

Type Systems

Lecture 10: Classical Logic and Continuation-Passing Style

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Proof (and Refutation) Terms

Propositions	$A ::= \top \mid A \wedge B \mid \perp \mid A \vee B \mid \neg A$
True contexts	$\Gamma ::= \cdot \mid \Gamma, x : A$
False contexts	$\Delta ::= \cdot \mid \Delta, u : A$
Values	$e ::= \langle \rangle \mid \langle e, e' \rangle \mid L e \mid R e \mid \text{not}(k)$ $\mid \mu u : A. c$
Continuations	$k ::= [] \mid [k, k'] \mid \text{fst } k \mid \text{snd } k \mid \text{not}(e)$ $\mid \mu x : A. c$
Contradictions	$c ::= \langle e \mid_A k \rangle$

Expressions — Proof Terms

$$\frac{x : A \in \Gamma}{\Gamma; \Delta \vdash x : A \text{ true}} \text{ HYP}$$

(No rule for \perp)

$$\frac{}{\Gamma; \Delta \vdash \langle \rangle : \top \text{ true}} \text{ TP}$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \quad \Gamma; \Delta \vdash e' : B \text{ true}}{\Gamma; \Delta \vdash \langle e, e' \rangle : A \wedge B \text{ true}} \wedge P$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true}}{\Gamma; \Delta \vdash L e : A \vee B \text{ true}} \vee P_1$$

$$\frac{\Gamma; \Delta \vdash e : B \text{ true}}{\Gamma; \Delta \vdash R e : A \vee B \text{ true}} \vee P_2$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \text{not}(k) : \neg A \text{ true}} \neg P$$

Continuations — Refutation Terms

$$\frac{x : A \in \Delta}{\Gamma; \Delta \vdash x : A \text{ false}} \text{HYP}$$

$$\text{(No rule for } \top \text{)} \quad \frac{}{\Gamma; \Delta \vdash [] : \perp \text{ false}} \perp\text{R}$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false} \quad \Gamma; \Delta \vdash k' : B \text{ false}}{\Gamma; \Delta \vdash [k, k'] : A \vee B \text{ false}} \vee\text{R}$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \text{fst } k : A \wedge B \text{ false}} \wedge\text{R}_1$$

$$\frac{\Gamma; \Delta \vdash k : B \text{ false}}{\Gamma; \Delta \vdash \text{snd } k : A \wedge B \text{ false}} \wedge\text{R}_2$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true}}{\Gamma; \Delta \vdash \text{not}(e) : \neg A \text{ false}} \neg\text{R}$$

Contradictions

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \quad \Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \langle e \mid_A k \rangle \text{ contr}} \text{ CONTR}$$

$$\frac{\Gamma; \Delta, u : A \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu u : A. c : A \text{ true}}$$

$$\frac{\Gamma, x : A; \Delta \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu x : A. c : A \text{ false}}$$

Operational Semantics

$$\langle \langle e_1, e_2 \rangle \mid_{A \wedge B} \text{fst } k \rangle \mapsto \langle e_1 \mid_A k \rangle$$

$$\langle \langle e_1, e_2 \rangle \mid_{A \wedge B} \text{snd } k \rangle \mapsto \langle e_2 \mid_B k \rangle$$

$$\langle L e \mid_{A \vee B} [k_1, k_2] \rangle \mapsto \langle e \mid_A k_1 \rangle$$

$$\langle R e \mid_{A \vee B} [k_1, k_2] \rangle \mapsto \langle e \mid_B k_2 \rangle$$

$$\langle \text{not}(k) \mid_{\neg A} \text{not}(e) \rangle \mapsto \langle e \mid_A k \rangle$$

$$\langle \mu u : A. c \mid_A k \rangle \mapsto [k/u]c$$

$$\langle e \mid_A \mu x : A. c \rangle \mapsto [e/x]c$$

Type Safety?

Preservation If $\cdot; \cdot \vdash c$ contr and $c \rightsquigarrow c'$ then $\cdot; \cdot \vdash c'$ contr.

