Randomised Algorithms
Lecture 2: Concentration Inequalities, Application to Balls-into-Bins

Thomas Sauerwald (tms41@cam.ac.uk)
All Graphs (Worst-Case)

- “naive” randomised algorithm from the first lecture
  - achieves approximation ratio of 2, that is \( \mathbb{E}[e_{\text{opt}}] \leq 2 \)
  - further results on the distribution of \( e(S, S^c) \) [Question 1.4,1.5]
  - not too hard to derandomise the algorithm [Question 1.3]
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- “more clever” randomised algorithm
  - combines the ideas of linear programming, randomised rounding (but also semi-definite programming)
  - achieves approximation ratio of \( \frac{1}{0.878} \approx 1.14 \) [book by Shmoys, Williamson]
More Remarks on MAX-CUT & Related Work (non-examinable)

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- “naive” randomised algorithm from the first lecture
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Special Graphs

- If $G$ is a random graph with edge probability $1/2$, then the naive algorithm achieves approximation ratio of $1 + o(1)$ [Question 2.9]

- For any $\epsilon > 0$, there is a randomised algorithm with running time $O(n^2)2^{O(1/\epsilon^2)}$ with $\mathbb{E}[e(S, S^c)] \geq e_{opt} - O(\epsilon n^2)$ [Mathieu, Schudy: “Yet Another Algorithm for Dense Max Cut: Go Greedy”, SODA’2008, pages 176–182]
How to Derive Chernoff Bounds

Application 1: Balls into Bins
General Recipe for Deriving Chernoff Bounds

Recipe

The three main steps in deriving Chernoff bounds for sums of independent random variables $X = X_1 + \cdots + X_n$ are:

1. Instead of working with $X$, we switch to the moment generating function $e^{\lambda X}$, $\lambda > 0$ and apply Markov's inequality $\mathbb{E}[e^{\lambda X}]$.
2. Compute an upper bound for $\mathbb{E}[e^{\lambda X}]$ (using independence).
3. Optimise value of $\lambda$ to obtain best tail bound.
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2. Compute an upper bound for $E[e^{\lambda X}]$ (using independence).
3. Optimise value of $\lambda$ to obtain best tail bound.
Chernoff Bound: Proof

Chernoff Bound (General Form, Upper Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then, for any $\delta > 0$ it holds that

$$
P[X \geq (1 + \delta)\mu] \leq \left[ \frac{e^{\delta}}{(1 + \delta)^{(1+\delta)}} \right]^\mu.
$$

Proof:
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Proof:

1. For $\lambda > 0$,

$$
\mathbb{P}[X \geq (1 + \delta)\mu] \leq \mathbb{P}[e^{\lambda X} \geq e^{\lambda (1+\delta)\mu}] \leq e^{-\lambda (1+\delta)\mu} \mathbb{E}\left[e^{\lambda X}\right]
$$

   $e^{\lambda X}$ is incr

   Markov
Chernoff Bound: Proof

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$$\mathbb{P}[X \geq (1 + \delta)\mu] \leq \left[\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right]^\mu.$$

Proof:

1. For $\lambda > 0$,

$$\mathbb{P}[X \geq (1 + \delta)\mu] \leq \mathbb{P}[e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}] \leq e^{-\lambda(1+\delta)\mu} \mathbb{E}[e^{\lambda X}]$$

2. $\mathbb{E}[e^{\lambda X}] = \mathbb{E}\left[e^{\lambda \sum_{i=1}^n X_i}\right] = \prod_{i=1}^n \mathbb{E}[e^{\lambda X_i}]$
Chernoff Bound: Proof

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = E[X] = \sum_{i=1}^{n} p_i$. Then, for any $\delta > 0$ it holds that

$$P[X \geq (1 + \delta)\mu] \leq \left[ \frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right]^\mu.$$

Proof:

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2. $E[e^{\lambda X}] = E[e^{\lambda \sum_{i=1}^{n} X_i}] = \prod_{i=1}^{n} E[e^{\lambda X_i}].$

