Randomised Algorithms
Lecture 4: Markov Chains and Mixing Times

Thomas Sauerwald (tms41@cam.ac.uk)
Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)
Applications of Markov Chains in Computer Science

Broadcasting

\[ A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
Applications of Markov Chains in Computer Science

Broadcasting

Clustering

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0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
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Applications of Markov Chains in Computer Science

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Applications of Markov Chains in Computer Science

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- Ranking Websites
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- Sampling and Optimisation
Applications of Markov Chains in Computer Science

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Matrix $A$:

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- Load Balancing
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Applications of Markov Chains in Computer Science

Broadcasting

Ranking Websites

Load Balancing

Clustering

Sampling and Optimisation

Particle Processes

$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
Markov Chains

We say that \((X_t)_{t=0}^\infty\) is a Markov Chain on State Space \(\Omega\) with Initial Distribution \(\mu\) and Transition Matrix \(P\) if:

1. For any \(x \in \Omega\),
   \[P[X_0 = x] = \mu(x)\].
2. The Markov Property holds: for all \(t \geq 0\) and any \(x_0, \ldots, x_t+1 \in \Omega\),
   \[P[X_{t+1} = x_{t+1} | X_t = x_t, \ldots, X_0 = x_0] = P[X_{t+1} = x_{t+1} | X_t = x_t]\] := \[P(x_t, x_{t+1})\].

For all \(0 \leq t_1 < t_2, x \in \Omega\),
\[P[X_{t_2} = x] = \sum_{y \in \Omega} P[X_{t_2} = x | X_{t_1} = y] \cdot P[X_{t_1} = y].\]
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P \left[ X_{t+1} = x_{t+1} \mid X_t = x_t, \ldots, X_0 = x_0 \right] = P \left[ X_{t+1} = x_{t+1} \mid X_t = x_t \right]
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From the definition one can deduce that (check!)
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\[:= P(x_t, x_{t+1}).\]

From the definition one can deduce that (check!)

- For all \(t, x_0, x_1, \ldots, x_t \in \Omega\),

\[
P \left[ X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0 \right]
\]

\[= \mu(x_0) \cdot P(x_0, x_1) \cdot \ldots \cdot P(x_{t-2}, x_{t-1}) \cdot P(x_{t-1}, x_t).\]
Markov Chains

Markov Chain (Discrete Time and State, Time Homogeneous)

We say that $(X_t)_{t=0}^\infty$ is a Markov Chain on State Space $\Omega$ with Initial Distribution $\mu$ and Transition Matrix $P$ if:

1. For any $x \in \Omega$, $P \left[ X_0 = x \right] = \mu(x)$.
2. The Markov Property holds: for all $t \geq 0$ and any $x_0, \ldots, x_{t+1} \in \Omega$,

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$$:= P(x_t, x_{t+1}).$$

From the definition one can deduce that (check!)

- For all $t, x_0, x_1, \ldots, x_t \in \Omega$,

$$P \left[ X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0 \right]$$

$$= \mu(x_0) \cdot P(x_0, x_1) \cdot \ldots \cdot P(x_{t-2}, x_{t-1}) \cdot P(x_{t-1}, x_t).$$

- For all $0 \leq t_1 < t_2, x \in \Omega$,

$$P \left[ X_{t_2} = x \right] = \sum_{y \in \Omega} P \left[ X_{t_2} = x \mid X_{t_1} = y \right] \cdot P \left[ X_{t_1} = y \right].$$
Example: the carbohydrate served with lunch in the college cafeteria.

