Randomised Algorithms

Lecture 4: Markov Chains and Mixing Times

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Lent 2023

UNIVERSITY OF CAMBRIDGE

Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Applications of Markov Chains in Computer Science

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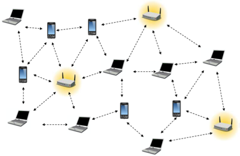
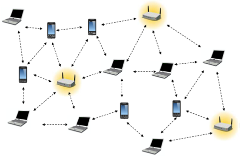


Broadcasting

Applications of Markov Chains in Computer Science



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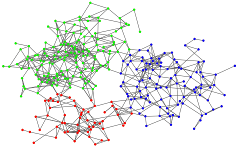

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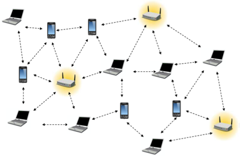

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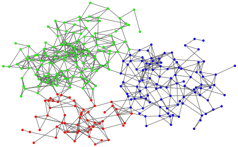

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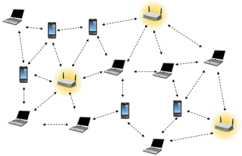

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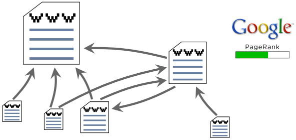

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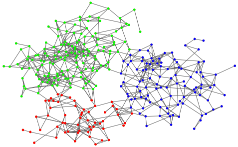

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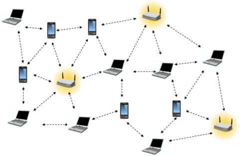

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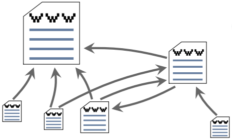

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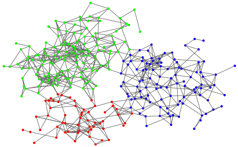

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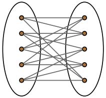
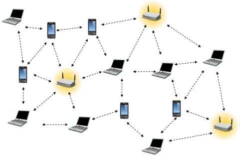


Sampling and Optimisation

Applications of Markov Chains in Computer Science



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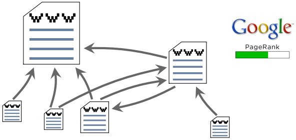

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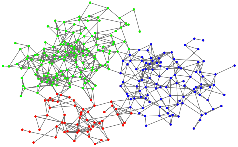

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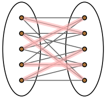

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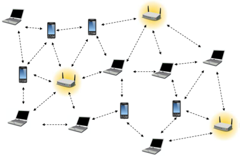

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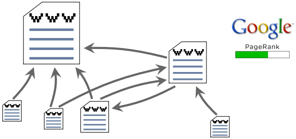

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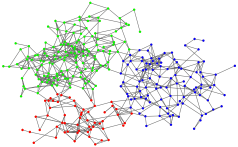

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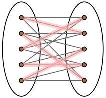

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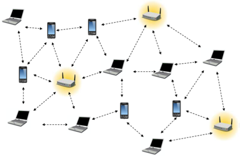

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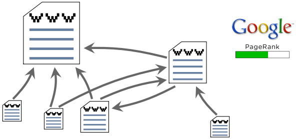

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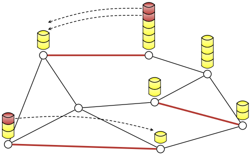
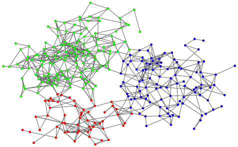


Load Balancing



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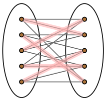

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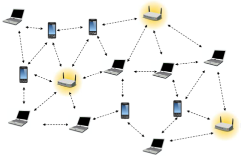

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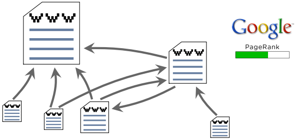

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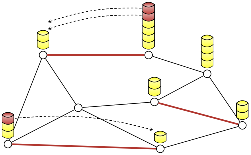

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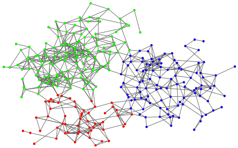

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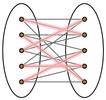

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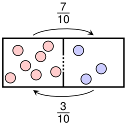

Particle Processes

Markov Chains

Markov Chain (Discrete Time and State, Time Homogeneous)

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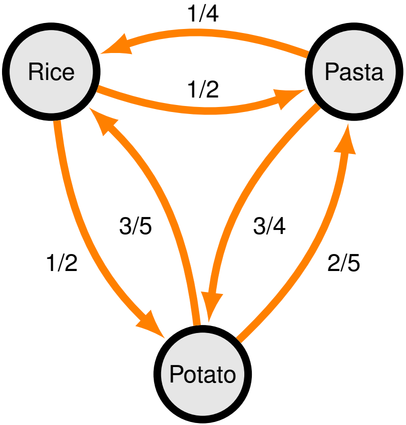
- For all $0 \leq t_1 < t_2, x \in \Omega$,

$$\mathbf{P}[X_{t_2} = x] = \sum_{y \in \Omega} \mathbf{P}[X_{t_2} = x \mid X_{t_1} = y] \cdot \mathbf{P}[X_{t_1} = y].$$

What does a Markov Chain Look Like?

Example: the carbohydrate served with lunch in the college cafeteria.



This has transition matrix:

	Rice	Pasta	Potato	
	0	$1/2$	$1/2$	Rice
$P =$	$1/4$	0	$3/4$	Pasta
	$3/5$	$2/5$	0	Potato

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$\Rightarrow$ can replace ρ by any (load) vector and view P as a balancing matrix!

Stopping and Hitting Times

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Example - College Carbs Stopping times:

✓ "We had rice yesterday"

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Some distinguish between $\tau_y^+ = \min\{t \geq 1 \colon X_t = y\}$ and $\tau_y = \min\{t \geq 0 \colon X_t = y\}$

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A Useful Identity

Hitting times are the solution to a set of linear equations:

$$h(x, y) \overset{\text{Markov Prop.}}{=} 1 + \sum_{z \in \Omega \setminus \{y\}} P(x, z) \cdot h(z, y) \qquad \forall x \neq y \in \Omega.$$

Outline

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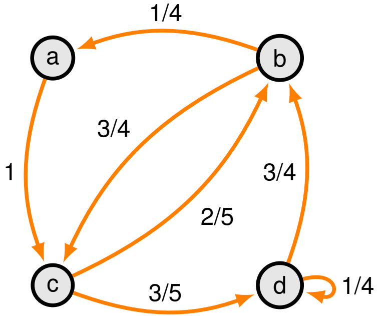

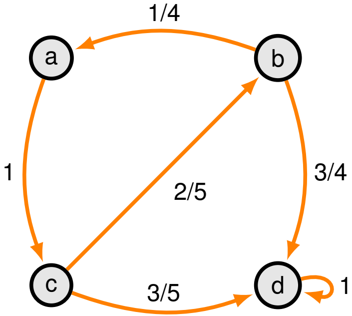

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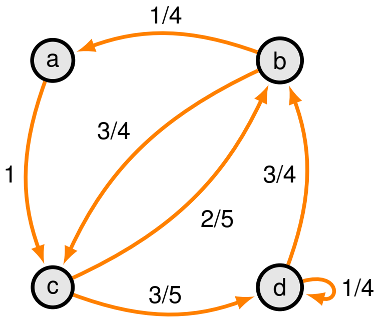

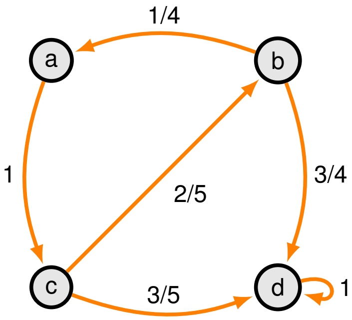

Exercise: Which of the two chains (if any) are irreducible?

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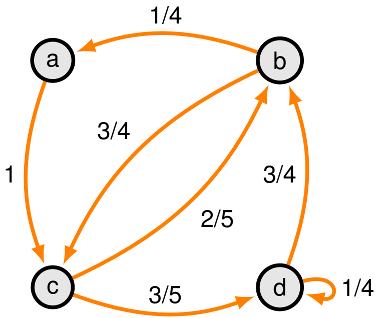

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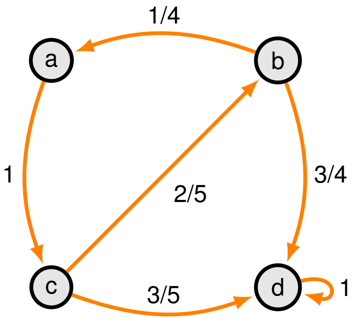

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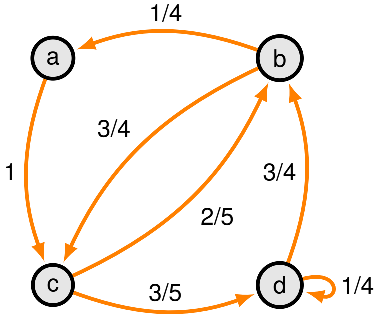

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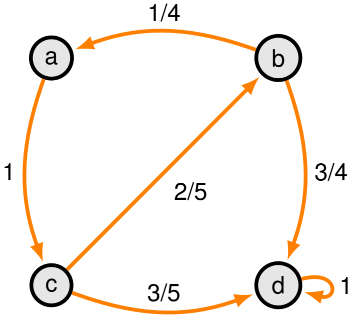

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Finite Hitting Time Theorem

For any states x and y of a finite irreducible Markov Chain $h(x, y) < \infty$.

Stationary Distribution

A probability distribution $\pi = (\pi(1), \ldots, \pi(n))$ is the stationary distribution of a Markov Chain if $\pi P = \pi$ (π is a left eigenvector with eigenvalue 1)

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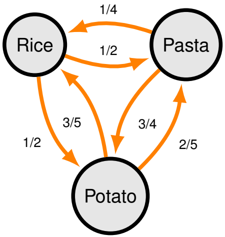

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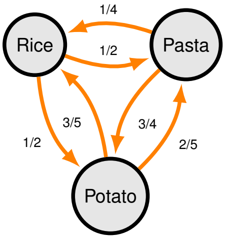

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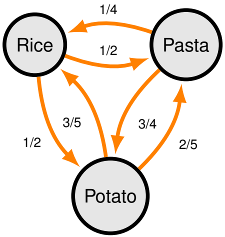

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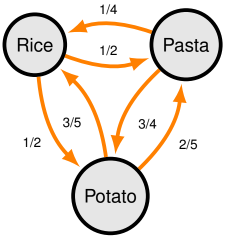

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Existence and Uniqueness of a Positive Stationary Distribution

Let P be finite, irreducible M.C., then there exists a unique probability distribution π on Ω such that $\pi = \pi P$ and $\pi(x) = 1/h(x,x) > 0$, $\forall x \in \Omega$.

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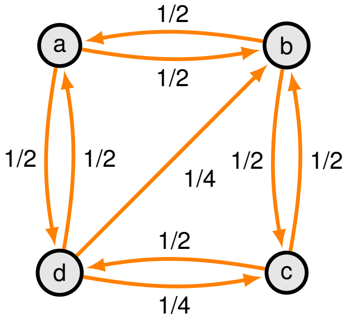

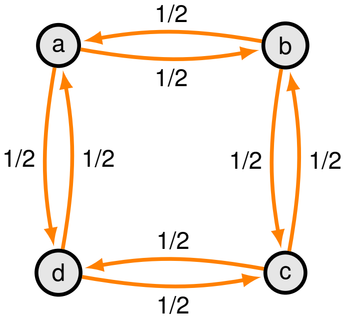

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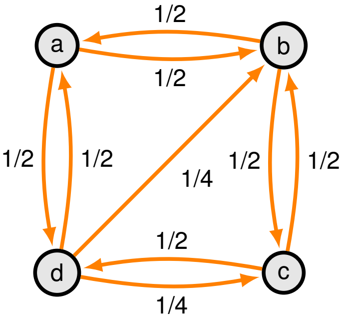

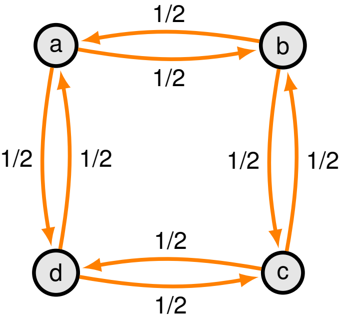

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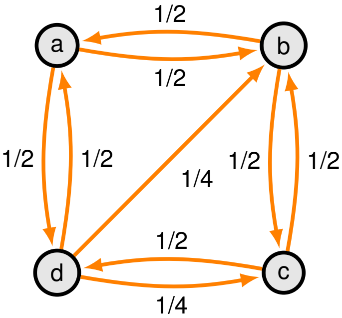

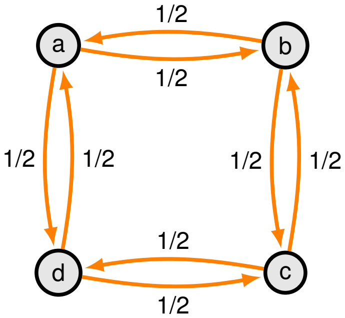

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- We will prove a simpler version of the Convergence Theorem after introducing Spectral Graph Theory.