3. $E[e^{\lambda X_i}] = e^\lambda p_i + (1 - p_i) = 1 + p_i(e^\lambda - 1) \leq e^{p_i(e^\lambda - 1)}$ when $1 + x \leq e^x$. 
Chernoff Bound: Proof

1. For $\lambda > 0$, 
$$P[X \geq (1 + \delta) \mu] = e^{\lambda x} \text{ is incr}.$$ 

2. 
$$E[e^{\lambda X}] = E[\prod_{i=1}^{n} e^{\lambda X_i}] = \text{indep} \prod_{i=1}^{n} E[e^{\lambda X_i}]$$

3. 
$$E[e^{\lambda X_i}] = e^{\lambda p_i} + (1 - p_i) = 1 + p_i (e^{\lambda} - 1) \leq e^{\lambda x} \leq e^{\lambda p_i} (e^{\lambda} - 1)$$

4. Putting all together 
$$P[X \geq (1 + \delta) \mu] \leq e^{\lambda (1 + \delta) \mu} \prod_{i=1}^{n} e^{\lambda p_i} (e^{\lambda} - 1) = e^{\lambda (1 + \delta) \mu} e^{\lambda \mu} (e^{\lambda} - 1)$$

5. Choose $\lambda = \log(1 + \delta) > 0$ to get the result.
Chernoff Bound: Proof

1. For $\lambda > 0$,

$$\Pr[ X \geq (1 + \delta)\mu ] = \Pr[ e^{\lambda X} \geq e^{\lambda (1 + \delta)\mu} ] \leq e^{-\lambda (1 + \delta)\mu} \mathbb{E}[ e^{\lambda X} ]$$

2. $\mathbb{E}[ e^{\lambda X} ] = \mathbb{E}[ e^{\lambda \sum_{i=1}^{n} X_i} ] = \prod_{i=1}^{n} \mathbb{E}[ e^{\lambda X_i} ]$

3. $\mathbb{E}[ e^{\lambda X_i} ] = e^{\lambda p_i} + (1 - p_i) = 1 + p_i(e^{\lambda} - 1) \leq e^{p_i(e^{\lambda} - 1)}$
1. For $\lambda > 0$,
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P[ X \geq (1 + \delta)\mu ] = \mathbb{P}[ e^{\lambda X} \geq e^{\lambda(1 + \delta)\mu} ] \leq e^{-\lambda(1 + \delta)\mu} \mathbb{E}[ e^{\lambda X} ]
\]

2. $\mathbb{E}[ e^{\lambda X} ] = \mathbb{E}[ e^{\lambda \sum_{i=1}^{n} X_i} ] = \prod_{i=1}^{n} \mathbb{E}[ e^{\lambda X_i} ]$

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P[ X \geq (1 + \delta)\mu ] \leq e^{-\lambda(1 + \delta)\mu} \prod_{i=1}^{n} e^{p_i(e^{\lambda} - 1)} = e^{-\lambda(1 + \delta)\mu} e^{\mu(e^{\lambda} - 1)}
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$$

2. \( \mathbf{E} \left[ e^{\lambda X} \right] = \mathbf{E} \left[ e^{\lambda \sum_{i=1}^{n} X_i} \right] = \prod_{i=1}^{n} \mathbf{E} \left[ e^{\lambda X_i} \right] \)

3. \( \mathbf{E} \left[ e^{\lambda X_i} \right] = e^{\lambda p_i + (1 - p_i)} = 1 + p_i(e^{\lambda} - 1) \leq e^{p_i(e^{\lambda} - 1)} \)

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P \left[ X \geq (1 + \delta) \mu \right] \leq e^{-\lambda (1 + \delta) \mu} \prod_{i=1}^{n} e^{p_i(e^{\lambda} - 1)} = e^{-\lambda (1 + \delta) \mu} e^{\mu(e^{\lambda} - 1)}
$$

5. Choose $\lambda = \log(1 + \delta) > 0$ to get the result.
We can also use Chernoff Bounds to show a random variable is **not too small** compared to its mean:

