \end{pmatrix} M = \vec{0}$, then applying the inverse of M we get $\vec{0} = \vec{0} M^{-1} = \begin{pmatrix} \lambda_1 & \cdots & \lambda_m \end{pmatrix} M M^{-1} = \begin{pmatrix} \lambda_1 & \cdots & \lambda_m \end{pmatrix}$. So since $\mathbb{R}^n$ is a vector space of dimension n, we must have $m \leq n$. By a symmetric argument, $n \leq m$.

Question 4

(a) If f is an isomorphism, its inverse f^{-1} is in particular a left inverse. If g is a left inverse for f, then for all $h, k \in \mathbf{C}(Z,X)$ we have $f \circ h = f \circ k \implies h = \mathrm{id}_X \circ h = g \circ f \circ h = g \circ f \circ k = \mathrm{id}_X \circ k = k$, so that f is a monomorphism.

(b) If $h, k \in \mathbf{C}(W,X)$ satisfy $(g \circ f) \circ h = (g \circ f) \circ k$, then $f \circ h = f \circ k$ since g is a monomorphism; and then $h = k$ since f is a monomorphism.

(c) If $h, k \in \mathbf{C}(W,X)$ satisfy $f \circ h = f \circ k$, then $(g \circ f) \circ h = (g \circ f) \circ k$ and since $g \circ f$ is a monomorphism, this implies $h = k$.

(d) The monomorphisms in **Set** are exactly the injective functions.

Proof. If $f \in \mathbf{Set}(X,Y)$ is injective, then for any $g, h \in \mathbf{Set}(Z,X)$, if $f \circ g = f \circ h$, then for all $z \in Z$ we have $f(g\,z) = f(h\,z)$, so $g\,z = h\,z$ (since f is injective); therefore g and h are equal functions.

Conversely, if $f \in \mathbf{Set}(X,Y)$ is a monomorphism, then for any $x, x' \in X$ let $\ulcorner x \urcorner, \ulcorner x \urcorner \in \mathbf{Set}(1,X)$ be the functions mapping the unique element of $1 = \{0\}$ to x and x' respectively. If $f\,x = f\,x'$, then $f \circ \ulcorner x \urcorner = f \circ \ulcorner x' \urcorner \in \mathbf{Set}(1,Y)$. Since f is a monomorphism, this implies $\ulcorner x \urcorner = \ulcorner x' \urcorner$ and hence $x = \ulcorner x \urcorner\, 0 = \ulcorner x' \urcorner\, 0 = x'$. So f is injective. □

Not every monomorphism in **Set** is split. For example, consider the unique morphism in $\mathbf{Set}(\emptyset, 1)$ (where $\emptyset$ denotes the empty set). This is injective (vacuously), but there is no function $1 \longrightarrow \emptyset$ in **Set**.

(e) Consider $2 = \{0,1\}$, $3 = \{0,1,2\}$ and the injective function $f \in \mathbf{Set}(2,3)$ with $f\,0 = 0$ and $f\,1 = 1$. There are two different left inverses for f, one mapping 2 to 0 and the other mapping 2 to 1.

(f) All morphisms in a pre-ordered set are monomorphisms, because there is at most one morphism between two objects. The only split monomorphisms are the isomorphisms (since if $f : p \longrightarrow q$ and $g : q \longrightarrow p$ then f and g are isomorphisms, since $g \circ f$ and $f \circ g$ are necessarily equal to the unique morphism, namely the identity, on p and q respectively).

Question 5

(a) Suppose $f \in \mathbf{Set}(X, Y)$ is surjective. If $g, h \in \mathbf{Set}(Y, Z)$ and $g \circ f = h \circ f$, then for all $y \in Y$, there exists $x \in X$ with $y = f\,x$ (since f is surjective) and hence $g\,y = g(f\,x) = (g \circ f)\,x = (h \circ f)\,x = h(f\,x) = h\,y$; therefore g and h are equal functions.