Convergence to Stationarity (Example)

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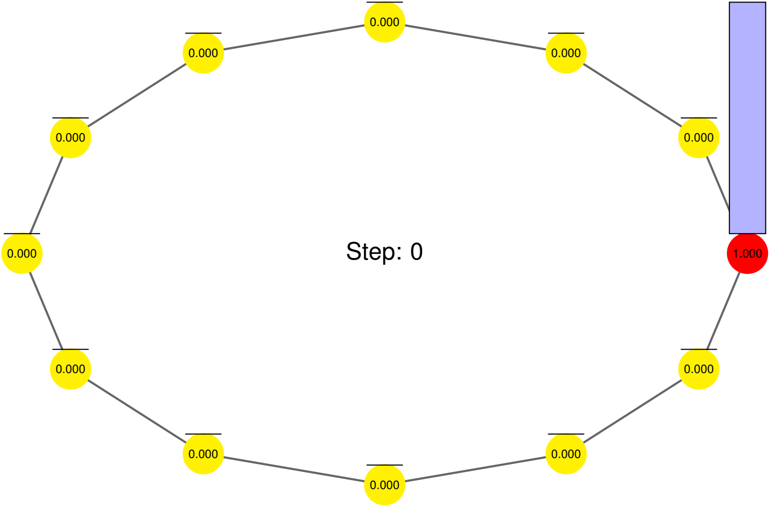

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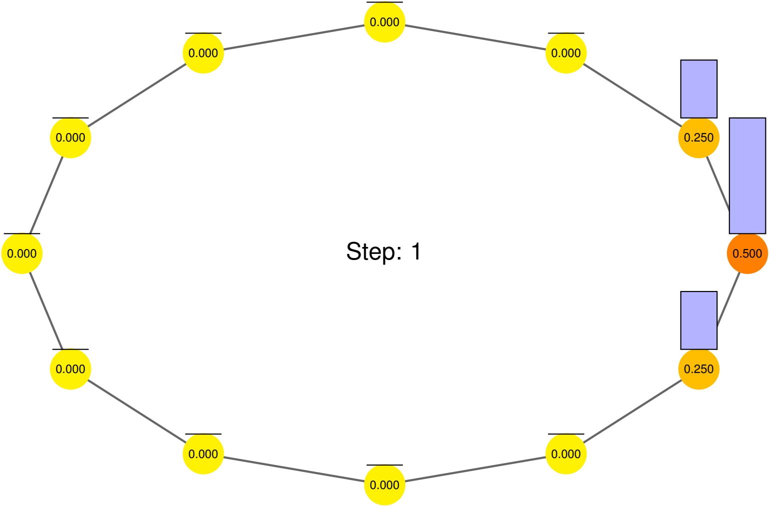

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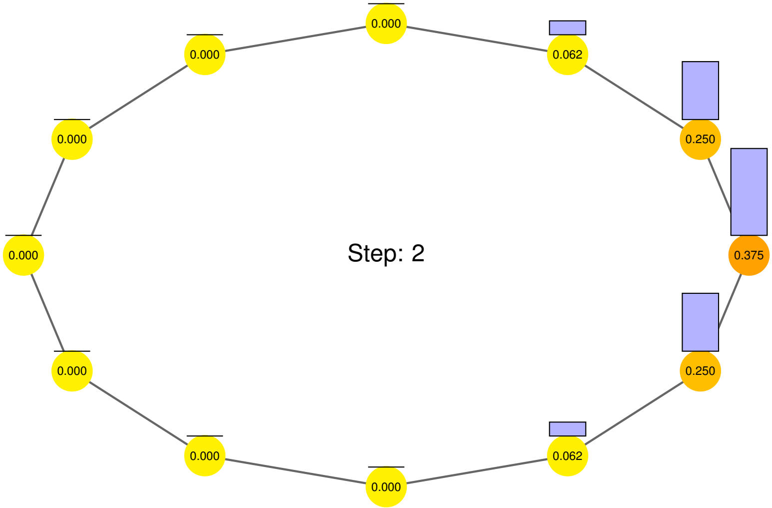

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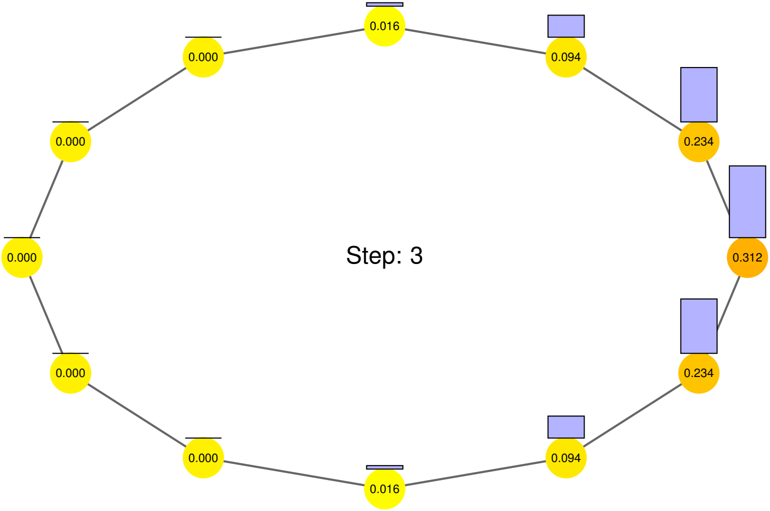

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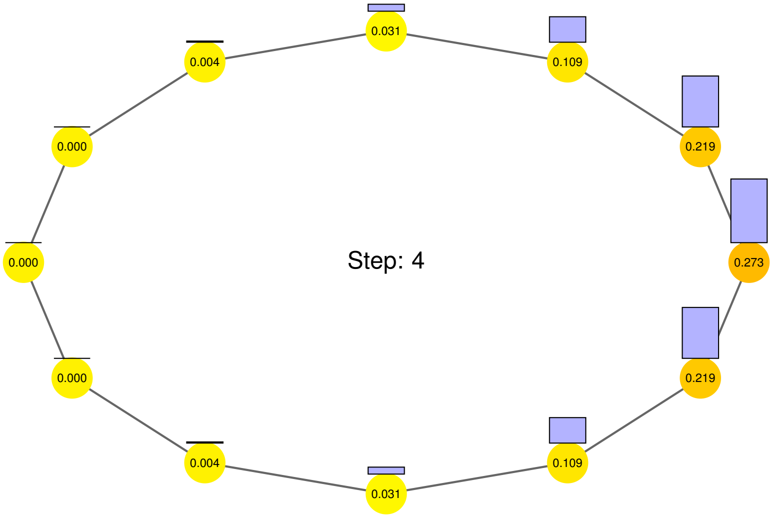

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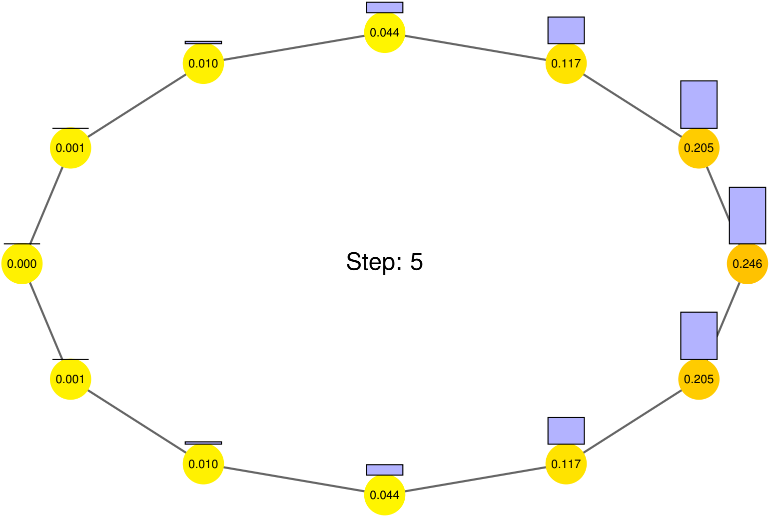

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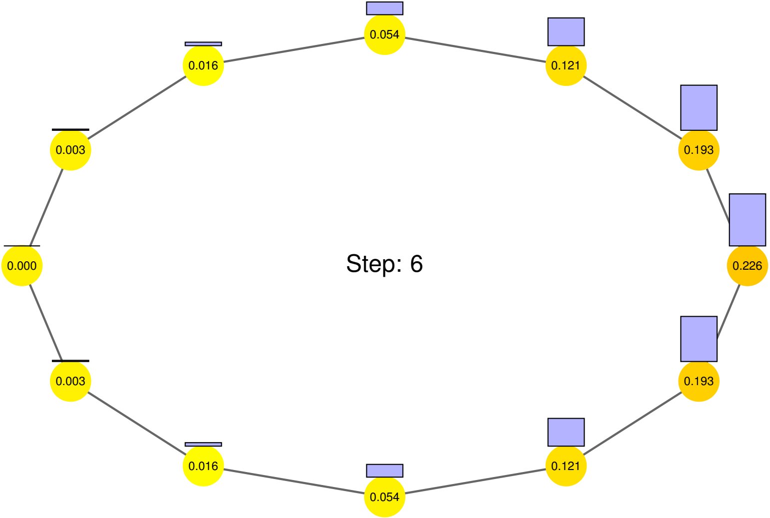

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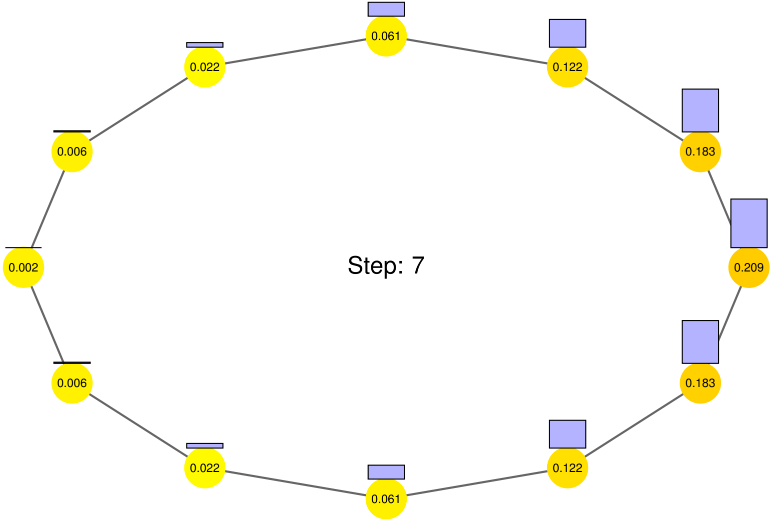

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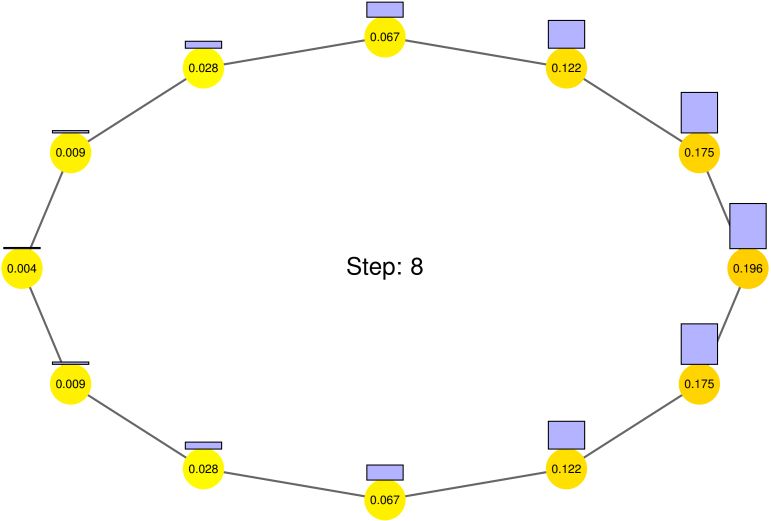

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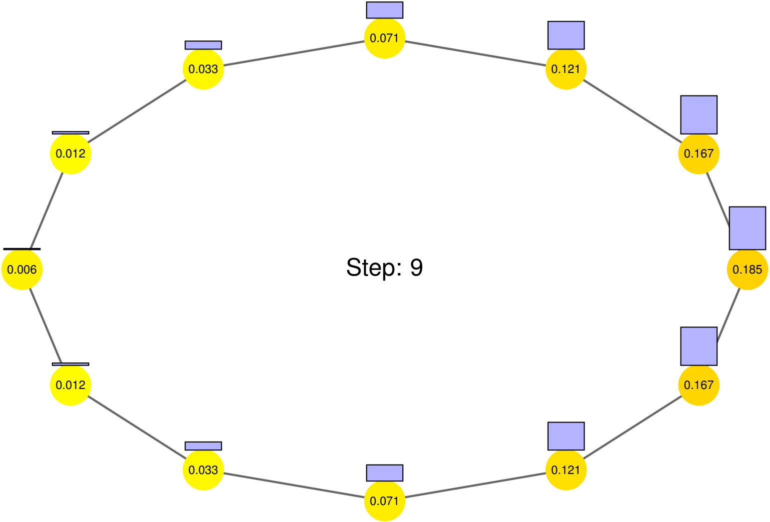

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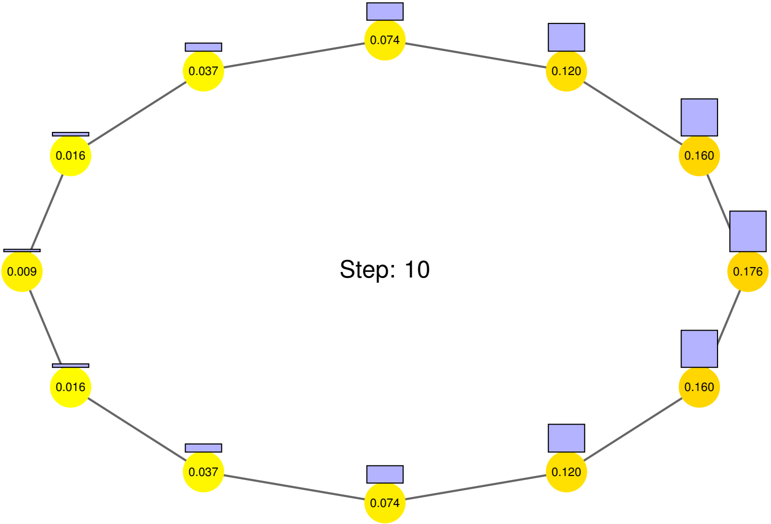