This has transition matrix:

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 \\
1/4 & 0 & 3/4 \\
3/5 & 2/5 & 0 \\
\end{bmatrix}
\]
The Transition Matrix $P$ of a Markov chain $(\mu, P)$ on $\Omega = \{1, \ldots n\}$ is given by

$$P = \begin{bmatrix}
P(1,1) & \ldots & P(1,n) \\
\vdots & \ddots & \vdots \\
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- $\rho^t = (\rho^t(1), \rho^t(2), \ldots, \rho^t(n))$: state vector at time $t$ (row vector).
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- Multiplying $\rho^t$ by $P$ corresponds to advancing the chain one step:

$$\rho^t(y) = \sum_{x \in \Omega} \rho^{t-1}(x) \cdot P(x, y) \quad \text{and thus} \quad \rho^t = \rho^{t-1} \cdot P.$$
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- The Markov Property and line above imply that for any $t \geq 0$

$$\rho^t = \rho \cdot P^{t-1} \quad \text{and thus} \quad P^t(x, y) = \mathbb{P} [X_t = y \mid X_0 = x].$$
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and thus $P^t(x, y) = P [ X_t = y \mid X_0 = x ]$.

Thus $\rho^t(x) = (\mu P^t)(x)$ and so $\rho^t = \mu P^t = (\mu P^t(1), \mu P^t(2), \ldots, \mu P^t(n))$. 
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- Everything boils down to deterministic vector/matrix computations.
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- Everything boils down to deterministic vector/matrix computations
  $\Rightarrow$ can replace $\rho$ by any (load) vector and view $P$ as a balancing matrix!
Stopping and Hitting Times

A non-negative integer random variable $\tau$ is a stopping time for $(X_t)_{t \geq 0}$ if for every $s \geq 0$ the event \{\(\tau = s\)\} depends only on $X_0, \ldots, X_s$. 

Example - College Carbs

Stopping times:

✓ “We had rice yesterday”;
\[ \tau := \min\{t \geq 1 : X_t - 1 = \text{“rice”}\} \]
× “We are having pasta next Thursday”

For two states $x, y \in \Omega$ we call $h(x, y)$ the hitting time of $y$ from $x$:

\[ h(x, y) := \mathbb{E}_x[\tau_y] = \mathbb{E}_x[\tau_y | X_0 = x] \]

where $\tau_y = \min\{t \geq 1 : X_t = y\}$.

Some distinguish between $\tau^+_y = \min\{t \geq 1 : X_t = y\}$ and $\tau_y = \min\{t \geq 0 : X_t = y\}$.

Hitting times are the solution to a set of linear equations:

\[ h(x, y) = \begin{cases} \mathbb{1} + \sum_{z \in \Omega \setminus \{y\}} p(x, z) \cdot h(z, y) & \forall x \neq y \in \Omega \\ 0 & \text{if } x = y \end{cases} \]

A Useful Identity

4. Markov Chains and Mixing Times © T. Sauerwald

Recap of Markov Chain Basics 7
Stopping and Hitting Times

A non-negative integer random variable $\tau$ is a **stopping time** for $(X_t)_{t \geq 0}$ if for every $s \geq 0$ the event $\{\tau = s\}$ depends only on $X_0, \ldots, X_s$.

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Stopping and Hitting Times

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where $\tau_y = \min \{t \geq 1 : X_t = y\}$.

---

**A Useful Identity**

Hitting times are the solution to a set of linear equations:

$$h(x, y) = 1 + \sum_{z \in \Omega \setminus \{y\}} P(x, z) \cdot h(z, y) \quad \forall x \neq y \in \Omega.$$
Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)
A Markov Chain is **irreducible** if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$. 
A Markov Chain is **irreducible** if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$. 

For any states $x$ and $y$ of a finite irreducible Markov Chain $h(x, y) < \infty$. 

### Finite Hitting Time Theorem
A Markov Chain is irreducible if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$.

Exercise: Which of the two chains (if any) are irreducible?
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A Markov Chain is irreducible if for every pair of states $x, y \in \Omega$ there is an integer $k \geq 0$ such that $P^k(x, y) > 0$.

Finite Hitting Time Theorem

For any states $x$ and $y$ of a finite irreducible Markov Chain $h(x, y) < \infty$. 
A probability distribution $\pi = (\pi(1), \ldots, \pi(n))$ is the stationary distribution of a Markov Chain if $\pi P = \pi$ ($\pi$ is a left eigenvector with eigenvalue 1)
Stationary Distribution

A probability distribution \( \pi = (\pi(1), \ldots, \pi(n)) \) is the stationary distribution of a Markov Chain if \( \pi P = \pi \) (\( \pi \) is a left eigenvector with eigenvalue 1).