Proof By *case analysis* on evaluation derivations!

(We don't even need induction!)

Type Preservation

$\langle\langle e_1, e_2 \rangle |_{A \wedge B} \text{fst } k \rangle \rightsquigarrow \langle e_1 |_A k \rangle$

Assumption

$$\frac{\overbrace{\Gamma; \Delta \vdash \langle e_1, e_2 \rangle : A \wedge B \text{ true}}^{(1)} \quad \overbrace{\Gamma; \Delta \vdash \text{fst } k : A \wedge B \text{ false}}^{(2)}}{\Gamma; \Delta \vdash \langle\langle e_1, e_2 \rangle |_{A \wedge B} \text{fst } k \rangle \text{ contr}}$$

Assumption

$$\frac{\overbrace{\Gamma; \Delta \vdash e_1 : A \text{ true}}^{(3)} \quad \Gamma; \Delta \vdash e_2 : B \text{ true}}{\Gamma; \Delta \vdash \langle e_1, e_2 \rangle : A \wedge B \text{ true}} \wedge P$$

Analysis of (1)

$$\frac{\overbrace{\Gamma; \Delta \vdash k : A \text{ false}}^{(4)}}{\Gamma; \Delta \vdash \text{fst } k : A \wedge B \text{ false}} \wedge R_1$$

Analysis of (2)

$\therefore \cdot \vdash \langle e_1 |_A k \rangle \text{ contr}$

By rule on (3), (

Progress?

Progress? If $\cdot; \cdot \vdash c$ contr then $c \rightsquigarrow c'$ (or c final).

Proof:

1. A closed term c is a contradiction
2. Hopefully, there aren't any contradictions!
3. So this theorem is vacuous (assuming classical logic is consistent)

Making Progress Less Vacuous

Propositions $A ::= \dots \mid \text{ans}$
Values $e ::= \dots \mid \text{halt}$
Continuations $k ::= \dots \mid \text{done}$

$\Gamma; \Delta \vdash \text{halt} : \text{ans true}$

$\Gamma; \Delta \vdash \text{done} : \text{ans false}$

Progress If $\cdot; \cdot \vdash c$ contr then $c \rightsquigarrow c'$ or $c = \langle \text{halt} \mid_{\text{ans}} \text{done} \rangle$.

Proof By induction on typing derivations

The Price of Progress

$$\frac{\frac{\Gamma; \Delta, A \vdash \text{ans true} \quad \Gamma; \Delta, A \vdash \text{ans false}}{\Gamma; \Delta, A \vdash \text{contr}}}{\Gamma; \Delta \vdash A \text{ true}} \quad \frac{\frac{\Gamma, A; \Delta \vdash \text{ans true} \quad \Gamma, A; \Delta \vdash \text{ans false}}{\Gamma, A; \Delta \vdash \text{contr}}}{\Gamma; \Delta \vdash A \text{ false}}}{\Gamma; \Delta \vdash \neg A \text{ true}}}{\Gamma; \Delta \vdash A \wedge \neg A \text{ true}}$$

- As a term:

$\langle \mu u : A. \langle \text{halt} \mid \text{done} \rangle, \text{not}(\mu x : A. \langle \text{halt} \mid \text{done} \rangle) \rangle$

- Adding a halt configuration makes classical logic inconsistent – $A \wedge \neg A$ is derivable

Embedding Classical Logic into Intuitionistic Logic

- Intuitionistic logic has a clean computational reading
- Classical logic *almost* has a clean computational reading
- Q: Is there any way to equip classical logic with computational meaning?
- A: Embed classical logic *into* intuitionistic logic

The Double Negation Translation

- Fix an intuitionistic proposition p
- Define “quasi-negation” $\sim X$ as $X \rightarrow p$
- Now, we can define a translation on types as follows:

$$(\neg A)^\circ = \sim A^\circ$$

$$\top^\circ = 1$$

$$(A \wedge B)^\circ = A^\circ \times B^\circ$$

$$\perp^\circ = p$$

$$(A \vee B)^\circ = \sim\sim(A^\circ + B^\circ)$$

Triple-Negation Elimination

In general, $\neg\neg X \rightarrow X$ is not derivable constructively. However, the following is derivable:

Lemma For all X , there is a function $\text{tne} : (\sim\sim\sim X) \rightarrow \sim X$

$$\begin{array}{c} \frac{\dots \vdash q : X \rightarrow p \quad \dots \vdash x : X}{k : \sim\sim\sim X, x : X, q : \sim X \vdash qx : p} \\ \dots \quad \frac{k : \sim\sim\sim X, x : X \vdash \lambda q. qx : \sim\sim X}{k : \sim\sim\sim X, x : X \vdash k(\lambda q. qx) : p} \\ \frac{k : \sim\sim\sim X \vdash \lambda x. k(\lambda q. qa) : \sim X}{\dots \vdash \underbrace{\lambda k. \lambda a. k(\lambda q. qa)}_{\text{tne}} : (\sim\sim\sim X) \rightarrow \sim X} \end{array}$$

Intuitionistic Double Negation Elimination

Lemma For all A , there is a term dne_A such that

$$\cdot \vdash \text{dne}_A : \sim\sim A^\circ \rightarrow A^\circ$$

Proof By induction on A .

$$\begin{aligned} \text{dne}_\top &= \lambda x. \langle \rangle \\ \text{dne}_{A \wedge B} &= \lambda p. \left\langle \begin{array}{l} \text{dne}_A (\lambda k. q (\lambda p. k (\text{fst } p))), \\ \text{dne}_B (\lambda k. q (\lambda p. k (\text{snd } p))) \end{array} \right\rangle \\ \text{dne}_\perp &= \lambda q. q (\lambda x. x) \\ \text{dne}_{A \vee B} &= \lambda q : \underbrace{\sim\sim\sim\sim (A^\circ \vee B^\circ)}_{(A \vee B)^\circ}. \text{tne } q \\ \text{dne}_{\neg A} &= \lambda q : \underbrace{\sim\sim (\sim A^\circ)}_{(\neg A)^\circ}. \text{tne } q \end{aligned}$$

Double Negation Elimination for \perp

$$\frac{\frac{\frac{}{q : (p \rightarrow p) \rightarrow p \vdash q : (p \rightarrow p) \rightarrow p}}{q : (p \rightarrow p) \rightarrow p, x : p \vdash x : p}}{q : (p \rightarrow p) \rightarrow p \vdash \lambda x : p. x : p}}{q : (p \rightarrow p) \rightarrow p \vdash q(\lambda x : p. x) : p}}{\cdot \vdash \lambda q : (p \rightarrow p) \rightarrow p. q(\lambda x : p. x) : ((p \rightarrow p) \rightarrow p) \rightarrow p}}{\cdot \vdash \lambda q : \sim\sim p. q(\lambda x : p. x) : \sim\sim p \rightarrow p}}{\cdot \vdash \lambda q : \sim\sim \perp^\circ. q(\lambda x : p. x) : \sim\sim \perp^\circ \rightarrow \perp^\circ}}$$

Theorem Classical terms embed into intuitionistic terms:

1. If $\Gamma; \Delta \vdash e : A$ true then $\Gamma^\circ, \sim\Delta \vdash e^\circ : A^\circ$.
2. If $\Gamma; \Delta \vdash k : A$ false then $\Gamma^\circ, \sim\Delta \vdash k^\circ : \sim A^\circ$.
3. If $\Gamma; \Delta \vdash c$ contr then $\Gamma^\circ, \sim\Delta \vdash c^\circ : p$.

Proof By induction on derivations – but first, we have to define the translation!