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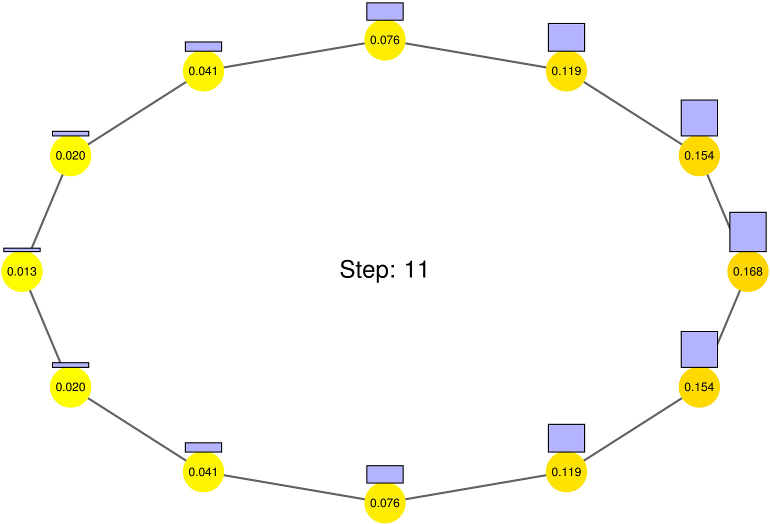

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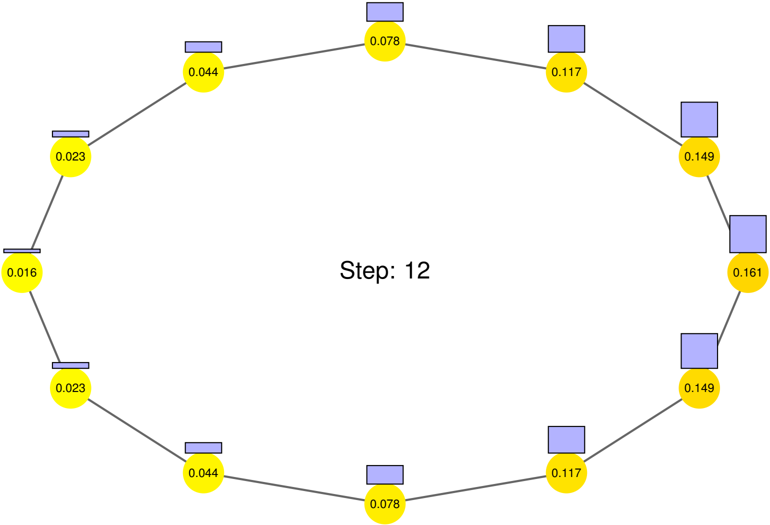

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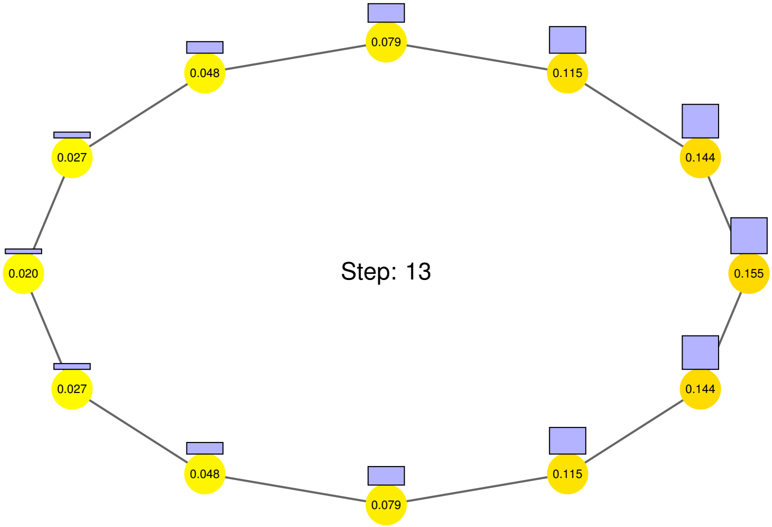

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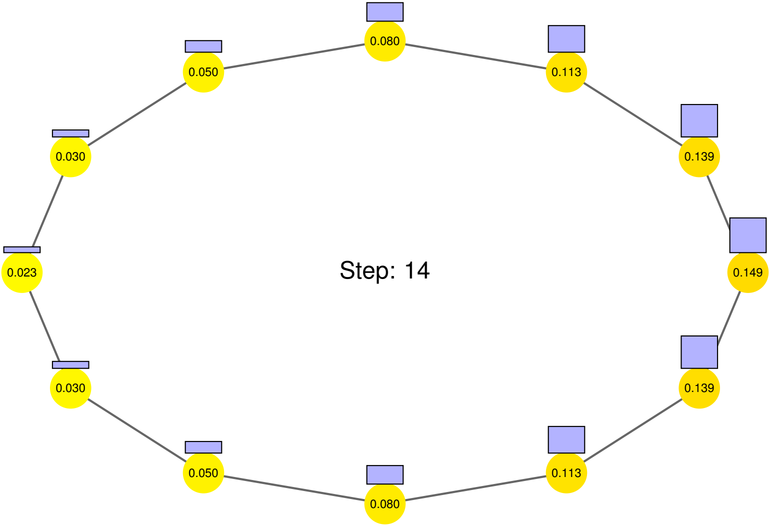

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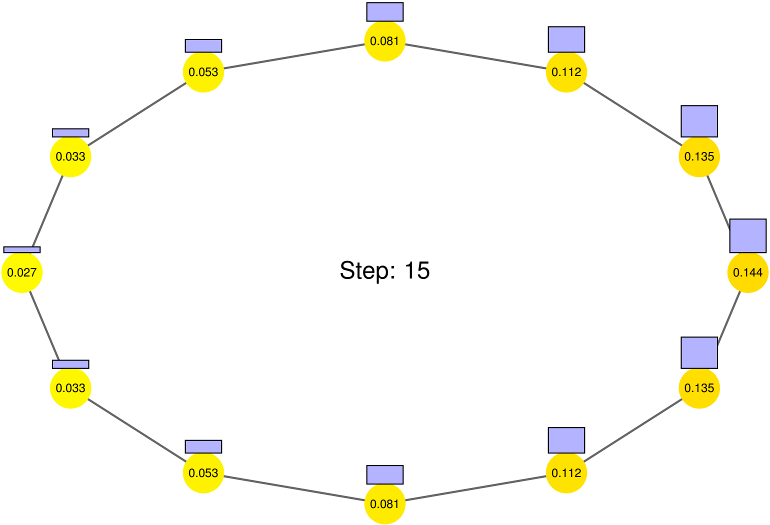

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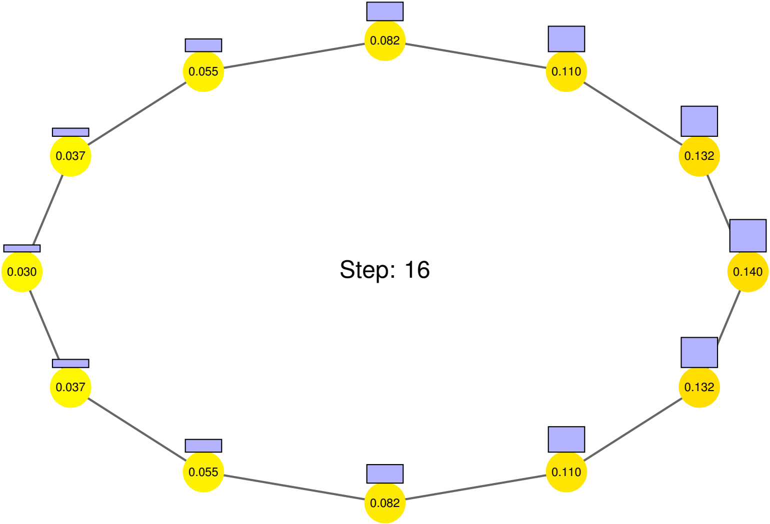

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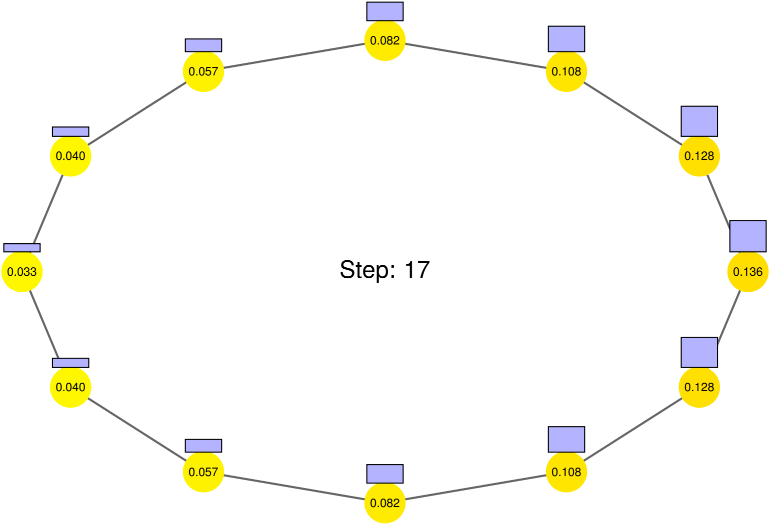

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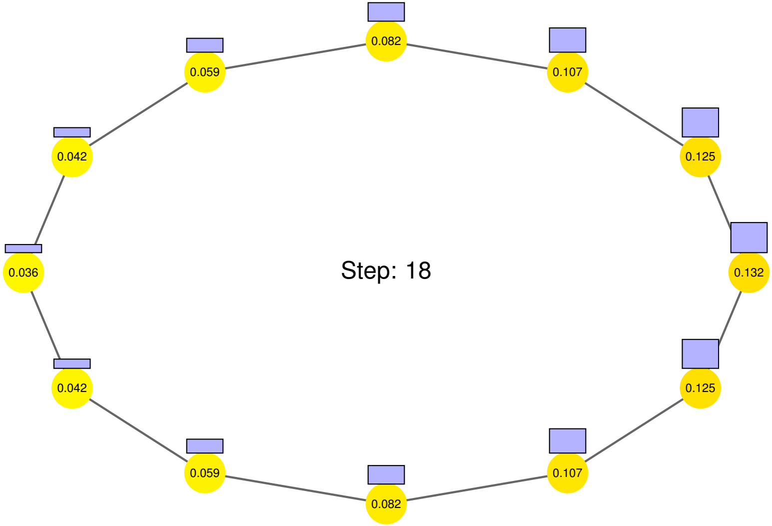

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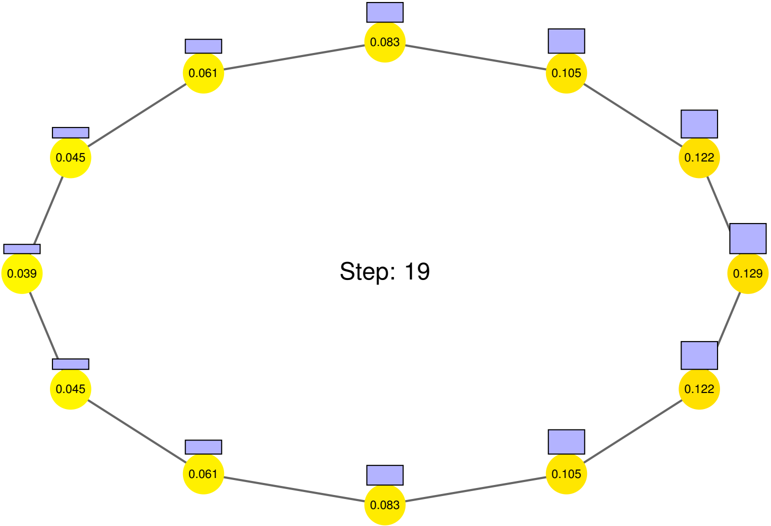

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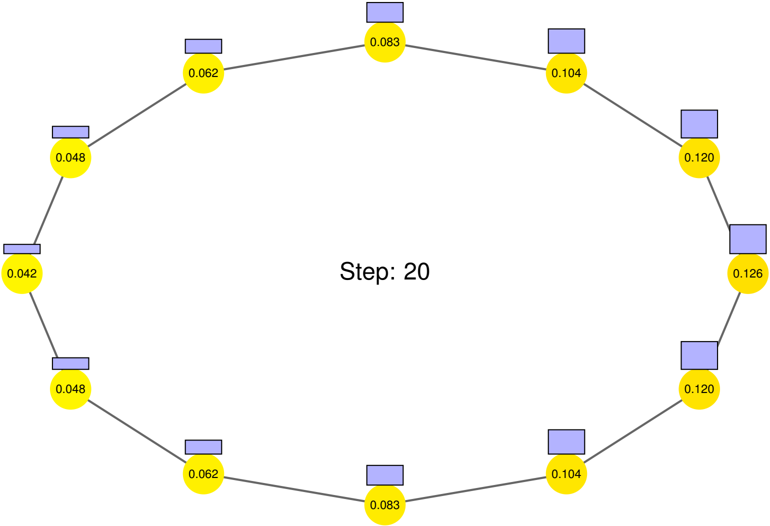