College carbs example:

\[
\begin{pmatrix}
4/13 & 4/13 & 5/13 \\
\pi & 1/4 & 1/2 & 1/2 \\
3/5 & 2/5 & 0 & 3/4 \\
\end{pmatrix} \cdot \begin{pmatrix}
0 & 1/2 & 1/2 \\
1/4 & 0 & 3/4 \\
3/5 & 2/5 & 0 \\
\pi & \end{pmatrix} = \begin{pmatrix}
4/13 & 4/13 & 5/13 \\
\pi & \end{pmatrix}
\]
Stationary Distribution

A probability distribution \( \pi = (\pi(1), \ldots, \pi(n)) \) is the **stationary distribution** of a Markov Chain if \( \pi P = \pi \) (\( \pi \) is a left eigenvector with eigenvalue 1).

**College carbs example:**

\[
\begin{pmatrix}
\frac{4}{13}, \\
\frac{4}{13}, \\
\pi
\end{pmatrix}
\begin{pmatrix}
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1/4 & 0 & 3/4 \\
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\end{pmatrix}
= \begin{pmatrix}
\frac{4}{13}, \\
\frac{4}{13}, \\
\pi
\end{pmatrix}
\]

- A Markov Chain reaches **stationary distribution** if \( \rho^t = \pi \) for some \( t \).
Stationary Distribution

A probability distribution \( \pi = (\pi(1), \ldots, \pi(n)) \) is the stationary distribution of a Markov Chain if \( \pi P = \pi \) (\( \pi \) is a left eigenvector with eigenvalue 1)

College carbs example:

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\begin{pmatrix}
\frac{4}{13}, \frac{4}{13}, \frac{5}{13} \\
\pi
\end{pmatrix}
\cdot
\begin{pmatrix}
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\end{pmatrix}
= \begin{pmatrix}
\frac{4}{13}, \frac{4}{13}, \frac{5}{13} \\
\pi
\end{pmatrix}
\]

- A Markov Chain reaches stationary distribution if \( \rho^t = \pi \) for some \( t \).
- If reached, then it persists: If \( \rho^t = \pi \) then \( \rho^{t+k} = \pi \) for all \( k \geq 0 \).
Stationary Distribution

A probability distribution \( \pi = (\pi(1), \ldots, \pi(n)) \) is the stationary distribution of a Markov Chain if \( \pi P = \pi \) (\( \pi \) is a left eigenvector with eigenvalue 1).

College carbs example:

\[
\begin{pmatrix}
\frac{4}{13} & \frac{4}{13} & \frac{5}{13} \\
\pi & & \\
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & 0 & \frac{3}{4} \\
\frac{3}{5} & \frac{2}{5} & 0 \\
\end{pmatrix}
= \begin{pmatrix}
\frac{4}{13} & \frac{4}{13} & \frac{5}{13} \\
\pi & & \\
\end{pmatrix}
\]

- A Markov Chain reaches stationary distribution if \( \rho^t = \pi \) for some \( t \).
- If reached, then it persists: If \( \rho^t = \pi \) then \( \rho^{t+k} = \pi \) for all \( k \geq 0 \).

Existence and Uniqueness of a Positive Stationary Distribution

Let \( P \) be finite, irreducible M.C., then there exists a unique probability distribution \( \pi \) on \( \Omega \) such that \( \pi = \pi P \) and \( \pi(x) = 1/h(x,x) > 0, \forall x \in \Omega. \)
Periodicity

- A Markov Chain is **aperiodic** if for all \( x \in \Omega \), \( \gcd\{ t \geq 1 : P^t(x, x) > 0 \} = 1 \).
Periodicity

- A Markov Chain is aperiodic if for all $x \in \Omega$, $\gcd\{t \geq 1 : P^t(x, x) > 0\} = 1$.
- Otherwise we say it is periodic.
A Markov Chain is aperiodic if for all $x \in \Omega$, $\gcd\{t \geq 1 : P^t(x, x) > 0\} = 1$. Otherwise we say it is periodic.
Periodicity

- A Markov Chain is aperiodic if for all \( x \in \Omega \), \( \gcd\{t \geq 1 : P^t(x, x) > 0\} = 1 \).
- Otherwise we say it is periodic.