Translating Value Contexts:

$$\begin{aligned}(\cdot)^\circ &= \cdot \\ (\Gamma, x : A)^\circ &= \Gamma^\circ, x : A^\circ\end{aligned}$$

Translating Continuation Contexts:

$$\begin{aligned}\sim(\cdot) &= \cdot \\ \sim(\Gamma, x : A) &= \sim\Gamma, x : \sim A^\circ\end{aligned}$$

Translating Contradictions

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \quad \Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \langle e \mid_A k \rangle \text{ contr}} \text{ CONTR}$$

Define:

$$\langle e \mid_A k \rangle = k^\circ e^\circ$$

Translating (Most) Expressions

$$x^\circ = x$$

$$\langle \rangle^\circ = \langle \rangle$$

$$\langle e_1, e_2 \rangle^\circ = \langle e_1^\circ, e_2^\circ \rangle$$

$$(L e)^\circ = \lambda k : \sim(A^\circ + B^\circ). k (L e^\circ)$$

$$(R e)^\circ = \lambda k : \sim(A^\circ + B^\circ). k (R e^\circ)$$

$$(\text{not}(k))^\circ = k^\circ$$

Translating (Most) Continuations

$$\begin{aligned}x^\circ &= x \\ []^\circ &= \lambda x : \text{ans. } x \\ [k_1, k_2]^\circ &= \lambda k : \sim\sim(A^\circ + B^\circ). \\ &\quad k (\lambda i : A^\circ + B^\circ. \\ &\quad\quad \text{case}(i, Lx \rightarrow k_1^\circ x, Ry \rightarrow k_2^\circ y)) \\ (\text{fst } k)^\circ &= \lambda p : (A^\circ \times B^\circ). k^\circ (\text{fst } p) \\ (\text{snd } k)^\circ &= \lambda p : (A^\circ \times B^\circ). k^\circ (\text{snd } p) \\ (\text{not}(e))^\circ &= \lambda k : \underbrace{\sim A^\circ}_{(-A)^\circ}. k e^\circ\end{aligned}$$

Translating Proof by Contradiction

$$\frac{\Gamma; \Delta, u : A \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu u : A. c : A \text{ true}}$$

- 1 $\Gamma^\circ, \sim(\Delta, u : A) \vdash c^\circ : p$ Assumption
- 2 $\Gamma^\circ, \sim\Delta, u : \sim A^\circ \vdash c^\circ : p$ Def. of \sim on contexts
- 3 $\Gamma^\circ, \sim\Delta \vdash \lambda u : \sim A^\circ. c^\circ : \sim A^\circ \rightarrow p$ $\rightarrow I$
- 4 $\Gamma^\circ, \sim\Delta \vdash \lambda u : \sim A^\circ. c^\circ : \sim\sim A^\circ$ Def. of \sim on types
- 5 $\Gamma^\circ, \sim\Delta \vdash \text{dne}_A(\lambda u : u : \sim A^\circ. c^\circ) : A^\circ$ $\rightarrow E$

So we define

$$(\mu u : A. c)^\circ = \text{dne}_A(\lambda u : \sim A^\circ. c^\circ)$$

Translating Refutation by Contradiction

$$\frac{\Gamma, x : A; \Delta \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu x : A. c : A \text{ false}}$$

1. We assume $\Gamma, x : A^\circ, \sim\Delta \vdash c^\circ : p$
2. So $\Gamma^\circ, x : A^\circ, \sim\Delta \vdash c^\circ : p$
3. So $\Gamma^\circ, \sim\Delta \vdash \lambda x : A^\circ. c^\circ : A^\circ \rightarrow p$
4. So $\Gamma^\circ, \sim\Delta \vdash \lambda x : A^\circ. c^\circ : \sim A^\circ$

So we define

$$(\mu x : A. c)^\circ = \lambda x : A^\circ. c^\circ$$

Consequences

- We now have a proof that every classical proof has a corresponding intuitionistic proof
- So classical logic is a *subsystem* of intuitionistic logic
- Because intuitionistic logic is consistent, so is classical logic
- Classical logic can inherit operational semantics from intuitionistic logic!

Many Different Embeddings

- Many different translations of classical logic were discovered many times
 - Gerhard Gentzen and Kurt Gödel
 - Andrey Kolmogorov
 - Valery Glivenko
 - Sigekatu Kuroda
- The key property is to show that $\sim\sim A^\circ \rightarrow A^\circ$ holds.