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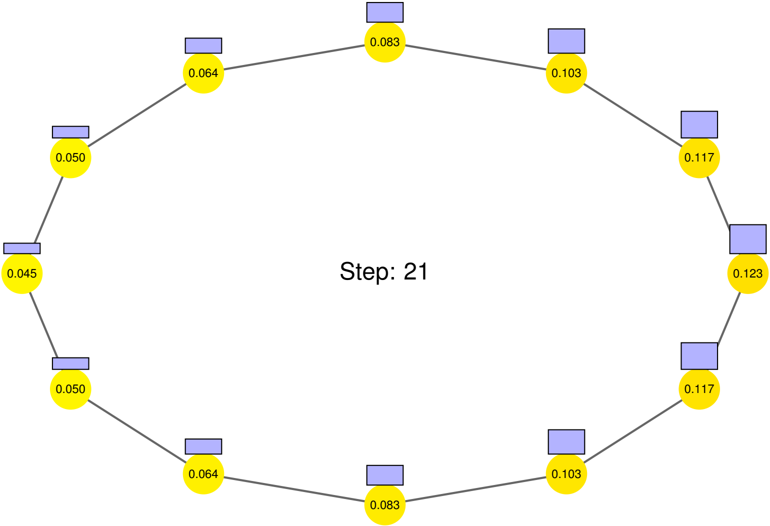

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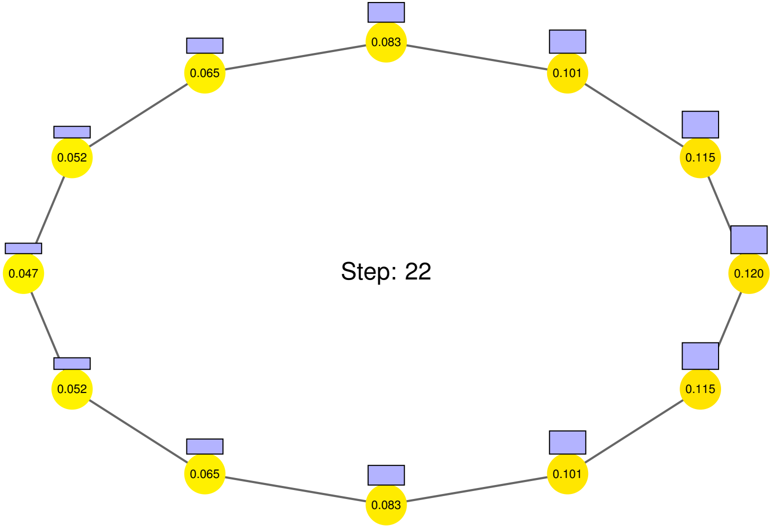

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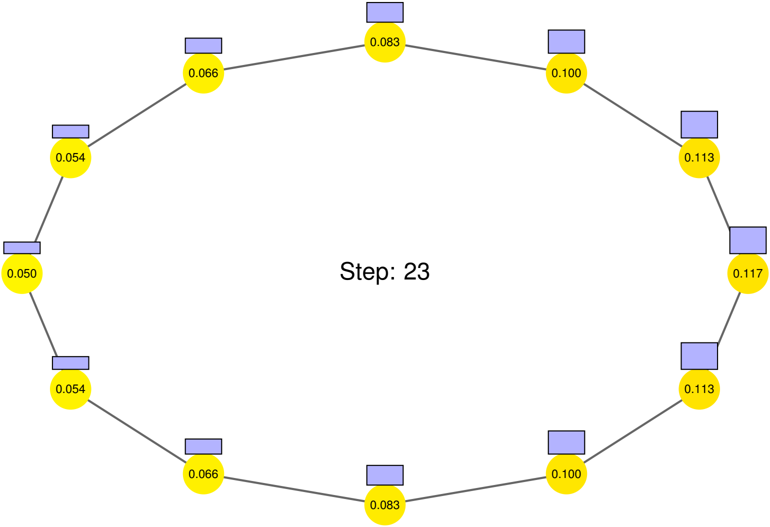

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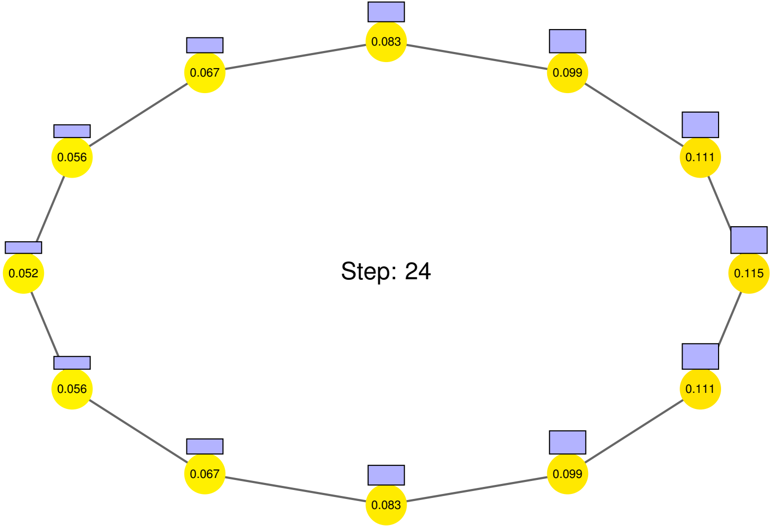

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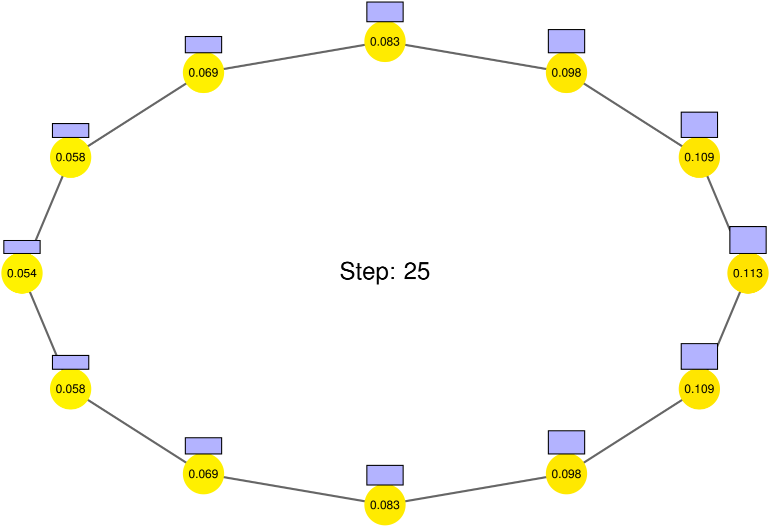

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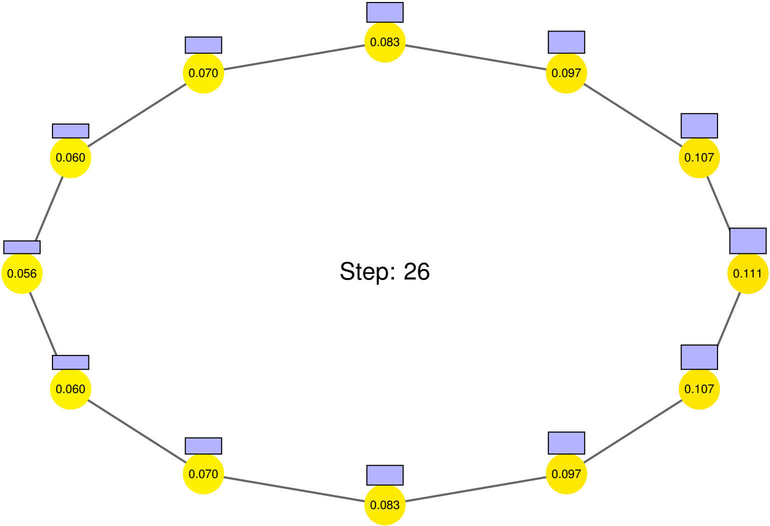

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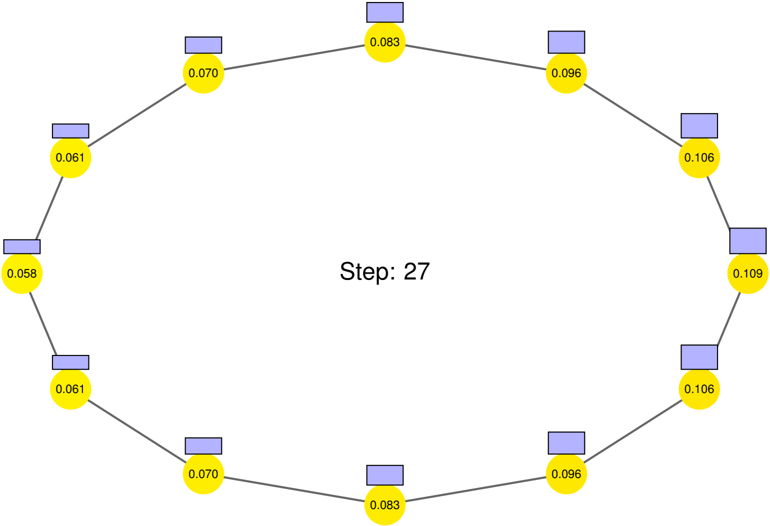

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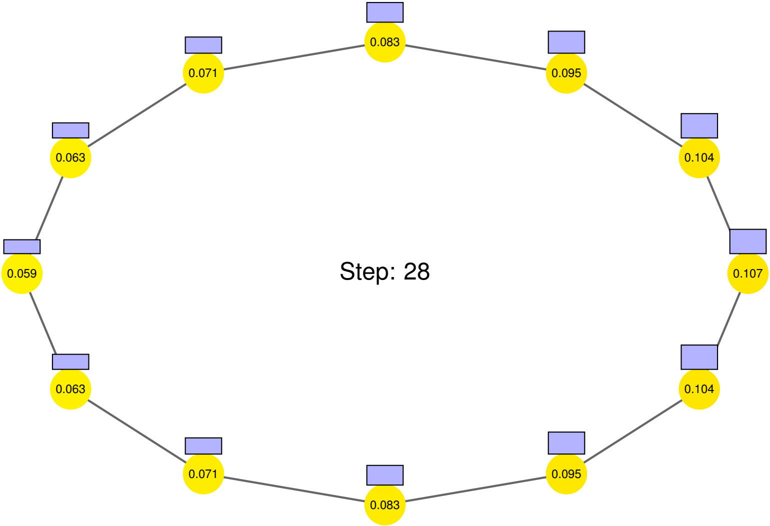

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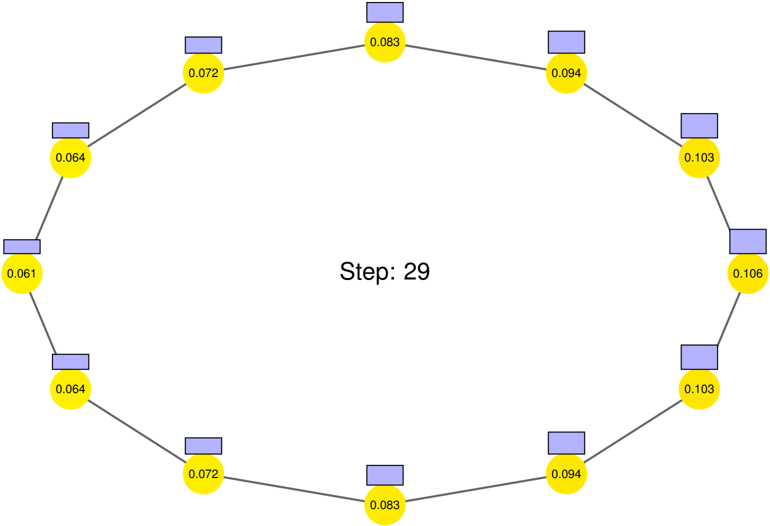

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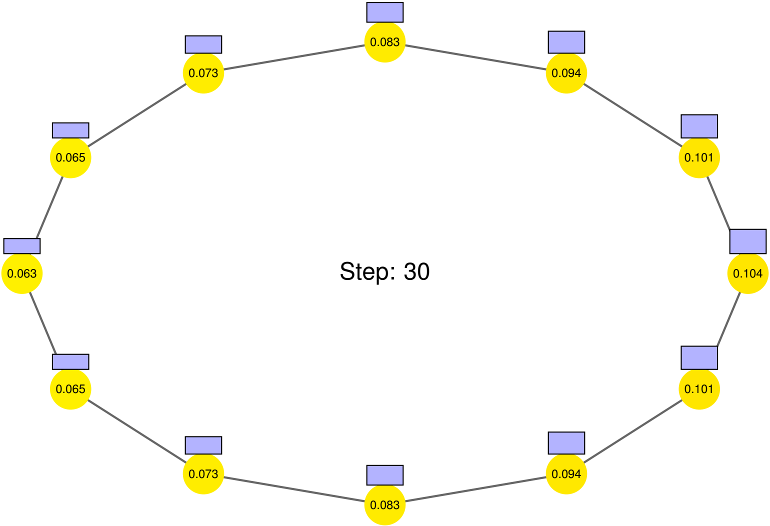

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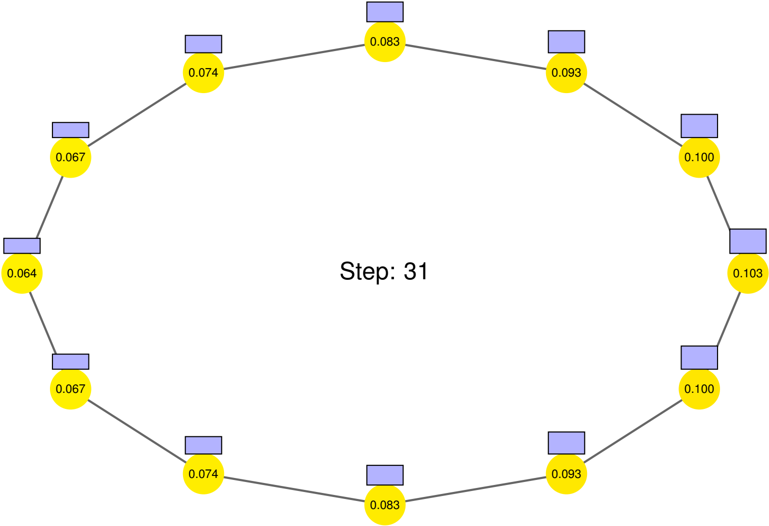

