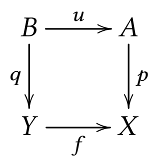
University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 4
by Andrew Pitts

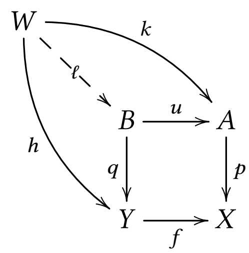
1. A *pullback square* in a category **C** is a commutative diagram of the form



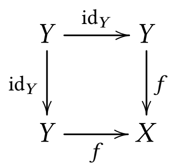
$$p \circ u = f \circ q \tag{1}$$

with the following universal property:

for all **C**-objects W and **C**-morphisms $Y \xleftarrow{h} W \xrightarrow{k} A$ satisfying $f \circ h = p \circ k$, there is a unique **C**-morphism $\ell : W \longrightarrow B$ satisfying $q \circ \ell = h$ and $u \circ \ell = k$



   (a) Let **C** be a category and $f : Y \longrightarrow X$ a morphism in **C**. Show that f is a monomorphism (see Exercise Sheet 1, question 4) if and only if



(2)

   is a pullback square in **C**.

   (b) If (1) is a pullback square and p is a monomorphism, show that q is a monomorphism.

   (c) If (1) is a pullback square and p is a isomorphism, show that q is a isomorphism.

   (d) Given an example of a pullback square (1) in the category **Set** of sets and functions, for which q is an isomorphism, but p is not a monomorphism. (Recall that in **Set**, monomorphisms and isomorphisms are given by the functions that are respectively injective and bijective.)

2. (a) Given morphisms $X' \xrightarrow{f} X$ and $Y \xrightarrow{g} Y'$ in a cartesian closed category **C**, show how to define a morphism $Y^X \longrightarrow (Y')^{X'}$ in **C**.

   (b) Given types A', A, B and B' in simply typed lambda calculus (STLC), give a term t satisfying

$$\diamond \vdash t : (A' \to A) \to (B \to B') \to (A \to B) \to (A' \to B')$$

   If the semantics in a cartesian closed category of A', A, B and B' are the objects X', X , Y and Y' respectively, what is the semantics of t?

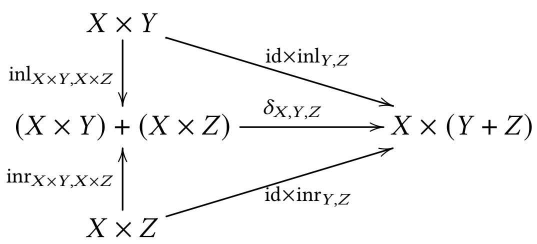
3. Let $\mathbf{C} = \mathbf{Set}^{\mathrm{op}}$ be the opposite category of the category **Set** of sets and functions.

   (a) State, without proof, what is the product in **C** of two objects X and Y.

   (b) Show by example that there are objects X and Y in **C** for which there is no exponential and hence that **C** is not a cartesian closed category.

4. [In this question I use the notation $X \xrightarrow{\mathrm{inl}_{X,Y}} X + Y \xleftarrow{\mathrm{inr}_{X,Y}} Y$ for the coproduct (Lecture 4) of two object X and Y in a category, since it will be clearer to make explicit the objects X and Y in the notation for the associated coproduct injections, $\mathrm{inl}_{X,Y}$ and $\mathrm{inr}_{X,Y}$.]

   A category **C** is *distributive* if it has all binary products and binary coproducts, and for all objects $X, Y, Z \in \mathbf{C}$, (using the defining property of the coproduct $X \times Y \xrightarrow{\mathrm{inl}_{X\times Y, X\times Z}} (X \times Y) + (X \times Z) \xleftarrow{\mathrm{inr}_{X\times Y, X\times Z}} X \times Z$), the unique morphism $\delta_{X,Y,Z} : (X \times Y) + (X \times Z) \to X \times (Y + Z)$ that makes the following diagram commute



(3)

   is an isomorphism.

   (a) Using the usual product and coproduct constructs in the category **Set** of sets and functions, show that it is a distributive category.

   (b) Give, with justification, an example of a category with binary products and coproducts that is not distributive.

   (c) If **C** is a distributive category and 0 is an initial object in **C**, prove that for all $X \in \mathbf{C}$, the unique morphism $0 \to X \times 0$ is an isomorphism.

5. A category **C** is called *locally finite* if for all $X, Y \in \mathrm{obj}\,\mathbf{C}$, the set of morphisms $\mathbf{C}(X, Y)$ is finite. **C** is said to be *finite* if it is both locally finite and $\mathrm{obj}\,\mathbf{C}$ is finite.

   (a) Prove that any finite category with binary products is a pre-order, that is, there is at most one morphism between any pair of objects. [Hint: if $f, g : X \to Y$ were distinct, use them to construct too large a number of morphisms from X to the product Y^n of Y with itself n (> 0) times, for some suitable some number n.]

   (b) Is every locally finite category with binary products a pre-order? (Either prove it, or give a counterexample.)