![Diagrams of Markov chains]

**Exercise:** Which of the two chains (if any) are aperiodic?
A Markov Chain is **aperiodic** if for all \( x \in \Omega \), \( \gcd\{t \geq 1 : P^t(x, x) > 0\} = 1 \).
Otherwise we say it is **periodic**.

**Exercise:** Which of the two chains (if any) are aperiodic?
Let $P$ be any finite, irreducible, aperiodic Markov Chain with stationary distribution $\pi$. Then for any $x, y \in \Omega$,

$$\lim_{t \to \infty} P^t(x, y) = \pi(y).$$
Let $P$ be any finite, irreducible, aperiodic Markov Chain with stationary distribution $\pi$. Then for any $x, y \in \Omega$,

$$\lim_{t \to \infty} P^t(x, y) = \pi(y).$$
Convergence Theorem

Let $P$ be any finite, irreducible, aperiodic Markov Chain with stationary distribution $\pi$. Then for any $x, y \in \Omega$,

$$\lim_{t \to \infty} P^t(x, y) = \pi(y).$$

- **Ergodic** = **Irreducible** + **Aperiodic**

- **mentioned before**: For finite irreducible M.C.’s $\pi$ exists, is unique and

$$\pi(y) = \frac{1}{h(y, y)} > 0.$$
Convergence Theorem

Let $P$ be any finite, irreducible, aperiodic Markov Chain with stationary distribution $\pi$. Then for any $x, y \in \Omega$,

$$\lim_{t \to \infty} P^t(x,y) = \pi(y).$$

- mentioned before: For finite irreducible M.C.’s $\pi$ exists, is unique and

$$\pi(y) = \frac{1}{h(y,y)} > 0.$$

- We will prove a simpler version of the Convergence Theorem after introducing Spectral Graph Theory.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$.
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

```

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
<td>0.250</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.250</td>
</tr>
</tbody>
</table>
```

Step: 1
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

![Diagram showing Markov chain transitions and stationary distribution values.

**Step: 2**
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 3

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.312</td>
</tr>
<tr>
<td>2</td>
<td>0.234</td>
</tr>
<tr>
<td>3</td>
<td>0.094</td>
</tr>
<tr>
<td>4</td>
<td>0.016</td>
</tr>
<tr>
<td>5</td>
<td>0.000</td>
</tr>
<tr>
<td>6</td>
<td>0.000</td>
</tr>
<tr>
<td>7</td>
<td>0.000</td>
</tr>
<tr>
<td>8</td>
<td>0.000</td>
</tr>
<tr>
<td>9</td>
<td>0.000</td>
</tr>
<tr>
<td>10</td>
<td>0.000</td>
</tr>
<tr>
<td>11</td>
<td>0.016</td>
</tr>
<tr>
<td>12</td>
<td>0.094</td>
</tr>
</tbody>
</table>
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

![Diagram showing Markov chain transitions with probabilities labeled on edges and vertex values at step 4]
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 5
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P_t(1, x)$. 

Step: 7
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 8
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 9
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 10
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 11
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 12
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 13
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 14
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 15

```
0.144 0.135 0.112 0.081 0.053 0.033 0.027
0.033 0.053 0.081 0.112 0.135 0.144
```

4. Markov Chains and Mixing Times © T. Sauerwald
Convergence to Stationarity (Example)

- Markov Chain: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 16
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 17
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 18
- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 20
Convergence to Stationarity (Example)

- Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 21
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 22
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

![Diagram of a Markov chain with probabilities for each step]
Convergence to Stationarity (Example)

- Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 24
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 25
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 26
Convergence to Stationarity (Example)

- **Markov Chain:** stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain:** stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 28
- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