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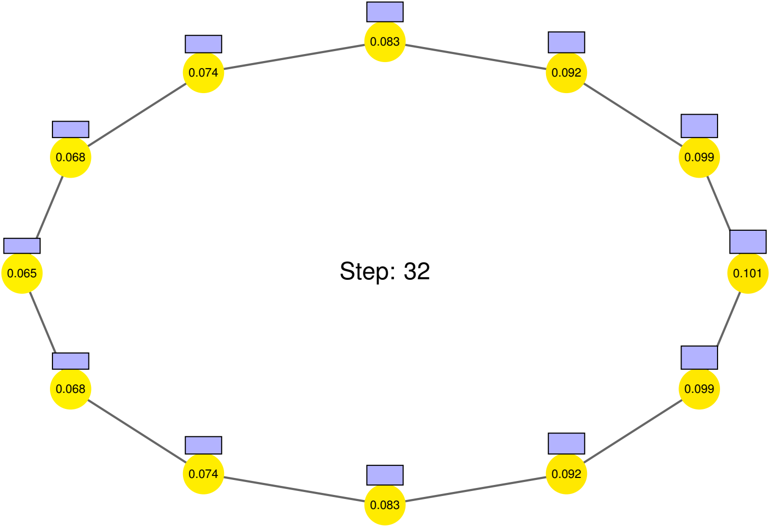

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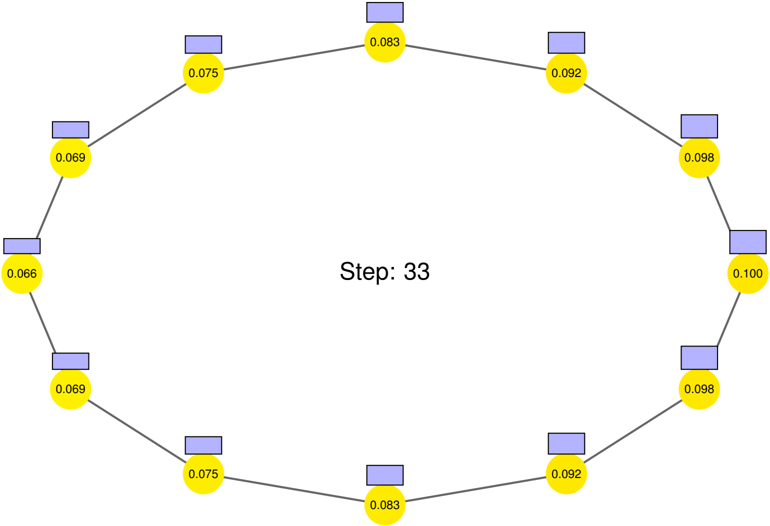

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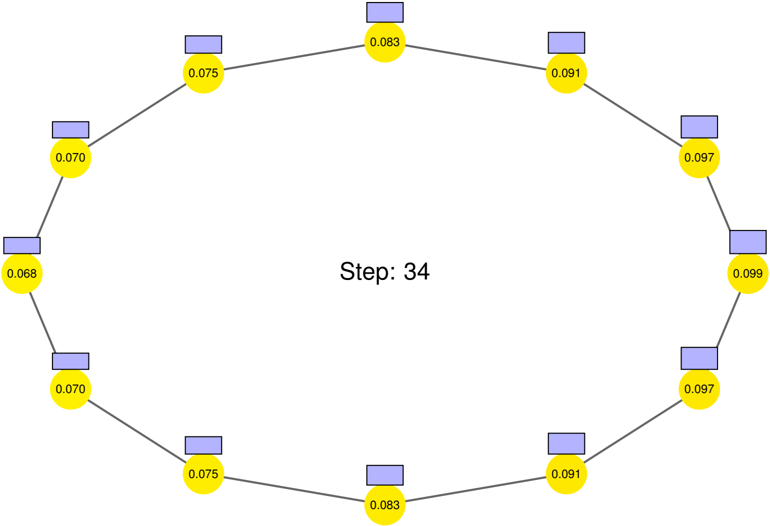

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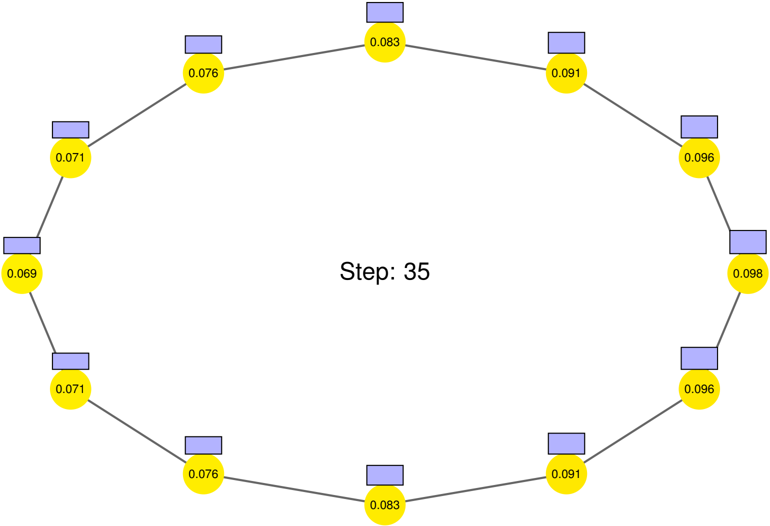

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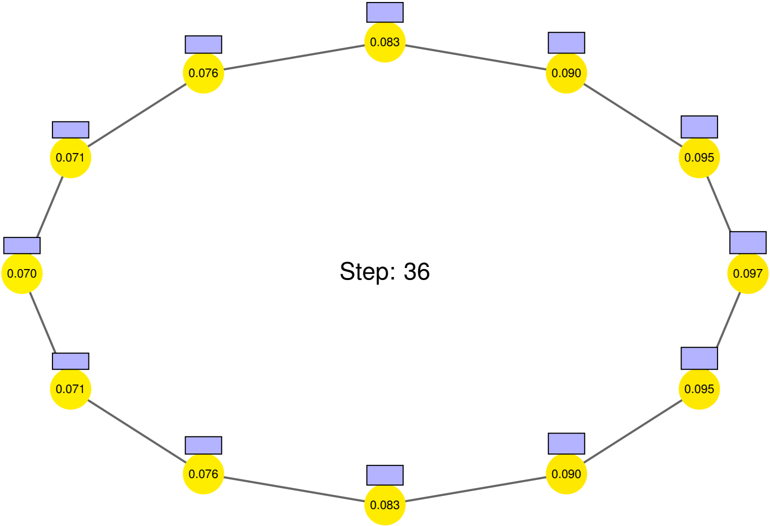

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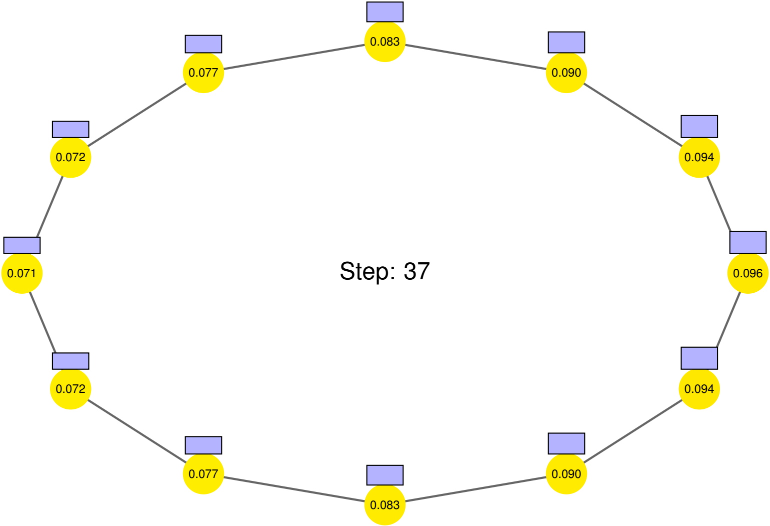

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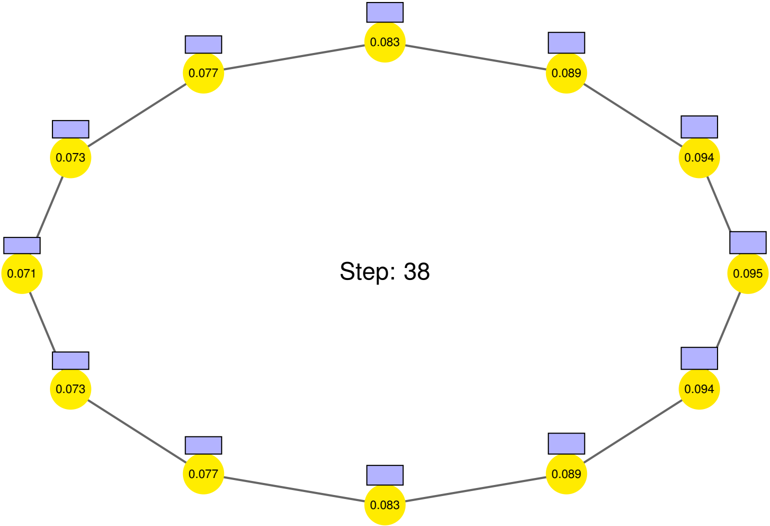

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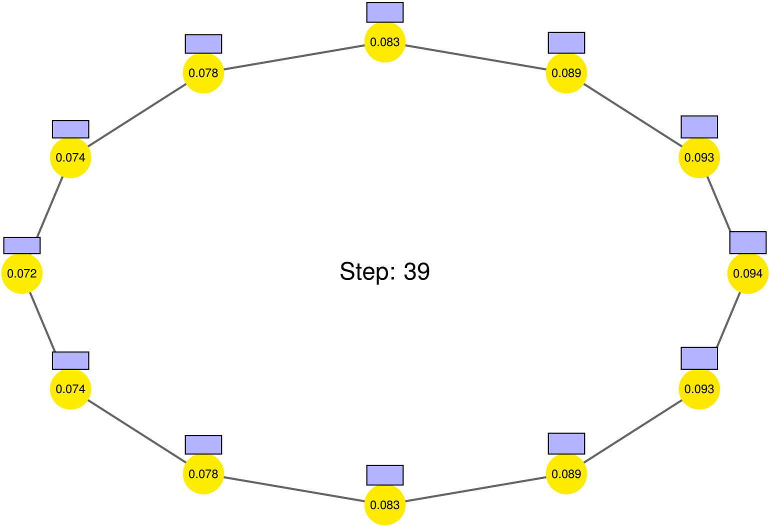

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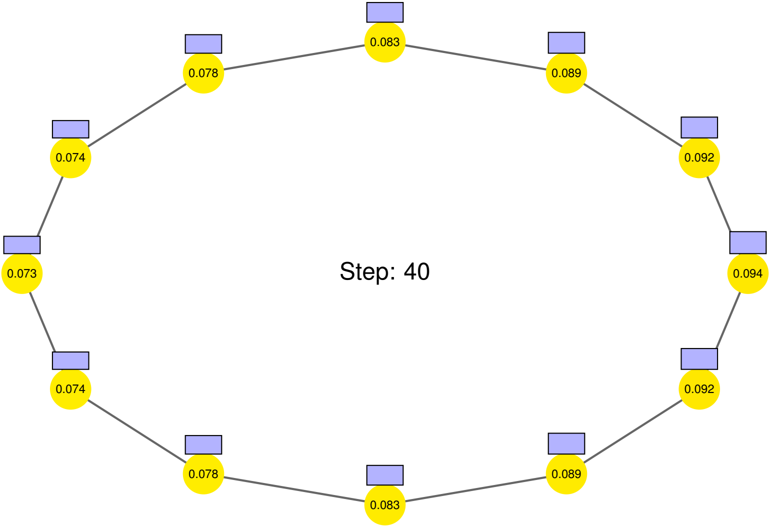

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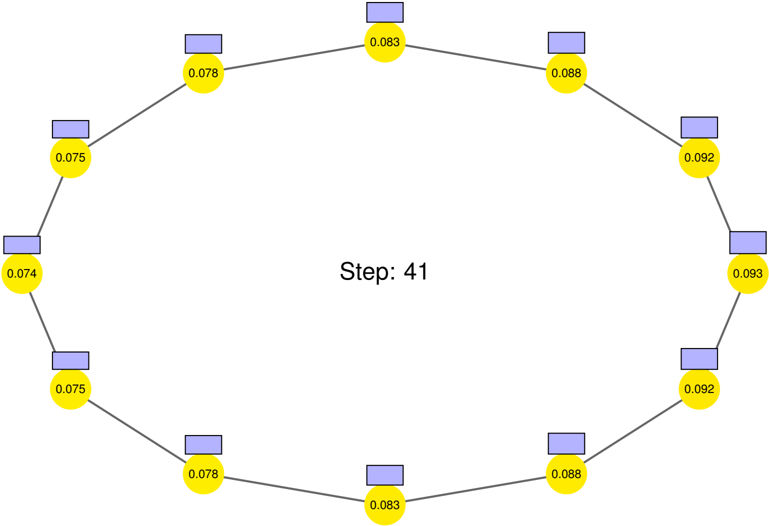