Conversely, suppose $f \in \mathbf{Set}(X, Y)$ is an epimorphism. For each $y \in Y$, consider the functions $g_y, h_y \in \mathbf{Set}(Y, \{0, 1\})$ that map y to 0 and to 1 respectively, and map all other elements of Y to 0. Since $g_y \neq h_y$ and f is an epimorphism, we must have $g_y \circ f \neq h_y \circ f$ and hence $g_y(f\,x) \neq h_y(f\,x)$, for some $x \in X$. Since g_y and h_y only take different values at y, it follows that $f\,x = y$. Therefore f is surjective.

(b) Since the opposite category P^{op} of a pre-ordered set P is again a pre-ordered set, we can re-use the answer to question (4f): all the morphisms of P are epimorphisms.

(c) In the pre-ordered set Q from question 1(b), the unique morphism $0 \longrightarrow 1$ is both a monomorphism (by 4(f)) and an epimorphism (by 5(b)), but not an isomorphism, because there is no morphism from 1 to 0.

Question 6

(a) $(1, 0, \mathrm{id}_1)$ is a terminal object, where $1 = \{0\}$.

(b) Consider the object $(\mathbb{N}, 0, \mathit{succ})$ where $\mathit{succ} \in \mathbf{Set}(\mathbb{N}, \mathbb{N})$ is the successor function, $\mathit{succ}\,n = n+1$. This is initial in $\mathbf{C}$, because for any object (X, x_0, x_s), the function $f : \mathbb{N} \longrightarrow X$ recursively defined by

$$\begin{aligned} f\,0 &= x_0 \\ f(n+1) &= x_s(f\,n) \end{aligned}$$

gives a morphism $f \in \mathbf{C}((\mathbb{N}, 0, \mathit{succ}), (X, x_0, x_s))$. It is the only such morphism, because if $g \in \mathbf{C}((\mathbb{N}, 0, \mathit{succ}), (X, x_0, x_s))$, then $g\,0 = x_0$ and for all $n \in \mathbb{N}$, $g(n+1) = (g \circ \mathit{succ})\,n = (x_s \circ g)\,n = x_s(g\,n)$; hence by induction on n, we have $(\forall n \in \mathbb{N})\; g\,n = f\,n$.

Question 7

(a) Each element $x \in X$ of a set $X \in \mathbf{Set}$ determines a point $\ulcorner x \urcorner : 1 \longrightarrow X$ in $\mathbf{Set}$, namely the function mapping the unique element of $1 = \{0\}$ to x. The mapping $x \mapsto \ulcorner x \urcorner$ is injective, since $\ulcorner x \urcorner\,0 = x$; furthermore for every $f \in \mathbf{Set}(X, Y)$, $f \circ \ulcorner x \urcorner = \ulcorner f\,x \urcorner$. So if $(\forall p \in \mathbf{Set}(1, X))\; f \circ p = g \circ p$, then $(\forall x \in X)\; f\,x = g\,x$, that is, $f = g$.

(b) $\mathbf{Set}^{\mathrm{op}}$ is not well-pointed. Note that the empty set $\emptyset$ is a terminal object in $\mathbf{Set}^{\mathrm{op}}$ (because it is initial in $\mathbf{Set}$) and that $\mathbf{Set}^{\mathrm{op}}(\emptyset, X) = \mathbf{Set}(X, \emptyset)$ is empty when $X \neq \emptyset$. Then for example $\mathrm{id}_{\mathbb{N}} \neq \mathit{succ} \in \mathbf{Set}^{\mathrm{op}}(\mathbb{N}, \mathbb{N})$, but $(\forall p \in \mathbf{Set}^{\mathrm{op}}(\emptyset, \mathbb{N})\; \mathrm{id}_{\mathbb{N}} \circ p = \mathit{succ} \circ p$ is vacuously true.