![Graph showing the convergence to stationarity](image-url)
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 30
Convergence to Stationarity (Example)

- **Markov Chain:** stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 33
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 34
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 35
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

![Diagram](image)
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P_t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 38
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

---

[Diagram showing a circle with vertices labeled 1 through 12, each vertex has a transition probability label, and the step count is 39.]
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 41
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

\[
\begin{array}{cccccccccccc}
0.074 & & & & & & & & & & & 0.092 \\
& 0.075 & & & & & & & & & & \\
& & 0.079 & & & & & & & & & \\
& & & 0.083 & & & & & & & & \\
& & & & 0.088 & & & & & & & \\
& & & & & 0.083 & & & & & & \\
& & & & & & 0.088 & & & & & \\
& & & & & & & 0.088 & & & & \\
& & & & & & & & 0.088 & & & \\
& & & & & & & & & 0.091 & & \\
& & & & & & & & & & 0.091 & \\
& & & & & & & & & & & 0.091 \\
\end{array}
\]
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

![Diagram showing the distribution of values at different steps](image-url)
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 44
Convergence to Stationarity (Example)

- Markov Chain: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 45
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with 1/2 and moves left (or right) w.p. 1/4
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$. 

Step: 46
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.

Step: 48
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Convergence to Stationarity (Example)

- **Markov Chain**: stays put with $1/2$ and moves left (or right) w.p. $1/4$
- At step $t$ the value at vertex $x \in \{1, 2, \ldots, 12\}$ is $P^t(1, x)$.
Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)
How Similar are Two Probability Measures?

### Loaded Dice

- You are presented three loaded (unfair) dice $A, B, C$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[A = x]$</td>
<td>1/3</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>$P[B = x]$</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>$P[C = x]$</td>
<td>1/6</td>
<td>1/6</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>9/24</td>
</tr>
</tbody>
</table>

Question 1: Which dice is the least fair? Most of you choose $A$. Why?

Question 2: Which dice is the most fair? Dice $B$ and $C$ seem "fairer" than $A$ but which is fairest?
How Similar are Two Probability Measures?

### Loaded Dice

- You are presented three loaded (unfair) dice $A$, $B$, $C$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[A = x]$</td>
<td>1/3</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>$P[B = x]$</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>$P[C = x]$</td>
<td>1/6</td>
<td>1/6</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>9/24</td>
</tr>
</tbody>
</table>

- **Question 1**: Which dice is the least fair?
You are presented three loaded (unfair) dice $A$, $B$, $C$:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[A = x]$</td>
<td>1/3</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>$P[B = x]$</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>$P[C = x]$</td>
<td>1/6</td>
<td>1/6</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>9/24</td>
</tr>
</tbody>
</table>

**Question 1:** Which dice is the least fair?

**Question 2:** Which dice is the most fair?
How Similar are Two Probability Measures?

**Loaded Dice**

- You are presented three loaded (unfair) dice $A, B, C$:

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(A = x) )</td>
<td>1/3</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>( P(B = x) )</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>( P(C = x) )</td>
<td>1/6</td>
<td>1/6</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>9/24</td>
</tr>
</tbody>
</table>

- Question 1: Which dice is the least fair?
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### How Similar are Two Probability Measures?

#### Loaded Dice

- You are presented three *loaded* (unfair) dice $A$, $B$, $C$:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<tbody>
<tr>
<td>$P[A = x]$</td>
<td>1/3</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>$P[B = x]$</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>$P[C = x]$</td>
<td>1/6</td>
<td>1/6</td>
<td>1/8</td>
<td>1/8</td>
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<td>9/24</td>
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</table>

- Question 1: Which dice is the least fair?

- Question 2: Which dice is the most fair?

---

![Graph of probability distributions](image)

- $P[\cdot = x]$

- $0.16$
- $0.33$
- $0.5$

---

4. Markov Chains and Mixing Times © T. Sauerwald

Total Variation Distance and Mixing Times
How Similar are Two Probability Measures?