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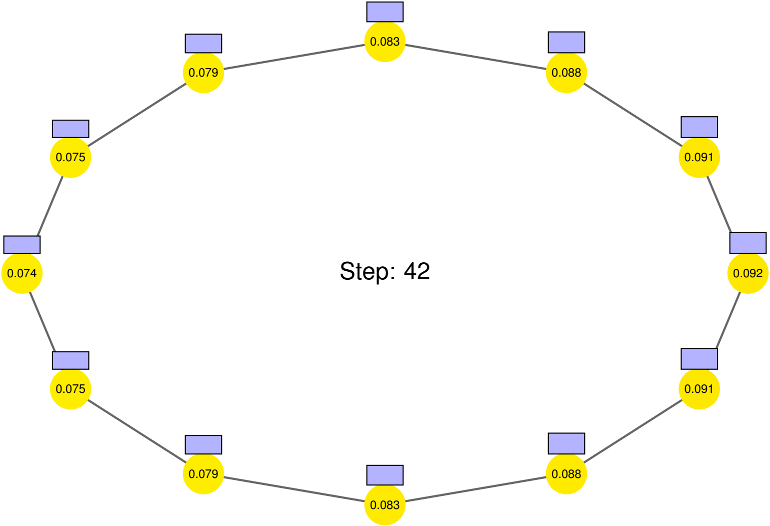

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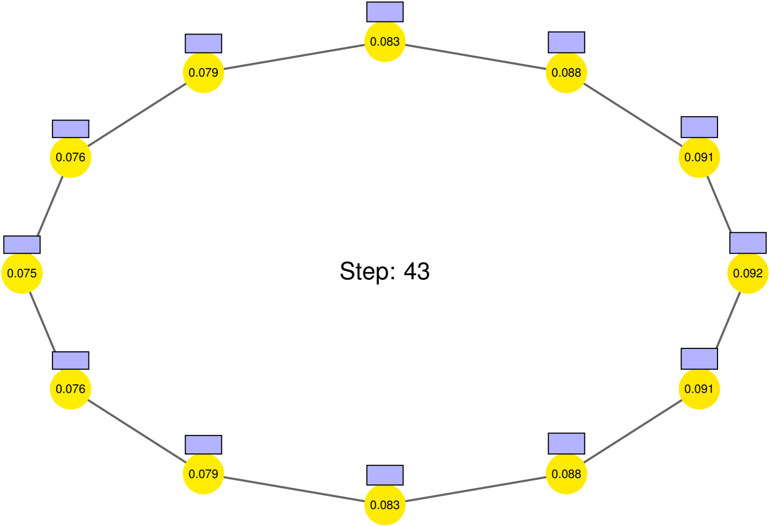

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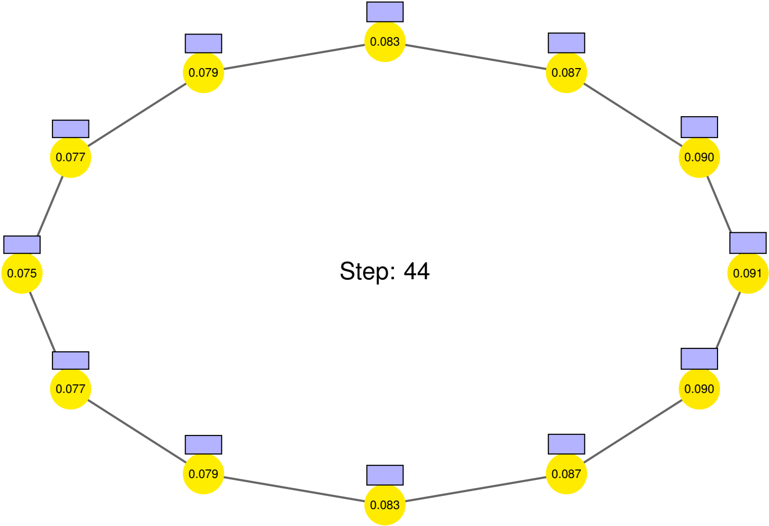

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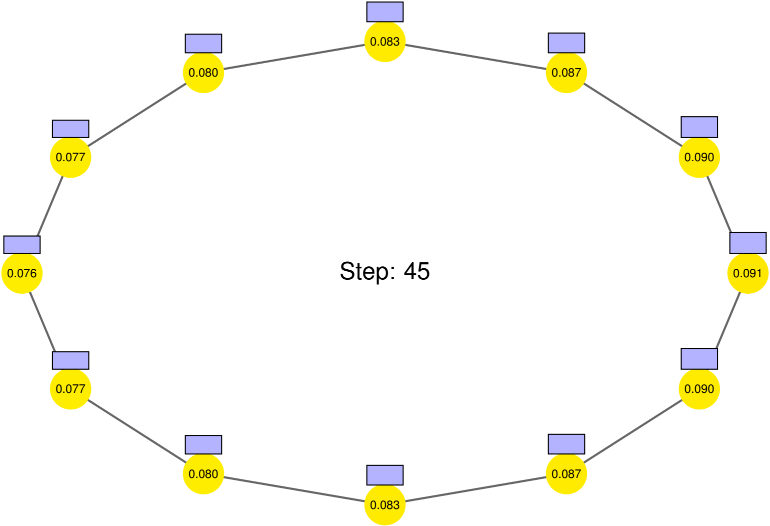

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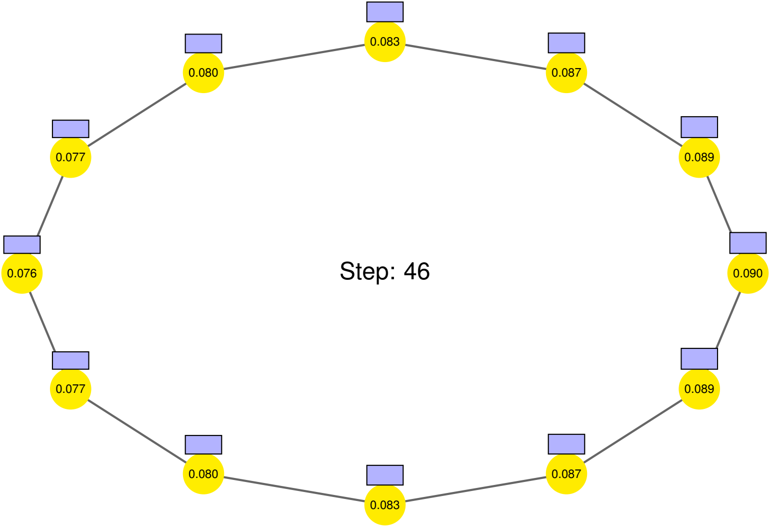

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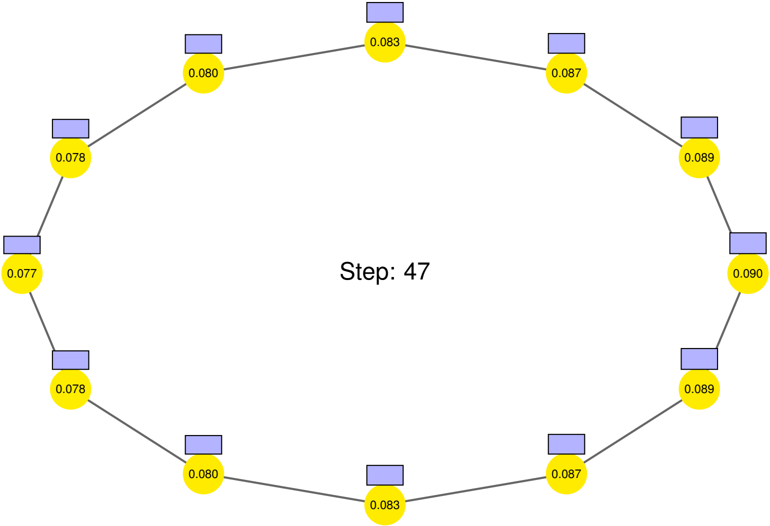

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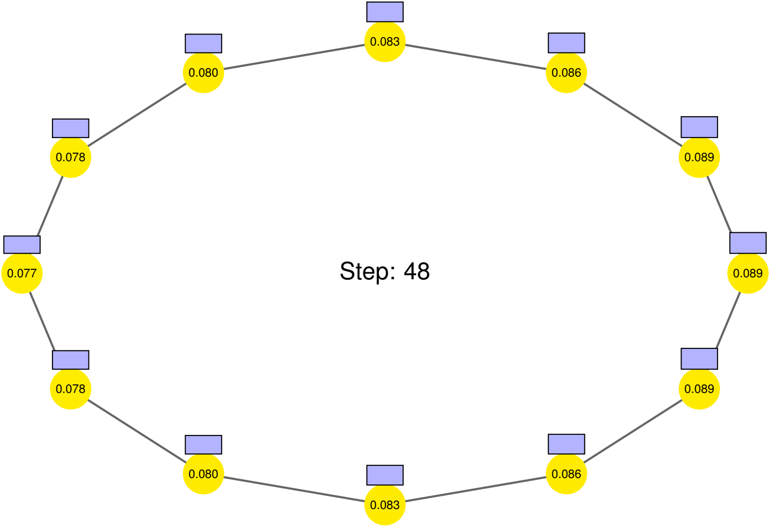

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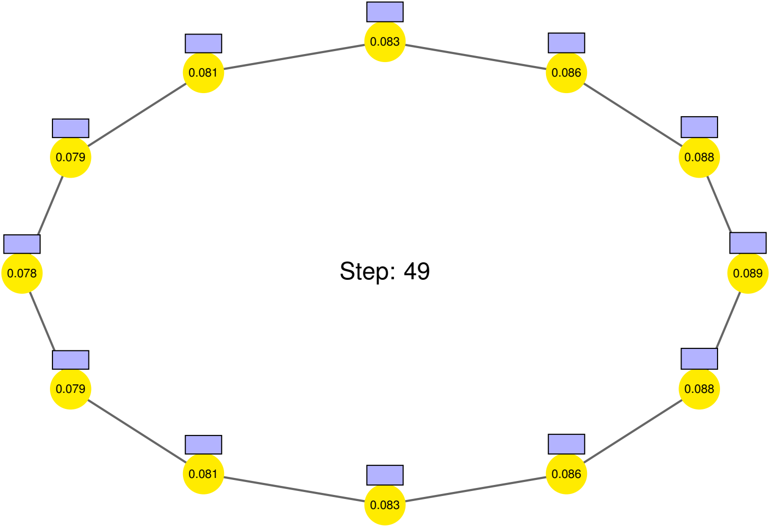

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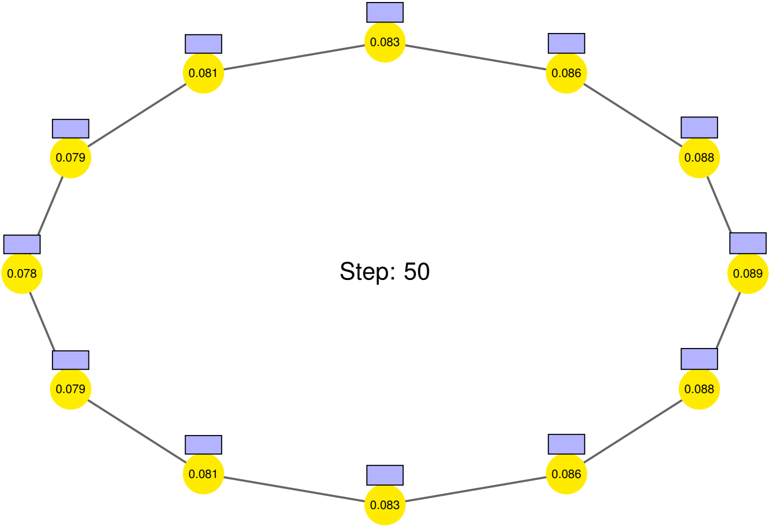

Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

How Similar are Two Probability Measures?

Loaded Dice

- You are presented three loaded (unfair) dice A, B, C:

x	1	2	3	4	5	6
$\mathbf{P}[A = x]$	1/3	1/12	1/12	1/12	1/12	1/3
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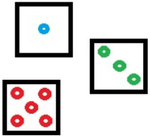

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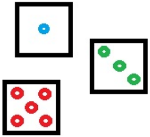

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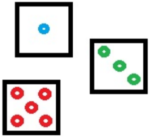

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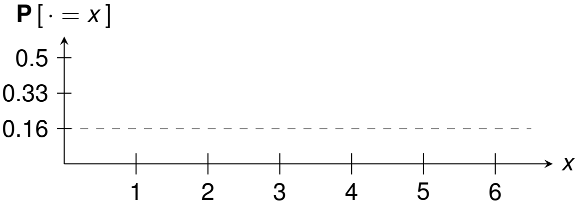

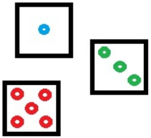

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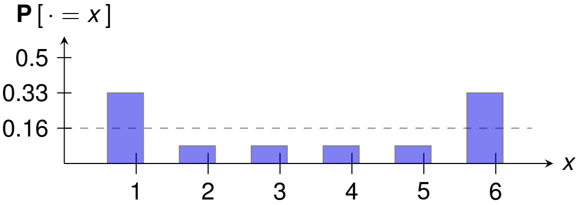

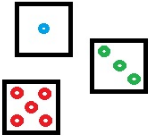

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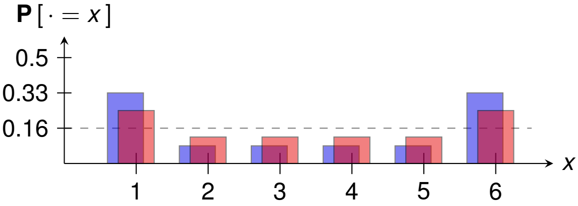

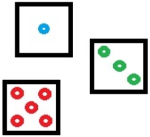

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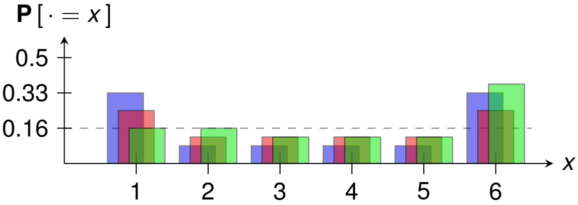

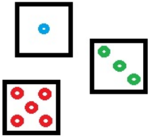

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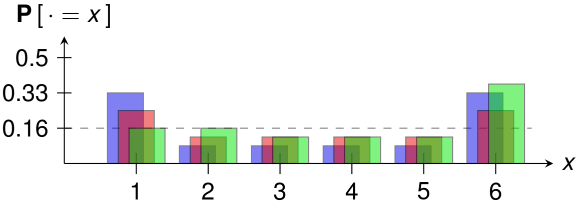

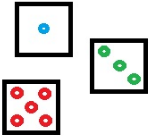

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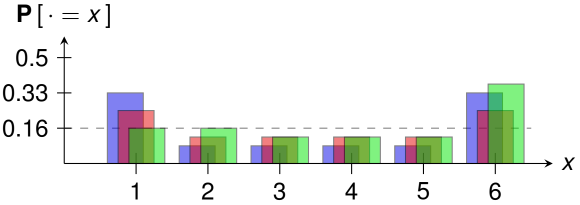

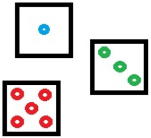

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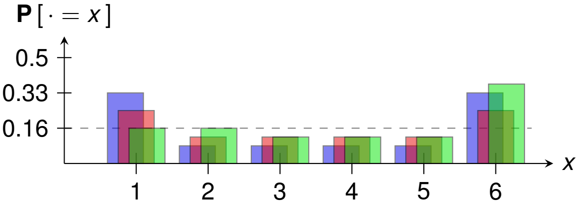

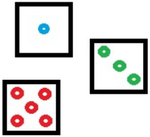

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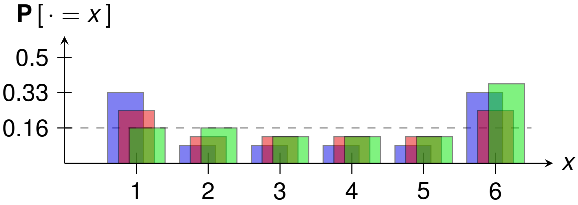
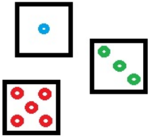
We need a formal "fairness measure" to compare probability distributions!





