University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 5
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1. For each set Σ, let $\mathrm{Pow}\,\Sigma$ be the set $\{S \mid S \subseteq \Sigma\}$ of all subsets of Σ. The operation of taking the union of two subsets and the empty subset together give the structure of a monoid on $\mathrm{Pow}\,\Sigma$. Writing $P\Sigma = (\mathrm{Pow}\,\Sigma, \_\cup\_, \emptyset)$ for this monoid, show that $\Sigma \mapsto P\Sigma$ is the object part of a functor $P : \mathbf{Set} \to \mathbf{Mon}$.

2. Let $F : \mathbf{Set} \to \mathbf{Mon}$ be the free monoid functor, given by finite lists and mapping over finite lists (Lecture 10). Show that the operation that sends each finite list to the finite set of elements that it contains gives a natural transformation $\theta : F \to P$, where P is the functor from question 1.

3. Given small categories $\mathbf{C}, \mathbf{D}$, functors $F, G : \mathbf{C} \to \mathbf{D}$ and a natural transformation $\theta : F \to G$, show that θ is an isomorphism in the functor category $\mathbf{D}^{\mathbf{C}}$ if and only if for all $X \in \mathrm{obj}\,\mathbf{C}$, the morphism $\theta_X \in \mathbf{D}(F X, G X)$ is an isomorphism in the category $\mathbf{D}$.

4. In this question you will prove that there is no *natural* way to choose an element from an arbitrary non-empty set. Let $P^+ : \mathbf{Set} \to \mathbf{Set}$ be the functor assigning to each set X its set $P^+X \triangleq \{S \subseteq X \mid S \neq \emptyset\}$ of non-empty subsets; the action of P^+ on morphisms in $\mathbf{Set}$ sends each function $f : X \to Y$ to the function $P^+f : P^+X \to P^+Y$, where for all $S \in P^+X$

$$(P^+f)S \triangleq \{f\,x \mid x \in S\}$$

(This does indeed make P^+ into a functor.)

Suppose that for each set X we are given a function $ch_X : P^+X \to X$ with the property that $ch_X(S) \in S$ for all $S \in P^+X$. Show that these functions cannot be the components of a natural transformation $P^+ \to \mathrm{Id}_{\mathbf{Set}}$ from P^+ to the identity functor on the category $\mathbf{Set}$ of sets and functions. [Hint: consider the naturality condition for the function $\tau : \{0, 1\} \to \{0, 1\}$ with $\tau(0) = 1$ and $\tau(1) = 0$.]

5. Suppose we are given categories $\mathbf{C}, \mathbf{D}, \mathbf{E}$, functors $F, G, H : \mathbf{C} \to \mathbf{D}$ and $I, J, K : \mathbf{D} \to \mathbf{E}$, and natural transformations
$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H \quad \text{and} \quad I \xrightarrow{\gamma} J \xrightarrow{\delta} K.$$

   (a) Using α and I, define a natural transformation $I\,\alpha : I \circ F \to I \circ G$.

   (b) Using F and γ, define a natural transformation $\gamma_F : I \circ F \to J \circ F$.

   (c) Using α and β, define a natural transformation $\beta \circ \alpha : F \to H$.

   (d) Using α and γ, define a natural transformation $\gamma * \alpha : I \circ F \to J \circ G$.

   (e) Show that the operations you defined in questions (5c) and (5d) satisfy: $(\delta * \beta) \circ (\gamma * \alpha) = (\delta \circ \gamma) * (\beta \circ \alpha)$. (This is called the *Interchange Law* for vertical and horizontal composition of natural transformations.)

6. Let $\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{G} \mathbf{C}$ and $(\theta_{X,Y} : \mathbf{D}(F X, Y) \cong \mathbf{C}((X, G Y) \mid X \in \mathrm{obj}\,\mathbf{C}, Y \in \mathrm{obj}\,\mathbf{D})$ be an adjunction between categories $\mathbf{C}$ and $\mathbf{D}$.

(a) Use θ to define natural transformations $\eta : \mathrm{Id}_\mathbf{C} \to G \circ F$ and $\varepsilon : F \circ G \to \mathrm{Id}_D$. (These are called respectively the *unit* and *counit* of the adjunction.)

(b) Prove that the natural transformations defined in part (6a) satisfy $\varepsilon_F \circ F\eta = \mathrm{id}_F$ and $G\varepsilon \circ \eta_G = \mathrm{id}_G$

$$
\begin{array}{ccc}
F \xrightarrow{\ F\eta\ } F \circ G \circ F & \qquad & G \xrightarrow{\ \eta_G\ } G \circ F \circ G \\
\end{array}
\tag{1}
$$

where the left triangle has maps id_F and ε_F down to F, and the right triangle has maps id_G and $G\varepsilon$ down to G.

where we are using notation as in questions (5a) and (5b). (These are called the *triangular identities* for the unit and counit of the adjunction.)

7. Given functors $\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{G} \mathbf{C}$ and natural transformations $\eta : \mathrm{Id}_\mathbf{C} \to G \circ F$ and $\varepsilon : F \circ G \to \mathrm{Id}_D$ satisfying the triangular identities (1), show that F is left adjoint to G.