# Randomised Algorithms

Lecture 2: Concentration Inequalities, Application to Balls-into-Bins

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Lent 2023

UNIVERSITY OF CAMBRIDGE

# More Remarks on MAX-CUT & Related Work (non-examinable)

## All Graphs (Worst-Case)

- "naive" randomised algorithm from the first lecture
  - achieves approximation ratio of 2, that is $\frac{e_{opt}}{\mathbf{E}[\, e(S,S^c)\, ]} \leq 2$
  - further results on the distribution of $e(S, S^c)$ *[Question 1.4,1.5]*
  - not too hard to derandomise the algorithm *[Question 1.3]*

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## Special Graphs

- If $G$ is a random graph with edge probability $1/2$, then the naive algorithm achieves approximation ratio of $1 + o(1)$ *[Question 2.9]*
- For any $\epsilon > 0$, there is a randomised algorithm with running time $O(n^2)2^{O(1/\epsilon^2)}$ with $\mathbf{E}\,[\,e(S,S^c)\,] \geq e_{opt} - O(\epsilon n^2)$
  *[Mathieu, Schudy: "Yet Another Algorithm for Dense Max Cut: Go Greedy", SODA'2008, pages 176–182]*

# Outline

How to Derive Chernoff Bounds

Application 1: Balls into Bins

# General Recipe for Deriving Chernoff Bounds

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3. Optimise value of $\lambda$ to obtain best tail bound

# Chernoff Bound: Proof

Chernoff Bound (General Form, Upper Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^{n} p_i$. Then, for any $\delta > 0$ it holds that

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5. Choose $\lambda = \log(1+\delta) > 0$ to get the result.

# Chernoff Bounds: Lower Tails

We can also use Chernoff Bounds to show a random variable is **not too small** compared to its mean:

Chernoff Bounds (General Form, Lower Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^n p_i$. Then, for any $0 < \delta < 1$ it holds that

$$\mathbf{P}[X \leq (1-\delta)\mu] \leq \left[\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu},$$

and thus, by substitution, for any $t < \mu$,

$$\mathbf{P}[X \leq t] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^{t}.$$

**Exercise on Supervision Sheet**

Hint: multiply both sides by $-1$ and repeat the proof of the Chernoff Bound

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- For all $t > 0$,

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All upper tail bounds hold even under a relaxed independence assumption: For all $1 \leq i \leq n$ and $x_1, x_2, \ldots, x_{i-1} \in \{0, 1\}$,

$$\mathbf{P}[X_i = 1 \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq p_i.$$

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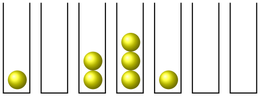
# Balls into Bins



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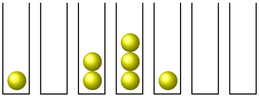

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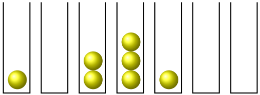

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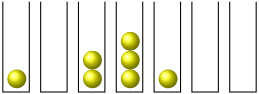

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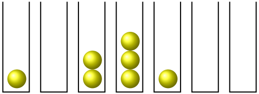

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**Exercise:** Think about the relation between the Balls into Bins Model and the Coupon Collector Problem.

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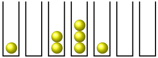

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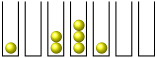

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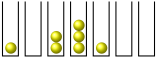

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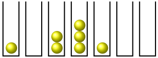

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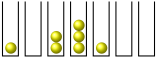

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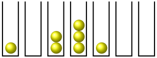

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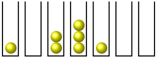

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here we could have used the "nicer" bounds as well!

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- Therefore whp, no bin receives at least $6 \log n$ balls

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- Therefore whp, no bin receives at least $6 \log n$ balls

whp stands for *with high probability*:
An event $\mathcal{E}$ (that implicitly depends on an input parameter $n$) occurs whp if
$\mathbf{P}\left[\mathcal{E}\right] \to 1$ as $n \to \infty$.
This is a very standard notation in randomised algorithms
but it may vary from author to author. Be careful!

# Balls into Bins: Bounding the Maximum Load (2/4)

- Let $\mathcal{E}_j := \{X(j) \geq 6 \log n\}$, that is, bin $j$ receives at least $6 \log n$ balls.
- We are interested in the probability that at least one bin receives at least $6 \log n$ balls $\Rightarrow$ this is the event $\bigcup_{j=1}^{n} \mathcal{E}_j$
- By the Union Bound,

$$\mathbf{P}\left[\bigcup_{j=1}^{n} \mathcal{E}_j\right] \leq \sum_{j=1}^{n} \mathbf{P}\left[\,\mathcal{E}_j\,\right] \leq n \cdot n^{-2} = n^{-1}.$$

- Therefore whp, no bin receives at least $6 \log n$ balls
- By pigeonhole principle, the max loaded bin receives at least $2 \log n$ balls. Hence our bound is pretty sharp.

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obtaining that $\mathbf{P}[\,X \geq t\,] \leq n^{-4/2} = n^{-2}$.

# Balls into Bins: Bounding the Maximum Load (4/4)

We just proved that

$$\mathbf{P}\left[\, X \geq 4 \log n / \log\log n \,\right] \leq n^{-2},$$

thus by the Union Bound, no bin receives more than $\Omega\left(\log n / \log\log n\right)$ balls with probability at least $1 - 1/n$. $\square$

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$\Rightarrow$ for $m = n$ this gives a maximum load of $\log_2 \log n + \Theta(1)$ w.p. $1 - 1/n$.

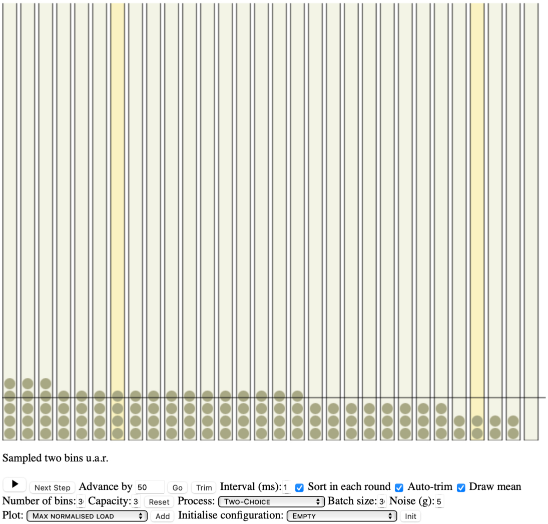
This is called the **power of two choices**: It is a common technique to improve the performance of randomised algorithms (covered in Chapter 17 of the textbook by Mitzenmacher and Upfal)

# ACM Paris Kanellakis Theory and Practice Award 2020


*For "the discovery and analysis of balanced allocations, known as the power of two choices, and their extensive applications to practice."*

*"These include i-Google's web index, Akamai's overlay routing network, and highly reliable distributed data storage systems used by Microsoft and Dropbox, which are all based on variants of the power of two choices paradigm. There are many other software systems that use balanced allocations as an important ingredient."*

# Simulation

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