The Gödel-Gentzen Translation

Now, we can define a translation on types as follows:

$$\begin{aligned}\neg A^\circ &= \sim A^\circ \\ \top^\circ &= 1 \\ (A \wedge B)^\circ &= A^\circ \times B^\circ \\ \perp^\circ &= p \\ (A \vee B)^\circ &= \sim(\sim A^\circ \times \sim B^\circ)\end{aligned}$$

- This uses a different de Morgan duality for disjunction

The Kolmogorov Translation

Now, we can define another translation on types as follows:

$$\begin{aligned}\neg A^\bullet &= \sim\sim\sim A^\bullet \\ A \supset B^\bullet &= \sim\sim(A^\bullet \rightarrow B^\bullet) \\ \top^\bullet &= \sim\sim 1 \\ (A \wedge B)^\bullet &= \sim\sim(A^\bullet \times B^\bullet) \\ \perp^\bullet &= \sim\sim \perp \\ (A \vee B)^\bullet &= \sim\sim(A^\bullet + B^\bullet)\end{aligned}$$

- Uniformly stick a double-negation in front of each connective.
- Deriving $\sim\sim A^\bullet \rightarrow A^\bullet$ is particularly easy:
 - The **tne** term will always work!

Implementing Classical Logic Axiomatically

- The proof theory of classical logic is elegant
- It is also very awkward to use:
 - Binding only arises from proof by contradiction
 - Difficult to write nested computations
 - Continuations/stacks are always explicit
- Functional languages make the stack implicit
- Can we make the continuations implicit?

The Typed Lambda Calculus with Continuations

Types	$X ::= 1 \mid X \times Y \mid 0 \mid X + Y \mid X \rightarrow Y \mid \neg X$
Terms	$e ::= x \mid \langle \rangle \mid \langle e, e \rangle \mid \text{fst } e \mid \text{snd } e$ $\mid \text{abort} \mid L e \mid R e \mid \text{case}(e, Lx \rightarrow e', Ry \rightarrow e'')$ $\mid \lambda x : X. e \mid e e'$ $\mid \text{throw}(e, e') \mid \text{letcont } x. e$
Contexts	$\Gamma ::= \cdot \mid \Gamma, x : X$

$$\frac{}{\Gamma \vdash \langle \rangle : 1} \text{1I}$$

$$\frac{\Gamma \vdash e : X \quad \Gamma \vdash e' : Y}{\Gamma \vdash \langle e, e' \rangle : X \times Y} \text{xI}$$

$$\frac{\Gamma \vdash e : X \times Y}{\Gamma \vdash \text{fst } e : X} \text{xE}_1$$

$$\frac{\Gamma \vdash e : X \times Y}{\Gamma \vdash \text{snd } e : Y} \text{xE}_1$$

Functions and Variables

$$\frac{X : X \in \Gamma}{\Gamma \vdash x : X} \text{HYP}$$

$$\frac{\Gamma, x : X \vdash e : Y}{\Gamma \vdash \lambda x : X. e : X \rightarrow Y} \rightarrow I$$

$$\frac{\Gamma \vdash e : X \rightarrow Y \quad \Gamma \vdash e' : X}{\Gamma \vdash e e' : Y} \rightarrow E$$

Sums and the Empty Type

$$\frac{\Gamma \vdash e : X}{\Gamma \vdash L e : X + Y} +I_1$$

$$\frac{\Gamma \vdash e : Y}{\Gamma \vdash R e : X + Y} +I_2$$

$$\frac{\Gamma \vdash e : X + Y \quad \Gamma, x : X \vdash e' : Z \quad \Gamma, y : Y \vdash e'' : Z}{\Gamma \vdash \text{case}(e, Lx \rightarrow e', Ry \rightarrow e'') : Z} +E$$