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</thead>
<tbody>
<tr>
<td>$\mathbb{P}[A = x]$</td>
<td>1/3</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>$\mathbb{P}[B = x]$</td>
<td>1/4</td>
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**Graph**

- $\mathbb{P}[\cdot = x]$
How Similar are Two Probability Measures?

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<tr>
<td>$P[A=x]$</td>
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<td>1/3</td>
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<td>1/8</td>
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<td>$P[C=x]$</td>
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4. Markov Chains and Mixing Times © T. Sauerwald

Total Variation Distance and Mixing Times

15
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<td>(P[B = x])</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
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<tr>
<td>(P[C = x])</td>
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<td>1/6</td>
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We need a formal “fairness measure” to compare probability distributions!
The **Total Variation Distance** between two probability distributions \( \mu \) and \( \eta \) on a countable state space \( \Omega \) is given by

\[
\| \mu - \eta \|_{tv} = \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \eta(\omega)|.
\]

Thus \( \| \mu - \eta \|_{tv} = \| \mu - \eta \|_{tv} \) and \( \| \mu - \eta \|_{tv} < \| \mu - \eta \|_{tv} \).

So \( \mu \) is the least "fair", however \( \eta \) and \( \zeta \) are equally "fair" (in TV distance).
Total Variation Distance

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Loaded Dice: let $D = \text{Unif}\{1, 2, 3, 4, 5, 6\}$ be the law of a fair dice:

$$\|D - A\|_{tv} = \frac{1}{2} \left( 2 \left| \frac{1}{6} - \frac{1}{3} \right| + 4 \left| \frac{1}{6} - \frac{1}{12} \right| \right) = \frac{1}{3}$$
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$$\|D - C\|_{tv} = \frac{1}{2} \left( 3 \left| \frac{1}{6} - \frac{1}{8} \right| + \left| \frac{1}{6} - \frac{9}{24} \right| \right) = \frac{1}{6}.$$
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Thus

$$\|D - B\|_{tv} = \|D - C\|_{tv} \quad \text{and} \quad \|D - B\|_{tv}, \|D - C\|_{tv} < \|D - A\|_{tv}.$$ 

So $A$ is the least “fair”, however $B$ and $C$ are equally “fair” (in TV distance).
Let $P$ be a finite Markov Chain with stationary distribution $\pi$. 
Let \( P \) be a finite Markov Chain with stationary distribution \( \pi \).

- Let \( \mu \) be a prob. vector on \( \Omega \) (might be just one vertex) and \( t \geq 0 \). Then

\[
P^t_{\mu} := \mathbb{P} \left[ X_t = \cdot \mid X_0 \sim \mu \right],
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- For any $\mu$, 

$$\|P^t_\mu - \pi\|_{tv} \leq \max_{x \in \Omega} \|P^t_x - \pi\|_{tv}.$$

Convergence Theorem (Implication for TV Distance)

We will see a similar result later after introducing spectral techniques!
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For any finite, irreducible, aperiodic Markov Chain

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Question: How fast do they converge?
Mixing Time of a Markov Chain

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The Mixing time $\tau_x(\epsilon)$ of a finite Markov Chain $P$ with stationary distribution $\pi$ is defined as

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\tau_x(\epsilon) = \min \left\{ t : \left\| P^t_x - \pi \right\|_{tv} \leq \epsilon \right\},
$$

This is how long we need to wait until we are "$\epsilon$-close" to stationarity. We often take $\epsilon = 1/4$, indeed let $t_{mix} := \tau(1/4)$. 

Mixing Time and Convergence - Total Variation Distance and Mixing Times

4. Markov Chains and Mixing Times © T. Sauerwald
Mixing Time of a Markov Chain

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Further Remarks on the Mixing Time (non-examinable)

- One can prove $\max_x \| P^t_x - \pi \|_{tv}$ is non-increasing in $t$ (this means if the chain is $\epsilon$-mixed at step $t$, then this also holds in future steps) \[Mitzenmacher, Upfal, 12.3\]

We chose $t_{mix} = \tau(\frac{1}{4})$, but other choices of $\epsilon$ are perfectly fine too (e.g., $t_{mix} = \tau(\frac{1}{e})$ is often used); in fact, any constant $\epsilon \in (0, \frac{1}{2})$ is possible.