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Thus

$$\|D - B\|_{tv} = \|D - C\|_{tv} \qquad \text{and} \qquad \|D - B\|_{tv}, \|D - C\|_{tv} < \|D - A\|_{tv}.$$

So A is the least “fair”, however B and C are equally “fair” (in TV distance).

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We will see a similar result later after introducing spectral techniques!

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- We often take $\epsilon = 1/4$, indeed let $t_{mix} := \tau(1/4)$

Further Remarks on the Mixing Time (non-examinable)

- One can prove $\max_x \left\| P_x^t - \pi \right\|_{tv}$ is non-increasing in t (this means if the chain is "ϵ-mixed" at step t, then this also holds in future steps) *[Mitzenmacher, Upfal, 12.3]*

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be the variation distance after t steps when starting from the worst state. Further, define

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These quantities are related by the following double inequality

$$d(t) \leq \overline{d}(t) \leq 2d(t).$$

Further, $\overline{d}(t)$ is sub-multiplicative, that is for any $s, t \geq 1$,

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Hence smaller constants $\epsilon < 1/4$ only increase the mixing time by some constant factor.

Outline

Recap of Markov Chain Basics

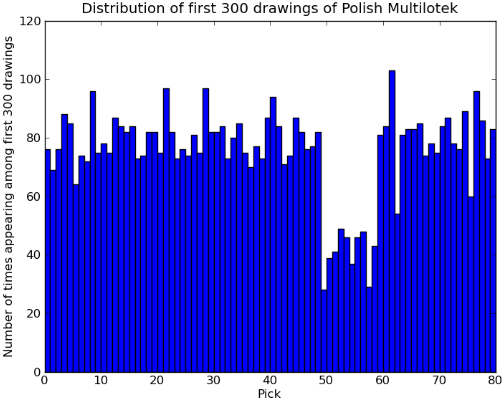
Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)

Experiment Gone Wrong...



Thanks to Krzysztof Onak (pointer) and Eric Price (graph)

Source: Slides by Ronitt Rubinfeld

What is Card Shuffling?

Source: wikipedia

How long does it take to shuffle a deck of 52 cards?

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Persi Diaconis (Professor of Statistics and former Magician)

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The Card Shuffling Markov Chain

TOPTORANDOMSHUFFLE (Input: A pile of n cards)

For $t = 1, 2, \ldots$
    Pick $i \in \{1, 2, \ldots, n\}$ uniformly at random
    Take the top card and insert it behind the i-th card

The Card Shuffling Markov Chain

TopToRandomShuffle (Input: A pile of n cards)

1: **For** $t = 1, 2, \ldots$
2: Pick $i \in \{1, 2, \ldots, n\}$ uniformly at random
3: Take the top card and insert it behind the i-th card

This is a slightly informal definition, so let us look at a small example...

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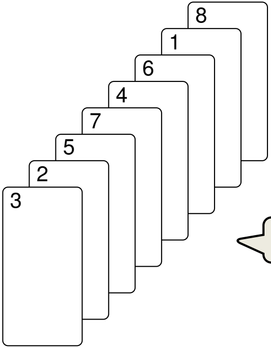
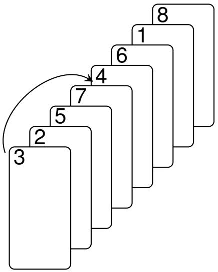


We will focus on this "small" set of cards ($n = 8$)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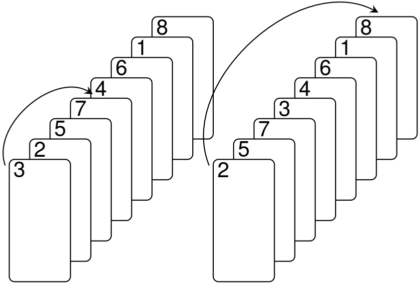
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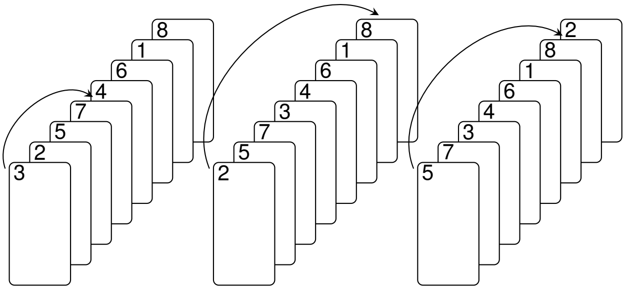
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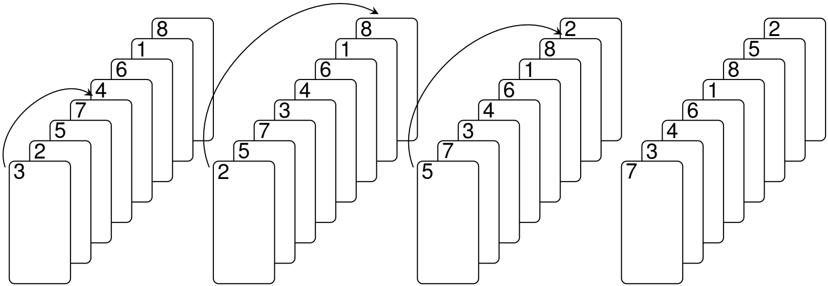
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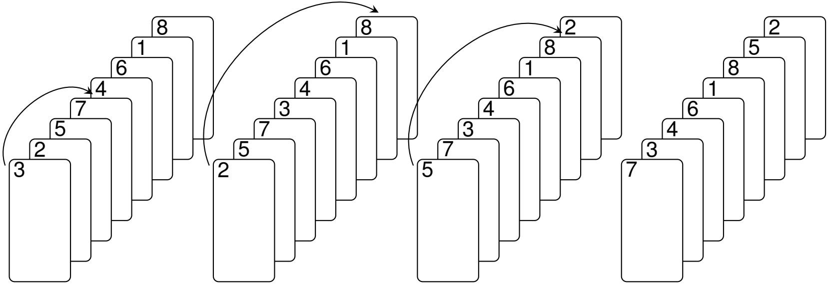
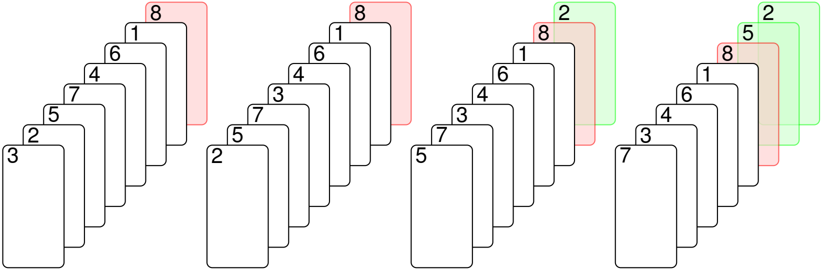


Even if we know which set of cards come after 8, every permutation is equally likely!



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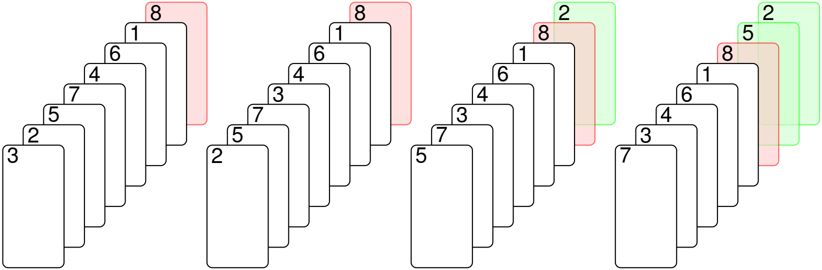

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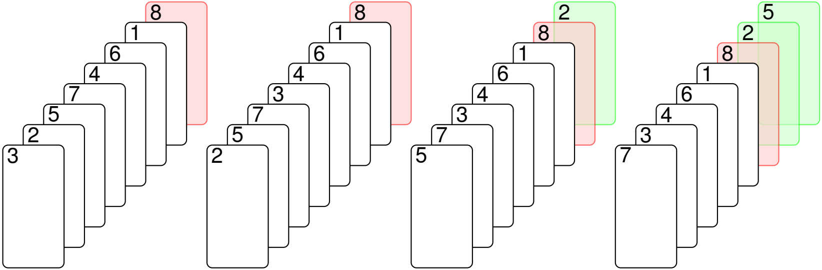

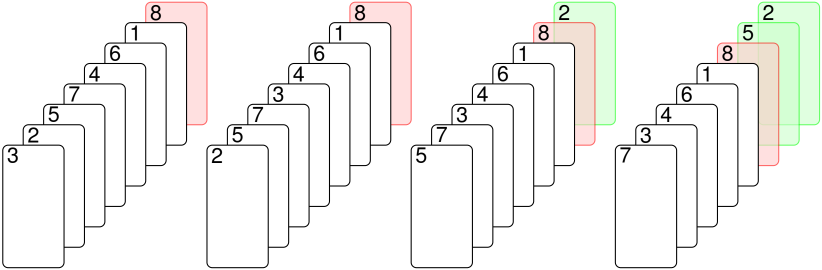

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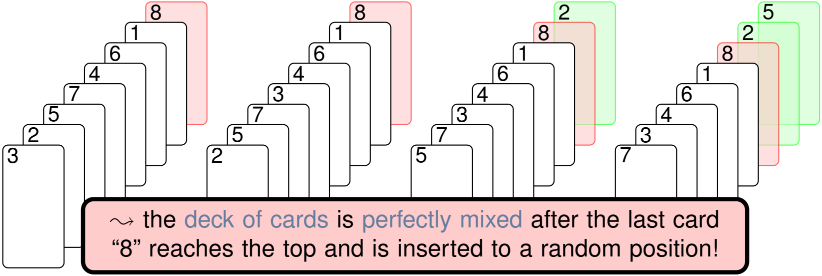
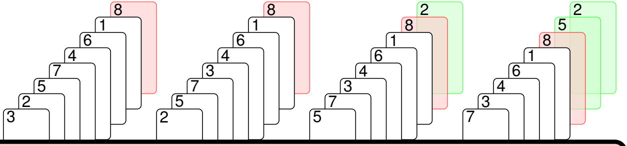


$\rightsquigarrow$ the deck of cards is perfectly mixed after the last card “8” reaches the top and is inserted to a random position!

Analysing the Mixing Time (Intuition)



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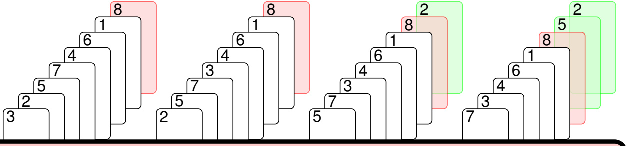

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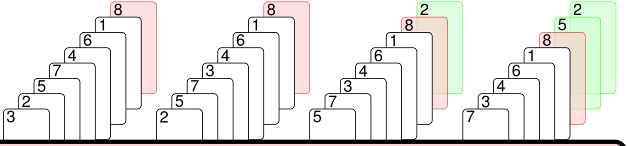

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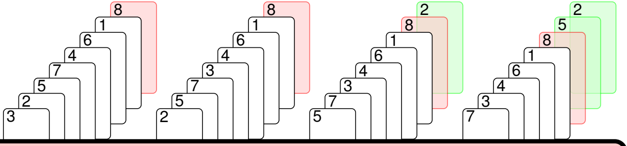

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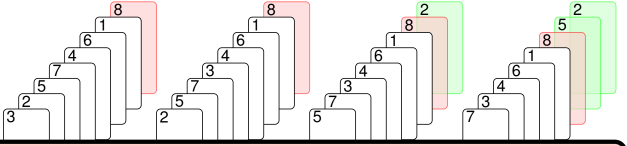

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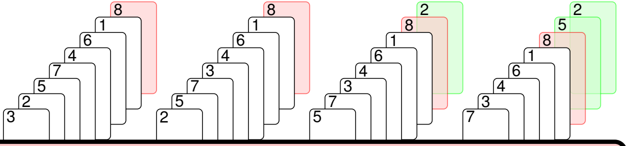

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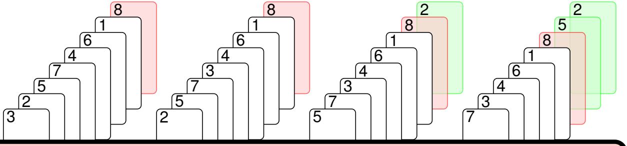

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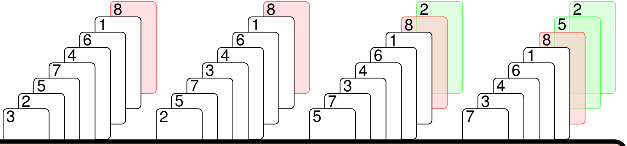

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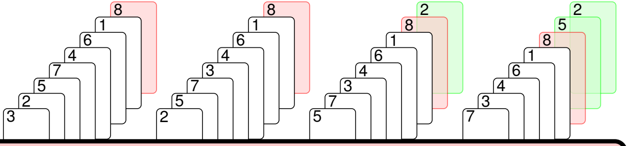

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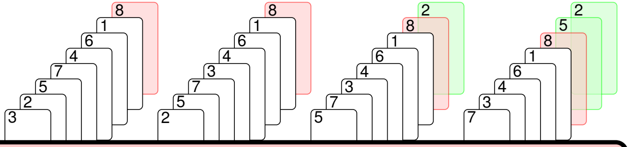

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Using the so-called coupling method, one could prove $t_{mix} \leq n \log n$.