Suppose \( X_1, \ldots, X_n \) are independent Bernoulli random variables with parameter \( p_i \). Let \( X = X_1 + \ldots + X_n \) and \( \mu = E[X] = \sum_{i=1}^{n} p_i \). Then, for any \( 0 < \delta < 1 \) it holds that

\[
P[X \leq (1 - \delta)\mu] \leq \left[ \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right]^\mu,
\]

and thus, by substitution, for any \( t < \mu \),

\[
P[X \leq t] \leq e^{-\mu} \left( \frac{e^\mu}{t} \right)^t.
\]

**Exercise on Supervision Sheet**

**Hint:** multiply both sides by \(-1\) and repeat the proof of the Chernoff Bound
Nicer Chernoff Bounds

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = E[X] = \sum_{i=1}^{n} p_i$. Then,

For all $t > 0$,

$$P[X \geq E[X] + t] \leq e^{-\frac{2t^2}{n}}$$

$$P[X \leq E[X] - t] \leq e^{-\frac{2t^2}{n}}$$

For $0 < \delta < 1$,

$$P[X \geq (1 + \delta)E[X]] \leq \exp\left(-\delta^2 E[X]^3\right)$$

$$P[X \leq (1 - \delta)E[X]] \leq \exp\left(-\delta^2 E[X]^2\right)$$

"Nicer" Chernoff Bounds

All upper tail bounds hold even under a relaxed independence assumption:

For all $1 \leq i \leq n$ and $x_1, x_2, \ldots, x_{i-1} \in \{0, 1\}$,

$$P[X_i = 1 | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq p_i.$$
Nicer Chernoff Bounds

“Nicer” Chernoff Bounds

Suppose \( X_1, \ldots, X_n \) are independent Bernoulli random variables with parameter \( p_i \). Let \( X = X_1 + \ldots + X_n \) and \( \mu = \mathbb{E}[X] = \sum_{i=1}^n p_i \). Then,
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  \]

- For $0 < \delta < 1$,
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P[X \geq (1 + \delta)\mathbb{E}[X]] \leq \exp\left(-\frac{\delta^2\mathbb{E}[X]}{3}\right)
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  \[
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How to Derive Chernoff Bounds

Application 1: Balls into Bins
Balls into Bins

You have $m$ balls and $n$ bins. Each ball is allocated in a bin picked independently and uniformly at random.
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  1. Bins are a hash table, balls are items
  2. Bins are processors and balls are jobs
  3. Bins are data servers and balls are queries
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  2. Bins are processors and balls are jobs
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**Exercise:** Think about the relation between the Balls into Bins Model and the Coupon Collector Problem.
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**Balls into Bins Model**

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- Focus on an arbitrary single bin. Let $X_i$ the indicator variable which is 1 iff ball $i$ is assigned to this bin. Note that $p_i = \mathbb{P}[X_i = 1] = 1/n$. 

By the Chernoff Bound, 

$$
\mathbb{P}[X \geq 6 \log n] \leq e^{-2 \log n \left( 2e \log n / 6 \log n \right)} = e^{-2} = n^{-2}.
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$$P[X \geq t] \leq e^{-\mu} (e^{\mu/t})^t$$

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We are interested in the probability that at least one bin receives at least $6 \log n$ balls $\Rightarrow$ this is the event $\bigcup_{j=1}^n \mathcal{E}_j$.

By the Union Bound,

$$
\Pr\left(\bigcup_{j=1}^n \mathcal{E}_j\right) \leq \sum_{j=1}^n \Pr(\mathcal{E}_j) \leq n \cdot n^{-2} = n^{-1}.
$$

Therefore whp, no bin receives at least $6 \log n$ balls.

By pigeonhole principle, the max loaded bin receives at least $2 \log n$ balls.

Hence our bound is pretty sharp.

whp stands for with high probability:

An event $E$ (that implicitly depends on an input parameter $n$) occurs whp if $\Pr(E) \to 1$ as $n \to \infty$.

This is a very standard notation in randomised algorithms but it may vary from author to author. Be careful!
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Balls into Bins: Bounding the Maximum Load (2/4)

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This is a very standard notation in randomised algorithms but it may vary from author to author. Be careful!
Question 2: How large is the maximum load if \( m = n \)?