Remark: This freedom on how to pick $\epsilon$ relies on the sub-multiplicative property of a (version) of the variation distance. First, let $d(t) := \max_x \| P^t_x - \pi \|_{tv}$ be the variation distance after $t$ steps when starting from the worst state. Further, define $d(t) := \max_{\mu, \nu} \| P^t \mu - P^t \nu \|_{tv}$. These quantities are related by the following double inequality $d(t) \leq d(t) \leq 2d(t)$. Further, $d(t)$ is sub-multiplicative, that is for any $s, t \geq 1$, $d(s + t) \leq d(s) \cdot d(t)$. Hence for any fixed $0 < \epsilon < \frac{1}{2}$ it follows from the above that $\tau(\epsilon) \leq \lceil \frac{\ln \epsilon}{\ln(2\delta)} \rceil \tau(\delta)$. In particular, for any $\epsilon < \frac{1}{4} \tau(\epsilon) \leq \lceil \log_2 \frac{1}{\epsilon} - 1 \rceil \tau(1/4)$. This 2 is the reason why we ultimately need $\epsilon < \frac{1}{2}$ in this derivation. On the other hand, see [Exercise (4/5).8] why $\epsilon < \frac{1}{2}$ is also necessary. Hence smaller constants $\epsilon < \frac{1}{4}$ only increase the mixing time by some constant factor.
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These quantities are related by the following double inequality

\[d(t) \leq \overline{d}(t) \leq 2d(t).\]

Further, $\overline{d}(t)$ is sub-multiplicative, that is for any $s, t \geq 1$,

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Hence for any fixed $0 < \varepsilon < \delta < 1/2$ it follows from the above that

\[\tau(\varepsilon) \leq \left\lceil \frac{\ln \varepsilon}{\ln(2\delta)} \right\rceil \tau(\delta).\]

In particular, for any $\varepsilon < 1/4$

\[\tau(\varepsilon) \leq \left\lfloor \log_2 \varepsilon^{-1} \right\rfloor \tau(1/4).\]
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- One can prove $\max_x \| P_t^x - \pi \|_{tv}$ is non-increasing in $t$ (this means if the chain is $\epsilon$-mixed at step $t$, then this also holds in future steps) \[\text{[Mitzenmacher, Upfal, 12.3]}\]

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Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Card Shuffling

Application 2: Markov Chain Monte Carlo (non-examin.)
Experiment Gone Wrong...

Distribution of first 300 drawings of Polish Multilotek

Thanks to Krzysztof Onak (pointer) and Eric Price (graph)

Source: Slides by Ronitt Rubinfeld
What is Card Shuffling?

Source: wikipedia

How long does it take to shuffle a deck of 52 cards?

Source: www.soundcloud.com

His research revealed beautiful connections between Markov Chains and Algebra.

Here we will focus on one shuffling scheme which is easy to analyse.

How quickly do we converge to the uniform distribution over all $n!$ permutations?
What is Card Shuffling?

How long does it take to shuffle a deck of 52 cards?

Persi Diaconis (Professor of Statistics and former Magician)

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The Card Shuffling Markov Chain

TOPTORANDOMSHUFFLE (Input: A pile of $n$ cards)

1: For $t = 1, 2, \ldots$
2: Pick $i \in \{1, 2, \ldots, n\}$ uniformly at random
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This is a slightly informal definition, so let us look at a small example...
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This is a slightly informal definition, so let us look at a small example...

We will focus on this “small” set of cards ($n = 8$)
Even if we know which set of cards come after 8, every permutation is equally likely! The deck of cards is perfectly mixed after the last card "8" reaches the top and is inserted to a random position!
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\[ \sim \text{the deck of cards is perfectly mixed after the last card "8" reaches the top and is inserted to a random position!} \]
Analysing the Mixing Time (Intuition)

A deck of cards is perfectly mixed after the last card “8” reaches the top and is inserted to a random position!