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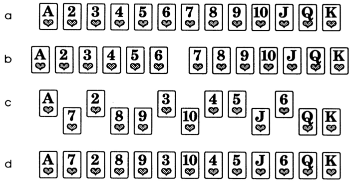

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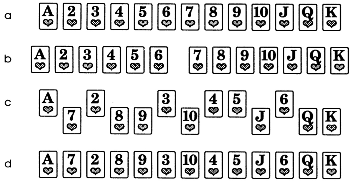

t	1	2	3	4	5	6	7	8	9	10
$\|\|P^t - \pi\|\|_{tv}$	1.000	1.000	1.000	1.000	0.924	0.614	0.334	0.167	0.085	0.043

Figure: Total Variation Distance for t riffle shuffles of 52 cards.

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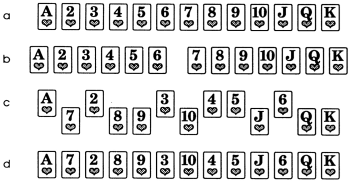

The Annals of Applied Probability
1992, Vol. 2, No. 2, 294–313

TRAILING THE DOVETAIL SHUFFLE TO ITS LAIR

By Dave Bayer[1] and Persi Diaconis[2]
Columbia University and Harvard University

We analyze the most commonly used method for shuffling cards. The main result is a simple expression for the chance of any arrangement after any number of shuffles. This is used to give sharp bounds on the approach to randomness: $\frac{3}{2}\log_2 n + \theta$ shuffles are necessary and sufficient to mix up n cards.
Key ingredients are the analysis of a card trick and the determination of the idempotents of a natural commutative subalgebra in the symmetric group algebra.

t	1	2	3	4	5	6	7	8	9	10
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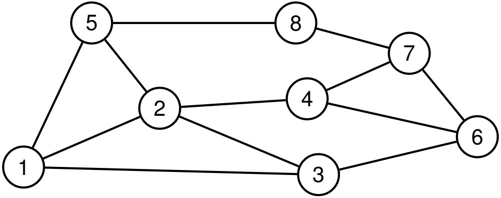
Outline

A Markov Chain for Sampling Independent Sets (1/2)



Independent Set

Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$.

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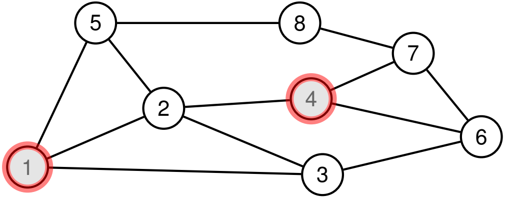

$S = \{1, 4\}$ is an independent set ✓

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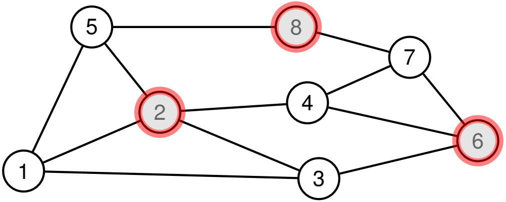

$S = \{2, 6, 8\}$ is an independent set ✓

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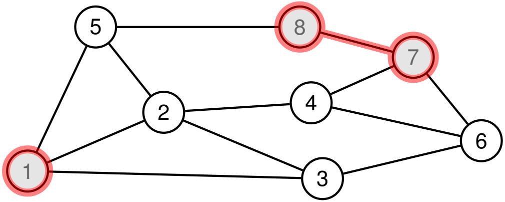

$S = \{1, 7, 8\}$ is **not** an independent set ×

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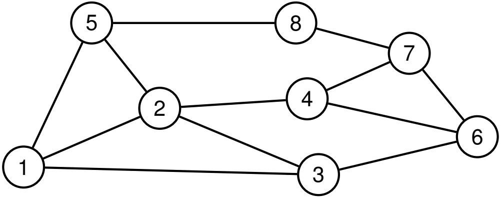

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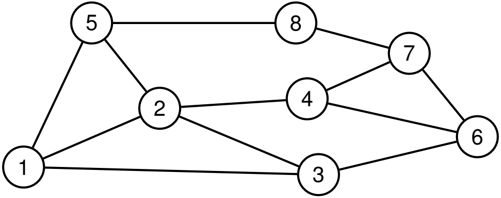

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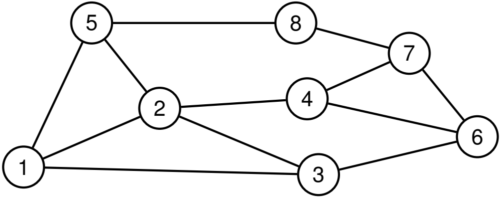

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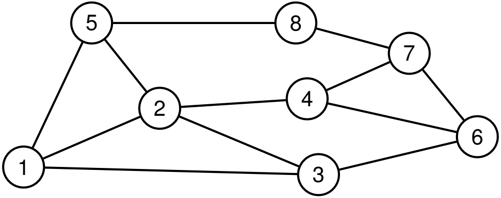

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We can use a generic Markov Chain Monte Carlo approach to tackle this problem!

A Markov Chain for Sampling Independent Sets (2/2)

INDEPENDENTSETSAMPLER

Let X_0 be an arbitrary independent set in G
For $t = 0, 1, 2, \ldots$:
  Pick a vertex $v \in V(G)$ uniformly at random
  If $v \in X_t$ **then** $X_{t+1} \leftarrow X_t \setminus \{v\}$
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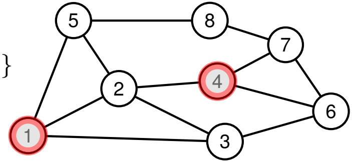
$X_0 = \{1, 4\}$



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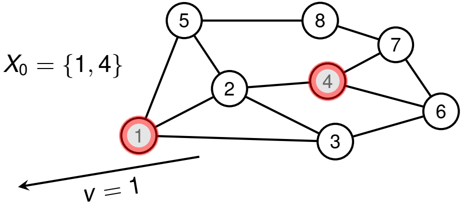

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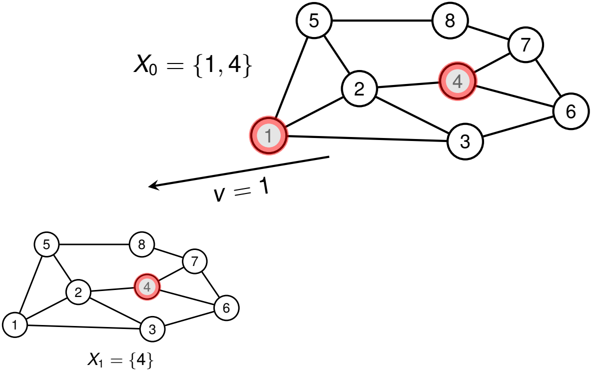

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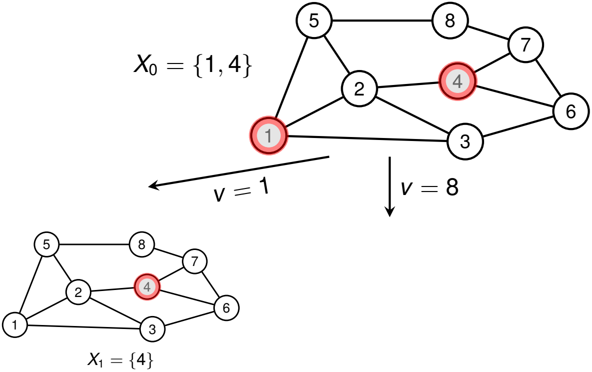

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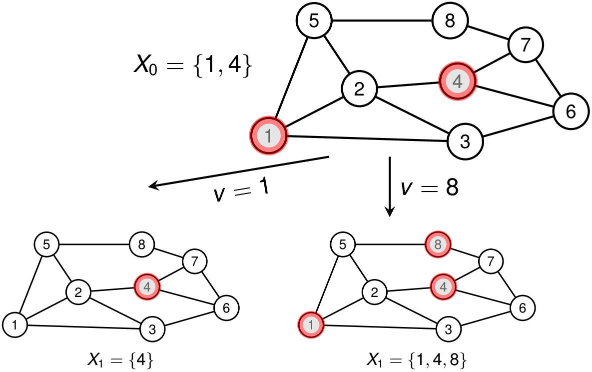

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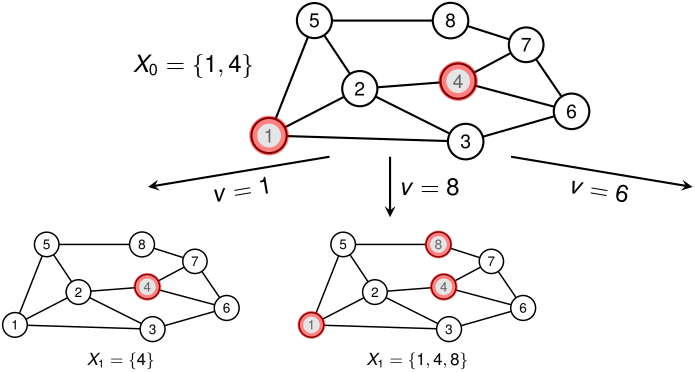

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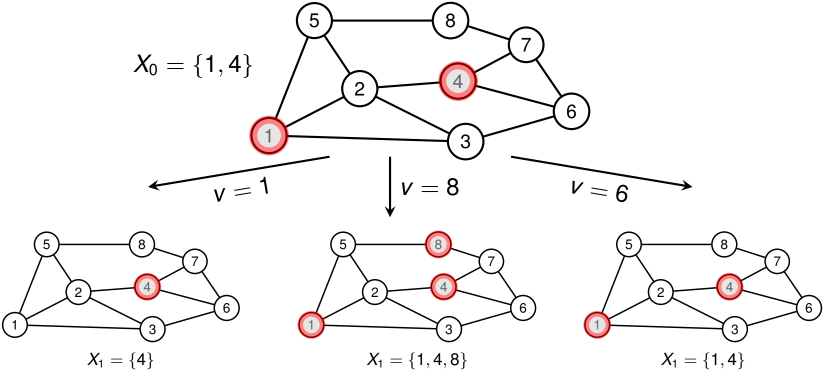

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- This is a local definition (no explicit definition of P!)

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- This chain is irreducible (every independent set is reachable)

A Markov Chain for Sampling Independent Sets (2/2)

INDEPENDENTSETSAMPLER

```
Let X_0 be an arbitrary independent set in G
For t = 0, 1, 2, ...:
    Pick a vertex v ∈ V(G) uniformly at random
    If v ∈ X_t then X_{t+1} ← X_t \ {v}
    elif v ∉ X_t and X_t ∪ {v} is an independent set then X_{t+1} ← X_t ∪ {v}
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```

Remark

- This is a local definition (no explicit definition of P!)
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- This chain is aperiodic (Check!)

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  elif $v \notin X_t$ **and** $X_t \cup \{v\}$ is an independent set **then** $X_{t+1} \leftarrow X_t \cup \{v\}$
  else $X_{t+1} \leftarrow X_t$

Remark

- This is a local definition (no explicit definition of P!)
- This chain is irreducible (every independent set is reachable)
- This chain is aperiodic (Check!)
- The stationary distribution is uniform, since $P_{u,v} = P_{v,u}$ (Check!)

A Markov Chain for Sampling Independent Sets (2/2)

INDEPENDENTSETSAMPLER

Let X_0 be an arbitrary independent set in G
For $t = 0, 1, 2, \ldots$:
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    If $v \in X_t$ **then** $X_{t+1} \leftarrow X_t \setminus \{v\}$
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Key Question: What is the mixing time of this Markov Chain?

A Markov Chain for Sampling Independent Sets (2/2)

INDEPENDENTSETSAMPLER

```
Let X_0 be an arbitrary independent set in G
For t = 0, 1, 2, ...:
    Pick a vertex v ∈ V(G) uniformly at random
    If v ∈ X_t then X_{t+1} ← X_t \ {v}
    elif v ∉ X_t and X_t ∪ {v} is an independent set then X_{t+1} ← X_t ∪ {v}
    else X_{t+1} ← X_t
```

Remark

- This is a local definition (no explicit definition of P!)
- This chain is irreducible (every independent set is reachable)
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Key Question: What is the mixing time of this Markov Chain?

not covered here, see the textbook by Mitzenmacher and Upfal