Using the Chernoff Bound:

\[
P\left[X \geq t\right] \leq e^{-\frac{1}{2} \left(\frac{e^t}{t}\right)}
\]

By setting \( t = 4 \log n / \log \log n \), we claim to obtain

\[
P\left[X \geq t\right] \leq \frac{n}{2}
\]

Indeed:

\[
\left(\frac{e^{\log \log n}}{4 \log n}\right)^{4 \log n / \log \log n} = e^{\frac{4 \log n \log \log n}{\log \log n} \cdot \log\left(\frac{e^{\log \log n}}{4 \log n}\right)}
\]

The term inside the exponential is

\[
4 \log n \log \log n \cdot \left(-\frac{1}{2} \log \log n\right)
\]

obtaining that

\[
P\left[X \geq t\right] \leq \frac{n}{2}
\]

This inequality only works for large enough \( n \).
Question 2: How large is the maximum load if $m = n$?

- Using the Chernoff Bound:

$$\mathbb{P}[X \geq t] \leq e^{-\mu}(e\mu/t)^t$$

$$\mathbb{P}[X \geq t] \leq e^{-1} \left(\frac{e}{t}\right)^t \leq \left(\frac{e}{t}\right)^t$$
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  $$P[X \geq t] \leq e^{-1} \left( \frac{e}{t} \right)^t \leq \left( \frac{e}{t} \right)^t$$

- By setting $t = 4 \log n / \log \log n$, we claim to obtain $P[X \geq t] \leq n^{-2}$. 
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- Using the Chernoff Bound:

\[
P[X \geq t] \leq e^{-t} \left( \frac{e}{t} \right)^{t} \leq \left( \frac{e}{t} \right)^{t}
\]

- By setting \( t = 4 \log n / \log \log n \), we claim to obtain

\[
P[X \geq t] \leq n^{-2}.
\]

- Indeed:

\[
\left( \frac{e \log \log n}{4 \log n} \right)^{4 \log n / \log \log n} = \exp \left( \frac{4 \log n}{\log \log n} \cdot \log \left( \frac{e \log \log n}{4 \log n} \right) \right)
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Question 2: How large is the maximum load if \( m = n \)?

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P[X \geq t] \leq e^{-\mu (e \mu / t)^t}
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By setting \( t = 4 \log n / \log \log n \), we claim to obtain \( P[X \geq t] \leq n^{-2} \).

Indeed:
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\left(\frac{e \log \log n}{4 \log n}\right)^{4 \log n / \log \log n} = \exp \left( \frac{4 \log n}{\log \log n} \cdot \log \left( \frac{e \log \log n}{4 \log n} \right) \right)
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The term inside the exponential is
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\frac{4 \log n}{\log \log n} \cdot (\log(e/4) + \log \log \log n - \log \log n)
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    obtaining that \( P[X \geq t] \leq n^{-4/2} = n^{-2} \). This inequality only works for large enough \( n \).
We just proved that
\[ P \left[ X \geq 4 \log n / \log \log n \right] \leq n^{-2}, \]
thus by the Union Bound, no bin receives more than \( \Omega \left( \log n / \log \log n \right) \) balls with probability at least \( 1 - 1/n \). \qed
Conclusions

- If the number of balls is $2 \log n$ times $n$ (the number of bins), then to distribute balls at random is a good algorithm.
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⇒ for $m = n$ this gives a maximum load of $\log_2 \log n + \Theta(1)$ w.p. $1 - 1/n$.

This is called the **power of two choices**: It is a common technique to improve the performance of randomised algorithms (covered in Chapter 17 of the textbook by Mitzenmacher and Upfal).
For “the discovery and analysis of balanced allocations, known as the power of two choices, and their extensive applications to practice.”

“These include i-Google’s web index, Akamai’s overlay routing network, and highly reliable distributed data storage systems used by Microsoft and Dropbox, which are all based on variants of the power of two choices paradigm. There are many other software systems that use balanced allocations as an important ingredient.”
Simulation

https://www.dimitrioslos.com/balls_and_bins/visualiser.html