How long does it take for the last card “8” to become top card?

At the last position, card “8” moves up with probability $\frac{1}{n}$ at each step.

At the second last position, card “8” moves up with probability $\frac{2}{n}$...

At the second position, card “8” moves up with probability $\frac{n-1}{n}$.

One final step to randomise card “8” (with probability 1).

This is a “reversed” coupon collector process with $n$ cards, which takes $n \log n$ in expectation.

Using the so-called coupling method, one could prove $t_{\text{mix}} \leq n \log n$. 
How long does it take for the last card “$n$” to become top card?
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The diagram shows a deck of cards being mixed. The topmost card is “8,” and it is randomly inserted into a new deck after being at the bottom. This process continues until the entire deck is mixed.
How long does it take for the last card “$n$” to become top card?

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Using the so-called coupling method, one could prove \( t_{mix} \leq n \log n \).
Riffle Shuffle

1. Split a deck of \( n \) cards into two piles (thus the size of each portion will be \( \text{Binomial} \)).

2. Riffle the cards together so that the card drops from the left (or right) pile with probability proportional to the number of remaining cards.

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<thead>
<tr>
<th>( t )</th>
<th>1.000</th>
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Figure: Total Variation Distance for \( t \) riffle shuffles of 52 cards.
1. **Split** a deck of \( n \) cards into two piles (thus the size of each portion will be **Binomial**).
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Outline

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Application 2: Markov Chain Monte Carlo (non-examin.)
Given an undirected graph $G = (V, E)$, an independent set is a subset $S \subseteq V$ such that there are no two vertices $u, v \in S$ with $\{u, v\} \in E(G)$. 

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**Independent Set**

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$S = \{1, 4\}$ is an independent set

How can we take a sample from the space of all independent sets? Naive brute-force would take an insane amount of time (and space)!

We can use a generic Markov Chain Monte Carlo approach to tackle this problem!
A Markov Chain for Sampling Independent Sets (1/2)

$S = \{2, 6, 8\}$ is an independent set

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A Markov Chain for Sampling Independent Sets (1/2)

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\[ S = \{1, 7, 8\} \text{ is not an independent set} \times \]
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A Markov Chain for Sampling Independent Sets (2/2)

**INDEPENDENTSAMPLER**

1: Let $X_0$ be an arbitrary independent set in $G$
2: For $t = 0, 1, 2, \ldots$
3: Pick a vertex $v \in V(G)$ uniformly at random
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Key Question: What is the mixing time of this Markov Chain? not covered here, see the textbook by Mitzenmacher and Upfal
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$X_0 = \{1, 4\}$

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4. Markov Chains and Mixing Times © T. Sauerwald Application 2: Markov Chain Monte Carlo (non-examin.)
A Markov Chain for Sampling Independent Sets (2/2)

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**Diagram:**

- $X_0 = \{1, 4\}$
- $v = 1$

---

4. Markov Chains and Mixing Times © T. Sauerwald  
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**Key Question:** What is the mixing time of this Markov Chain?

The algorithm is illustrated with a graph and examples of the state sequence $X_0, X_1, X_2$ for different choices of $v$. The graph shows vertices and edges, with one vertex being added to the independent set at each step.
A Markov Chain for Sampling Independent Sets (2/2)

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A Markov Chain for Sampling Independent Sets (2/2)

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Remark

- This is a local definition (no explicit definition of $P$!)

This chain is irreducible (every independent set is reachable)
This chain is aperiodic (Check!)
The stationary distribution is uniform, since

Remark

Key Question: What is the mixing time of this Markov Chain?

not covered here, see the textbook by Mitzenmacher and Upfal
**INDEPENDENT SET Sampler**

1: Let \( X_0 \) be an arbitrary independent set in \( G \)

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A Markov Chain for Sampling Independent Sets (2/2)

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- The stationary distribution is uniform, since $P_{u,v} = P_{v,u}$ (Check!)
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**Key Question:** What is the mixing time of this Markov Chain?
A Markov Chain for Sampling Independent Sets (2/2)

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