(no intro for 0)

$$\frac{\Gamma \vdash e : 0}{\Gamma \vdash \text{abort } e : Z} 0E$$

Continuation Typing

$$\frac{\Gamma, u : \neg X \vdash e : X}{\Gamma \vdash \text{letcont } u : X. e : X} \text{CONT}$$

$$\frac{\Gamma \vdash e : \neg X \quad \Gamma \vdash e' : X}{\Gamma \vdash \text{throw}_Y(e, e') : Y} \text{THROW}$$

Examples

Double-negation elimination:

$$\text{dne}_X : \neg\neg X \rightarrow X$$
$$\text{dne}_X \triangleq \lambda k : \neg\neg X. \text{letcont } u : \neg X. \text{throw}(k, u)$$

The Excluded Middle:

$$t : X \vee \neg X$$
$$t \triangleq \text{letcont } u : \neg(X \vee \neg X).$$
$$\text{throw}(u, \text{R}(\text{letcont } q : \neg\neg X. \\ \text{throw}(u, \text{L}(\text{dne}_X q))))$$

Continuation-Passing Style (CPS) Translation

Type translation:

$$\begin{aligned}\neg X^\bullet &= \sim\sim\sim X^\bullet \\ X \rightarrow Y^\bullet &= \sim\sim(X^\bullet \rightarrow Y^\bullet) \\ 1^\bullet &= \sim\sim 1 \\ (X \times Y)^\bullet &= \sim\sim(X^\bullet \times Y^\bullet) \\ 0^\bullet &= \sim\sim 0 \\ (X + Y)^\bullet &= \sim\sim(X^\bullet + Y^\bullet)\end{aligned}$$

Translating contexts:

$$\begin{aligned}(\cdot)^\bullet &= \cdot \\ (\Gamma, x : A)^\bullet &= \Gamma^\bullet, x : A^\bullet\end{aligned}$$

The CPS Translation Theorem

Theorem If $\Gamma \vdash e : X$ then $\Gamma^\bullet \vdash e^\bullet : X^\bullet$.

Proof: By induction on derivations – we “just” need to define e^\bullet .

The CPS Translation

$$\begin{aligned}x^\bullet &= \lambda k. k x \\ \langle \rangle^\bullet &= \lambda k. k \langle \rangle \\ \langle e_1, e_2 \rangle^\bullet &= \lambda k. e_1^\bullet (\lambda x. e_2^\bullet (\lambda y. k (x, y))) \\ (\text{fst } e)^\bullet &= \lambda k. e^\bullet (\lambda p. k (\text{fst } p)) \\ (\text{snd } e)^\bullet &= \lambda k. e^\bullet (\lambda p. k (\text{snd } p)) \\ (L e)^\bullet &= \lambda k. e^\bullet (\lambda x. k (L x)) \\ (R e)^\bullet &= \lambda k. e^\bullet (\lambda y. k (R y)) \\ \text{case}(e, Lx \rightarrow e_1, Ry \rightarrow e_2)^\bullet &= \lambda k. e^\bullet (\lambda v. \text{case}(v, \\ &\quad Lx \rightarrow e_1^\bullet k \\ &\quad Ry \rightarrow e_2^\bullet k)) \\ (\lambda x : X. e)^\bullet &= \lambda k. k (\lambda x : X^\bullet. e^\bullet) \\ (e_1 e_2)^\bullet &= \lambda k. e_1^\bullet (\lambda f. e_2^\bullet (\lambda x. k (fx)))\end{aligned}$$

The CPS Translation for Continuations

$$(\text{letcont } u : \neg X. e)^\bullet = \lambda k. [(\lambda q. q k)/u](e^\bullet)$$

$$\text{throw}(e_1, e_2)^\bullet = \text{tne}(e_1^\bullet) e_2^\bullet$$

- The rest of the CPS translation is bookkeeping to enable these two clauses to work!

Questions

1. Give the embedding (ie, the e° and k° translations) of classical into intuitionistic logic for the Gödel-Gentzen translation. You just need to give the embeddings for sums, since that is the only case different from lecture.
2. Using the intuitionistic calculus extended with continuations, give a typed term proving *Peirce's law*:

$$((X \rightarrow Y) \rightarrow X) \rightarrow X$$