University of Cambridge
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Category Theory
Exercise Sheet 1
by Andrew Pitts

1. (a) Show that the sets $2 = \{0, 1\}$ and $3 = \{0, 1, 2\}$ are not isomorphic in the category **Set** of sets and functions.

   (b) Let P be the pre-ordered set with underlying set $\{0, 1\}$ and pre-order: $0 \leq 0$, $1 \leq 1$. Let Q be the pre-ordered set with the same underlying set and pre-order: $0 \leq 0$, $0 \leq 1$, $1 \leq 1$. Show that P and Q are not isomorphic in the category **Preord** of pre-ordered sets and monotone functions.

   (c) Why are the sets $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (integers) and $\mathbb{Q}$ (rational numbers) isomorphic in **Set**? Regarding them as pre-ordered sets via the usual ordering on numbers, show that they are not isomorphic in **Preord**. [Hint: recall that $\mathbb{Q}$ has the property that for any two distinct elements there is a third distinct element lying between them in the ordering.]

2. Let **C** be a category and let $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$ be morphisms in **C**.

   (a) Prove that if f and g are both isomorphisms, with inverses f^{-1} and g^{-1} respectively, then $g \circ f$ is an isomorphism and its inverse is $f^{-1} \circ g^{-1}$.

   (b) Prove that if f and $g \circ f$ are both isomorphisms, then so is g.

   (c) If $g \circ f$ is an isomorphism, does that necessarily imply that either of f or g are isomorphisms?

3. Let **Mat** be a category whose objects are all the non-zero natural numbers $1, 2, 3, \ldots$ and whose morphisms $M \in \mathbf{Mat}(m, n)$ are $m \times n$ matrices with real number entries. If composition is given by matrix multiplication, what are the identity morphisms? Give an example of an isomorphism in **Mat** that is not an identity. Can two object m and n be isomorphic in **Mat** if $m \neq n$?

4. Let **C** be a category. A morphism $f : X \to Y$ in **C** is called a *monomorphism*, if for every object $Z \in \mathbf{C}$ and every pair of morphisms $g, h : Z \to X$ we have

$$f \circ g = f \circ h \;\Rightarrow\; g = h$$

   It is called a *split monomorphism* if there is some morphism $g : Y \to X$ with $g \circ f = \mathrm{id}_X$, in which case we say that g is a *left inverse* for f.

   (a) Prove that every isomorphism is a split monomorphism and that every split monomorphism is a monomorphism.

   (b) Prove that if $f : X \to Y$ and $g : Y \to Z$ are monomorphisms, then $g \circ f : X \to Z$ is a monomorphism.

   (c) Prove that if $f : X \to Y$ and $g : Y \to Z$ are morphisms in **C**, and $g \circ f$ is a monomorphism, then f is a monomorphism.

(d) Characterize the monomorphisms in the category **Set** of sets and functions. Is every monomorphism in **Set** a split monomorphism?

(e) By considering the category **Set**, show that a split monomorphism can have more than one left inverse.

(f) Regarding a pre-ordered set $(P, \leq)$ as a category, which of its morphisms are monomorphisms and which are split monomorphisms?

5. The dual of *monomorphism* is called *epimorphism*: a morphism $f : X \to Y$ in $\mathbf{C}$ is an epimorphism iff $f \in \mathbf{C}^{\mathrm{op}}(Y, X)$ is a monomorphism in $\mathbf{C}^{\mathrm{op}}$.

   (a) Show that $f \in \mathbf{Set}(X, Y)$ is an epimorphism iff f is a surjective function.

   (b) Regarding a pre-ordered set $(P, \leq)$ as a category, which of its morphisms are epimorphisms?

   (c) Give an example of a category containing a morphism that is both an epimorphism and a monomorphism, but not an isomorphism. [Hint: consider your answers to (4f) and (5b).]

6. Let $\mathbf{C}$ be the category the following category:

   - $\mathbf{C}$-objects are triples (X, x_0, x_s) where $X \in \mathbf{Set}$, $x_0 \in X$ and $x_s \in \mathbf{Set}(X, X)$;
   - $\mathbf{C}$-morphisms $f \in \mathbf{C}((X, x_0, x_s), (Y, y_0, y_s))$ are functions $f \in \mathbf{Set}(X, Y)$ satisfying $f\, x_0 = y_0$ and $f \circ x_s = y_s \circ f$;
   - composition and identities are as for the category **Set**.

   (a) Show that $\mathbf{C}$ has a terminal object.

   (b) Show that $\mathbf{C}$ has an initial object whose underlying set is the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of natural numbers.

7. In a category $\mathbf{C}$ with a terminal object 1, a morphism $p : 1 \to X$ is called a *point* (or *global element*) of the object X. $\mathbf{C}$ is said to be *well-pointed* if for all objects $X, Y \in \mathbf{C}$, two morphisms $f, g : X \to Y$ are equal if their compositions with all points of X are equal:

$$(\forall p \in \mathbf{C}(1, X),\ f \circ p = g \circ p) \implies f = g \tag{1}$$

   (a) Show that **Set** is well-pointed.

   (b) Is the opposite category $\mathbf{Set}^{\mathrm{op}}$ well-pointed? [Hint: observe that the left-hand side of the implication in (1) is vacuously true in the case that $\mathbf{C}(1, X)$ is empty.]